EFFECTS OF TURBULENT MIXING ON THE CRITICAL BEHAVIOR

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Effects of strongly anisotropic turbulent mixing on the critical behavior are studied by means of the renormalization group. Existence of new nonequilibrium types of critical regimes (universality classes) is established.

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Various systems of very different physical nature exhibit interesting singular behavior in the vicinity of their critical points. Their correlation functions reveal self-similar behavior with universal critical dimensions: they depend only on few global characteristics of the system (like symmetry or space dimensionality). Quantitative description of critical behavior is provided by the renormalization group (RG). In the RG approach, possible types of critical regimes (universality classes) are associated with infrared (IR) attractive fixed points of renormalizable field theoretic models. Most typical phase transitions belong to the universality class of the O_n-symmetric ψ^4 model of an n-component scalar order parameter. Universal characteristics of the critical behavior depend only on n and the space dimensionality d and can be calculated in the form of the expansion in $\varepsilon = 4-d$ or within other systematic perturbation schemes; see the monograph [1] and the literature cited therein.

It has long been realized that the behavior of a real system near its critical point is extremely sensitive to external disturbances, geometry of the experimental setup, gravity, presence of impurities and so on; see e.g. the monograph [2] for the general discussion and the references. What is more, some disturbances (randomly distributed impurities or turbulent mixing) can produce completely new types of critical behavior with rich and rather exotic properties, like e.g. expansion in $\sqrt{\varepsilon}$ rather than in ε; see e.g. [3].

These issues become even more important for the nonequilibrium phase transitions, because the ideal conditions of a "pure" stationary critical state can hardly be achieved in real chemical or biological systems, and the effects of various disturbances can never be completely excluded. In particular, intrinsic turbulence effects can hardly be avoided in chemical catalytic reactions or forest fires. One can also speculate that atmospheric turbulence can play important role for the spreading of an infectious disease by flying insects or birds. Effects of different kinds of turbulent and laminar motion on the critical behavior were studied e.g. in [3]–[9].

In this paper we study effects of strongly anisotropic turbulent mixing on the critical behavior of two paradigmatic models: the equilibrium model A, which describes purely relaxational dynamics of a nonconserved scalar order parameter (see e.g. [1]), and the Gribov model, which describes the nonequilibrium phase transition between the absorbing and fluctuating states in a reaction-diffusion system [10]. The velocity is modelled by the d-dimensional generalization of the random shear flow introduced in [11] within the context of passive scalar advection.

In the Langevin formulation the models are defined by stochastic differential equations for the order parameter $\psi = \psi(t, \mathbf{x})$:

$$\partial_t \psi = \lambda \left\{ (-\tau + \partial^2)\psi - V(\psi) \right\} + \zeta = 0, \tag{1}$$

where $\partial_t = \partial/\partial t$, ∂^2 is the Laplace operator, $\lambda > 0$ is the kinematic (diffusion) coefficient and $\tau \propto (T - T_c)$ is the deviation of the temperature (or its analog) from the critical value. The nonlinearity has the form $V(\psi) = u\psi^3/3!$ for the model A and $V(\psi) = g\psi^2/2$ for the Gribov process; g and $u > 0$ being the coupling constants. The Gaussian random noise $\zeta = \zeta(t, \mathbf{x})$ with zero mean is specified by the pair correlation function:

$$\langle \zeta(t, \mathbf{x})\zeta(t', \mathbf{x}')\rangle = 2\lambda\delta(t - t')\delta^{(d)}(\mathbf{x} - \mathbf{x}') \tag{2}$$

for the model A and

$$\langle \zeta(t, \mathbf{x})\zeta(t', \mathbf{x}')\rangle = g\lambda\, \psi(t, \mathbf{x})\, \delta(t - t')\delta^{(d)}(\mathbf{x} - \mathbf{x}') \tag{3}$$

for the Gribov process. The factor ψ in front of the correlator (1) guarantees that in the absorbing state the fluctuations cease entirely, while the factor 2λ in (2) ensures the correspondence to the static ψ^4 model.

