Comments on: "Optical solitons in a parabolic law media with fourth order dispersion" [Appl. Math. Comput. 208 (2009) 209-302]

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Abstract. Biswas and Milovic [Appl. Math. Comput. 208 (2009) 209-302] have found the optical one-soliton solutions of a fourth order dispersive cubic-quintic nonlinear Schrödinger equation. In this comment, we first show there are mistakes in the paper and demonstrate that the obtained solutions do not satisfy the considered equation. And then we reconstruct a series of exact solutions by means of the ansatz method and F-expansion method.

1. Analysis of the solutions given in Ref.[1]

As is well known, the investigation for solition solutions of nonlinear Schrödinger equation is always an important and attractive topic. Very recently, Biswas and Milovic[1] considered the higher order dispersive cubic-quintic nonlinear Schrödinger equation,

$$i\,q_t \,+\, a\,q_{xx} \,-\, b\,q_{xxxx} \,+\, c\,(|q|^2 \,+\, d\,|q|^4)\,q \,=\, 0. \tag{1}$$

One optical soliton solution of Eq.(1) was constructed. However, we find there are mistakes in the paper[1] and the obtained solution does not satisfy Eq.(1).

Biswas et al.[1] assumed that Eq.(1) has an exact solution in the form

$$q(x,t) \,=\, P(x,t)\,e^{i\,(-\kappa\,x+\omega\,t+\theta)}, \tag{2}$$

where $P(x,t)$ is a real function to be determined later, and κ, ω are real constants. Substituting Eq.(2) into Eq.(1) yields a complex differential equation of $P(x,t)$, in which the real and imaginary parts read, respectively,

$$\frac{\partial P}{\partial t} \,-\, 2\kappa(a+2b\kappa^2)\,\frac{\partial P}{\partial x} \,+\, 4b\,\kappa\,\frac{\partial^3 P}{\partial x^3} \,=\, 0, \tag{3}$$

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$$(\omega + a\,\kappa^2 + b\kappa^4)\,P - cP^3 - c\,d\,P^5 - (a + 6b\kappa^2)\frac{\partial^2 P}{\partial x^2} + b\,\frac{\partial^4 P}{\partial x^4} = 0. \tag{4}$$

In Ref.[1], the solution of Eqs.(3)-(4) was supposed as

$$P = \frac{A}{(\lambda + \cosh\tau)^p}, \quad \tau = B(x - \nu\,t), \tag{5}$$

Substituting Eq.(5) into Eqs.(3)-(4), the authors obtained two expressions with respect to $\cosh\tau$ and $\sinh\tau$. We have noticed that there are many mistakes about the expressions (12)-(13) given in Ref.[1]. Then equating the coefficients of $1/(\lambda + \cosh\tau)^{p+j}(j = 0, \cdots, 4)$ of the obtained expressions, the values of A, B, ω, λ and ν were determined. Together with the transforation (2), the optical soliton solution was constructed,

$$q(x, t) = \frac{A}{\lambda + \cosh(B(x - \nu\,t))}\,\mathrm{e}^{i\,(-\kappa\,x + \omega\,t + \theta)}, \tag{6}$$

where A, B, ω, λ and ν were given by Eqs.(16)-(21) of Ref.[1].

However, the "solution" (6) does not satisfy Eq.(1). We can note this fact without substituting (6) into Eq.(1). The solitons are the results of a delicate balance between dispersion and nonlinearity, thus it is impossible that the linear partial differential equation (3) admits the bell type solitary wave (5).

To be on the save side we have substituted Eq.(5) with $p = 1$ into Eq.(3) and have obtained the following expression,

$$E_1 = \left[\frac{\nu + 2a\kappa + 4b\kappa^3 - 4b\kappa B^2}{(\lambda + \cosh\tau)^2} + \frac{24b\kappa B^2 \cosh\tau}{(\lambda + \cosh\tau)^3} - \frac{24b\kappa B^2 \sinh^2\tau)}{(\lambda + \cosh\tau)^4}\right] A\,B\sinh\tau.$$

We can see that this expression is equal to zero only in two cases. One is $A = 0$ or $B = 0$, and the other is $\kappa = \nu = 0$. This means that the "solution" (6) obtained by Biswas et al. in [1] is not correct.