Coupling with the velocity field $\mathbf{v} = \{v_i(t, \mathbf{x})\}$ is introduced by the replacement $\partial_t \to \nabla_t = \partial_t + v_i\partial_i$, where ∇_t is the Lagrangian (Galilean covariant) derivative.

Let $\mathbf{n}$ be a unit constant vector that determines distinguished direction ("direction of the flow"). Then any vector can be decomposed into the components perpendicular and parallel to the flow, for example, $\mathbf{x} = \mathbf{x}_\perp + \mathbf{n}x_\parallel$ with $\mathbf{x}_\perp \cdot \mathbf{n} = 0$. The velocity field will be taken in the form $\mathbf{v} = \mathbf{n}v(t, \mathbf{x}_\perp)$, where $v(t, \mathbf{x}_\perp)$ is a scalar function independent of $x_\parallel$. Then the incompressibility condition $\partial_i v_i = \partial_\parallel v(t, \mathbf{x}_\perp) = 0$ is automatically satisfied.

For $v(t, \mathbf{x}_\perp)$ we assume a Gaussian distribution with zero mean and the pair correlation function of the form:

$$\langle v(t, \mathbf{x}_\perp)v(t', \mathbf{x}'_\perp)\rangle = D\, \delta(t - t') \int \frac{d\mathbf{k}_\perp}{(2\pi)^{d-1}} \exp\left\{ i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)\right\} \frac{1}{k_\perp^{d-1+\xi}}. \tag{4}$$

Here $k_\perp = |\mathbf{k}_\perp|$, $D > 0$ is a constant amplitude factor and ξ an arbitrary exponent, which (along with the conventional $\varepsilon = 4 - d$) will play the part of a formal RG expansion parameter. The IR regularization is provided by the cutoff $k_\perp > m$ in (4). The natural interval for the exponent is $0 < \xi < 2$ with the most realistic Kolmogorov value $\xi = 4/3$; the "Batchelor limit" $\xi \to 2$ corresponds to smooth velocity.

According to the general theorem (see e.g. [1]), stochastic problems described above can be reformulated as field theoretic models of extended set of fields $\Phi = \{\psi, \psi^\dagger, \mathbf{v}\}$. The role of the coupling constants is played by the parameters

$$u, g^2 \sim \Lambda^{4-d}, \quad w = D/\lambda \sim \Lambda^\xi, \tag{5}$$

where Λ is some typical ultraviolet (UV) momentum scale. From (5) it follows that the both models become logarithmic (all the coupling constants are simultaneously dimensionless) at $d = 4$ and $\xi = 0$. Thus the UV divergences in the Green functions manifest themselves as poles in $\varepsilon = 4 - d$, ξ and, in general, their linear combinations. The divergences can be removed by the standard renormalization procedure. In order to ensure *multiplicative* renormalizability of the model, it is necessary to split the Laplacian in (1) into the parallel and perpendicular parts $\partial^2 \to \partial_\perp^2 + f\partial_\parallel^2$ by introducing a new parameter $f > 0$ (in the anisotropic case, these two terms are renormalized in a different way). Then the corresponding RG equations are derived; their IR attractive fixed points determine the IR asymptotic scaling regimes of the models. Detailed analysis of the model A can be found in [8] and for the Gribov model it will be given elsewhere. Below we only present the results of the explicit one-loop calculation (leading order in ξ and ε), which appear similar for the both models. There are four fixed points:

1) Gaussian (free) fixed point: $g_* = u_* = w_* = 0$, IR attractive for $\varepsilon < 0$, $\xi < 0$.

2) $g_* = u_* = 0$ (exact result to all orders), $w_* \sim \xi$; IR attractive for $\xi > 0$ and $\xi > 2\varepsilon$. In this regime, the nonlinearity $V(\psi)$ in the stochastic equation (1) becomes irrelevant, and we arrive at the model of a linear convection-diffusion equation for a passive scalar field ψ.