2. New optical solitary wave solution of Eq.(1)

In the following, we adopt the ansatz solution of Li et al.[2] in the form

$$q(x, t) = E(x, t)\,\mathrm{e}^{i\,(k\,x - \omega\,t + \theta)}, \tag{7}$$

where $E(x, t)$ is the complex envelope function, and k, ω are real constants. Substituting Eq.(7) into Eq.(1) and removing the exponential term, we can rewrite Eq.(1) as

$$i\,E_t + 2ik\,(a + 2b\,k^2)\,E_x + (a + 6\,b\,k^2)\,E_{xx} - 4i\,b\,k\,E_{xxx} - bE_{xxxx}$$
$$+ (\omega - a\,k^2 - b\,k^4)\,E + c\,|E|^2 E + c\,d\,|E|^4 E = 0. \tag{8}$$

We now take the complex envelope ansatz function $E(x,t)$ as

$$E(x,t) = i\beta + \lambda \tanh(\xi), \ \xi = px + st, \tag{9}$$

where β, λ, p, s are real constants. Substituting Eq.(9) into Eq.(8) and setting the coefficients of $\tanh(\xi)^j (j = 0, \cdots, 5)$ to zero, one obtains the following algebraic equations:

$$\lambda (dc\lambda^4 - 24 p^4 b) = 0, \quad \lambda(\lambda^3 \beta cd + 24 bp^3 k) = 0,$$

$$\lambda(s - 2\lambda \beta^3 cd - \lambda \beta c + 2 kpa + 32 bp^3 k + 4 bpk^3) = 0,$$

$$\lambda (\lambda^2 c + 2\lambda^2\beta^2 cd + 40 p^4 b + 2 p^2 a + 12 k^2 p^2 b) = 0,$$

$$\lambda (2 p^2 a + k^2 a - \beta^4 cd - \beta^2 c + k^4 b + 16 p^4 b - \omega + 12 k^2 p^2 b) = 0,$$

$$\beta bk^4 - \beta \omega - \beta^3 c - 2 ap\lambda k - \beta^5 cd - s\lambda - 4 bp\lambda k^3 - 8 bp^3 k\lambda + \beta ak^2 = 0.$$

Solving it we obtain one set of nontrivial solution,

$$s = 8bpk(k^2 + p^2), \quad \beta = -\frac{k\lambda}{p}, \quad \omega = 2 p^2 a + 3 k^2 a + 37 k^4 b + 52 k^2 p^2 b + 16 p^4 b,$$

$$c = -\frac{2p^2(30 bk^2 + 20 bp^2 + a)}{\lambda^2}, \quad d = -\frac{12bp^2}{\lambda^2(30 bk^2 + 20 bp^2 + a)}. \tag{10}$$

From (7), (9) and (10), we obtain the optical solitary wave of Eq.(1),

$$q_1(x,t) = \left(-\frac{i k\lambda}{p} + \lambda \tanh(p x + 8bpk(k^2 + p^2) t)\right) e^{i\left(k x - (2 p^2 a + 3 k^2 a + 37 k^4 b + 52 k^2 p^2 b + 16 p^4 b) t + \theta\right)},$$

where p, k are determined by the last two identities of Eq.(10). From (9) and (10), the amplitude of the complex envelope function $E(x,t)$ reads,

$$|E(x,t)| = \left\{\frac{k^2\lambda^2}{p^2} + \lambda^2 \tanh^2(p x + 8bpk(k^2 + p^2) t)\right\}^{1/2},$$

which may approach nonzero when the time variable approaches infinity.

3. A series of exact solutions for Eq.(1) by using F-expansion method

We suppose that the solution of (1) is of the form

$$q(x,t) = P(\tau) e^{i\eta}, \ \tau = B(x - \nu t), \ \eta = (-\kappa x + \omega t + \theta), \tag{11}$$

where $P(\tau)$ is a real function, and B, ν, κ, ω are real constants to be determined. Substituting Eq.(11) to Eq.(1) and separating the real and imaginary parts, one may obtain the following equations,

$$- B(\nu + 2a\kappa + 4b\kappa^3) P' + 4b \kappa B^3 P''' = 0, \tag{12}$$

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$$(\omega + a\,\kappa^2 + b\kappa^4)\,P \; - \; c\,P^3 \; - \; c\,d\,P^5 \; - \; B^2(a + 6b\kappa^2)\,P'' \; + \; b\,B^4\,P'''' \; = \; 0. \qquad (13)$$

The linear ordinary differential equation (12) has no solitary wave solutions, thus we have to take $\kappa = \nu = 0$. In this case Eq.(12) is satisfied identically, and Eq.(13) becomes,

$$\omega\,P \; - \; cP^3 \; - \; c\,d\,P^5 \; - \; a\,B^2\,P'' \; + \; b\,B^4\,P'''' \; = \; 0. \qquad (14)$$