3) $w_* = 0$ (exact), $g_*^2 \sim u_* \sim \varepsilon$; IR attractive for $\varepsilon > 0$, $\xi < 0$ for the model A and $\varepsilon > 0$, $\xi < \varepsilon/12$ for the Gribov model. In this regime, effects of the turbulent mixing are irrelevant, the isotropy violated by the velocity ensemble is restored and the leading terms of the IR behavior coincide exactly with those of the standard models (1)–(3).

4) The most interesting fixed point with nontrivial positive values of g_*, u_* and w_*. It is IR attractive for $\varepsilon > 0$, $\xi > 0$ for the model A and $\varepsilon > 0$, $\xi > \varepsilon/12$ for the Gribov model. It corresponds to new nontrivial IR scaling regimes, in which both nonlinearities in the stochastic equations and the turbulent mixing are important; the corresponding critical dimensions depend essentially on the both RG expansion

parameters ε and ξ and are calculated as double series in these parameters. This behavior reveals strong anisotropy and belongs to new, completely nonequilibrium, universality classes. The realistic values $\xi = 4/3$ and $d = 2$ or 3 belong to these classes.

The regions of IR stability for these fixed points are shown in Fig. 1. For comparison, the boundaries for the isotropic stirring [9] are given. In the one-loop approximation, all the boundaries are given by straight lines; there are neither gaps nor overlaps between the different regions. The latter fact is valid to all orders of the RG expansion, although the boundaries between the regions 3 and 4 (determined by the nonlinearity $V(\psi)$) and 2 and 4 (determined by the velocity statistics) will become curved.

Let us illustrate the consequences of the RG analysis for the spreading of a cloud of the "agent" in the turbulent environment. The mean-square displacement $R_i(t)$ in the i-th direction at time $t > 0$ of the agent's particle, which started from the origin $\mathbf{x}' = 0$ at time $t' = 0$, is given by the relation

$$R_i^2(t) = \int d\mathbf{x}\, x_i^2\, G(t, \mathbf{x}), \quad G(t, \mathbf{x}) = \langle \psi(t, \mathbf{x}) \psi^{\dagger}(0, \mathbf{0}) \rangle, \quad x = |\mathbf{x}|. \tag{6}$$

The linear response function $G(t, \mathbf{x})$ has the following asymptotic representation

$$G(t, \mathbf{x}) = x_{\perp}^{-\Delta_{\psi} - \Delta_{\psi^{\dagger}}} F\left(x_{\perp} t^{-1/\Delta_{\omega}},\, x_{\|} t^{-\Delta_{\|}/\Delta_{\omega}}, \tau x_{\perp}^{\Delta_{\tau}} \right), \tag{7}$$

where $x_{\perp} = |\mathbf{x}_{\perp}|$ and F is the scaling function. The set of critical dimensions Δ_{*} for the fields and parameters is determined by the fixed point. In particular, for the Gribov model at the fixed point 4 in the one-loop approximation one has $\Delta_{\omega} = 2 - (2\varepsilon - \xi)/23$, $\Delta_{\|} = 1 + (12\xi - \varepsilon)/23$, $\Delta_{\psi} = \Delta_{\psi^{\dagger}} = d/2 + (14\xi - 5\varepsilon)/46$, $\Delta_{\tau} = 2 + 3(\xi - 2\varepsilon)/23$. Substituting (7) into (6) at $\tau = 0$ (that is, directly at criticality) gives the power laws

$$R_{\perp}^2(t) \propto t^{\alpha_{\perp}}, \quad R_{\|}^2(t) \propto t^{\alpha_{\|}}, \tag{8}$$

where the exponents $\alpha_{\perp}$, $\alpha_{\|}$ are simply related to the dimensions Δ_{*}. For the fixed point 3 (linear passive advection) one obtains exact results $\alpha_{\perp} = 1$, $\alpha_{\|} = 1 + \xi/2$. For the transverse direction this gives the ordinary diffusion law $R_{\perp}(t) \propto t^{1/2}$, while in the direction of the flow the spreading is accelerated and for $\xi = 4/3$ takes on the form $R_{\|}(t) \propto t^{5/3}$, or equivalently $dR_{\|}^2/dt \propto R_{\|}^{4/5}$. This "4/5" law differs from the classical "4/3" Richardson's law for the isotropic case. For the regime 4 the exponents in (8) depend on the both parameters ξ and ε.

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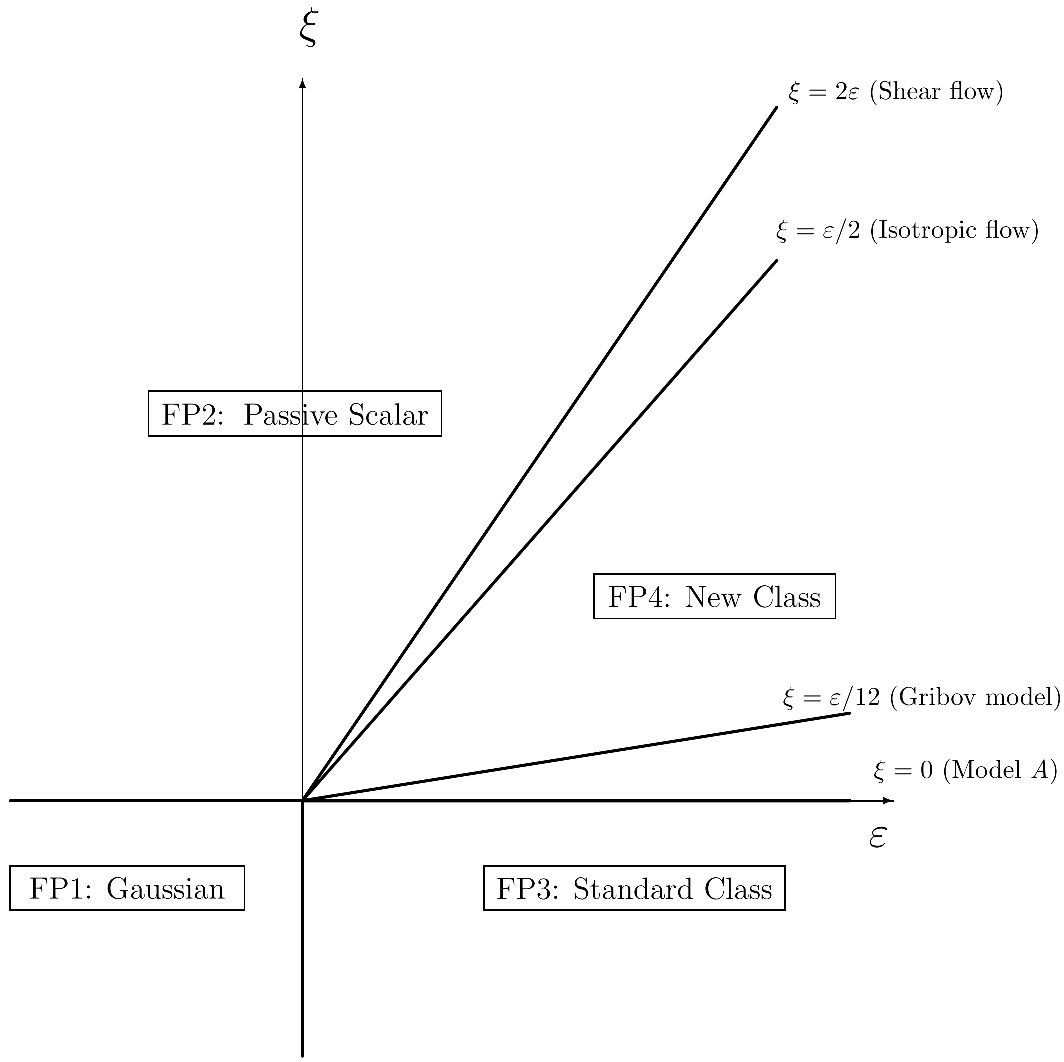

Figure 1: Regions of stability of the fixed points in various models.