Eq.(14) can be solved by using the F-expansion method[3]-[6]. According to the F-expansion method, we suppose,

$$P(\tau) \; = \; \sum_{i=0}^{n} A_i\,F^i(\tau), \; A_n \neq 0, \qquad (15)$$

where $A_i(i = 0, \cdots, n)$ are real constants to be determined, the integer n is determined by balancing the linear highest order term and nonlinear term. And $F(\tau)$ in (15) satisfies,

$$\frac{d\,F(\tau)}{d\,\tau} \; = \; h_0 \; + \; h_2\,F(\tau)^2 \; + \; h_4\,F(\tau)^4, \qquad (16)$$

where h_0, h_2, h_4 are real constants. By balancing the linear highest order derivative term P'''' with nonlinear term P^5 in Eq.(14), we find $n = 1$. Thus Eq.(15) becomes,

$$P(\tau) \; = \; A_0 \; + \; A_1\,F(\tau). \qquad (17)$$

Substituting Eq.(17) into Eqs.(14) along with Eq.(16), collecting all terms with the same power of $F^j(\tau)(j = 0, \cdots, 5)$, and equating the coefficients of these terms yields a set of algebraic equations with respect to A_0, A_1, B, ω, a, b, c, d, h_0, h_2, h_4:

$$dcA_1{}^4A_0 \; = \; 0, \quad \omega\,A_0 - cA_0{}^3 - dcA_0{}^5 \; = \; 0,$$

$$10\,dcA_1{}^2A_0{}^3 + 3\,cA_1{}^2A_0 \; = \; 0, \quad 24\,bA_1B^4h_4{}^2 \; - \; d\,cA_1{}^5 \; = \; 0,$$

$$20\,bA_1B^4h_2h_4 - 10\,dcA_1{}^3A_0{}^2 - cA_1{}^3 - 2\,A_1B^2ah_4 \; = \; 0,$$

$$\omega\,A_1 - 3\,cA_1A_0{}^2 - A_1B^2ah_2 + 12\,bA_1B^4h_4h_0 + bA_1B^4h_2{}^2 - 5\,dcA_1A_0{}^4 \; = \; 0.$$

Solving the above algebraic equations, we have a set of nontrivial solution,

$$A_0 \; = \; 0, \quad A_1 \; = \; \pm\sqrt{\frac{12bB^2h_4}{d(10bB^2h_2 - a)}},$$

$$\omega = B^2\,(ah_2 - 12\,bB^2h_4h_0 - bB^2h_2{}^2), \quad c \; = \; \frac{d\,(10\,bB^2h_2 - a)^2}{6b}. \qquad (18)$$

When $h_0 = 1, h_2 = -(1 + m^2), h_4 = m^2$, Eq.(16) has the solution $F(\tau) = \mathrm{sn}\,\tau$. From Eq.(11) and Eq. (17), Eq.(1) has the solution,

$$q_2(x, t) = \pm\sqrt{-\frac{12B^2bm^2}{d(10B^2b + 10B^2bm^2 + a)}}\;\mathrm{sn}(B\,x, m)\,\mathrm{e}^{i\,(-B^2(B^2b + B^2bm^4 + 14B^2bm^2 + am^2 + a)\,t + \theta)},$$

where B is determined by $d(10bB^2 + 10m^2bB^2 + a)^2 - 6bc = 0$.

When $h_0 = 1 - m^2, h_2 = 2m^2 - 1, h_4 = -m^2$, Eq.(16) has the solution $F(\tau) = \text{cn}\tau$. Inserting it into (17) and using the transformation (11), Eq.(1) has the solution,

$$q_3(x,t) = \pm \sqrt{\frac{12B^2bm^2}{d(10\,B^2b - 20\,B^2bm^2 + a)}}\,\text{cn}(B\,x, m)\,\text{e}^{i\,(B^2(16B^2bm^2 - 16B^2bm^4 - B^2b - a + 2am^2)\,t + \theta)},$$

where B is determined by $d(a + 10bB^2 - 20bB^2m^2)^2 - 6bc = 0$.

Some solitary wave solutions can be obtained if the modulus m approaches to 1. For example, when $m \to 1$, the solution $q_2(x,t)$ degenerates to the kink type envelope wave solution,

$$q_4(x,t) = \pm \sqrt{-\frac{12b\,B^2}{d(20\,B^2b + a)}}\,\tanh(B\,x)\,\text{e}^{i\,(-2\,B^2(8\,B^2b + a)\,t + \theta)},$$

where B is determined by $d(a + 20bB^2)^2 - 6bc = 0$.

When $m \to 1$, the solution $q_3(x,t)$ degenerates to the bell type envelope wave solution,

$$q_5(x,t) = \pm \sqrt{\frac{12B^2b}{d(a - 10\,B^2b)}}\,\text{sech}(B\,x)\,\text{e}^{i\,(B^2(a - B^2b)\,t + \theta)},$$

where B is determined by $d(a - 10bB^2)^2 - 6bc = 0$.

As pointed out in Ref.[3], Eq.(16) has many other Jacobi elliptic function solutions in terms of $\text{dn}(\xi)$, $\text{ns}(\xi)$, $\text{nd}(\xi)$, $\text{nc}(\xi)$, $\text{sc}(\xi)$, $\text{cs}(\xi)$, $\text{sd}(\xi)$, $\text{ds}(\xi)$, $\text{cd}(\xi)$, $\text{dc}(\xi)$ as well as the corresponding solitary wave and trigonometric function solutions. For simplicity, such types of solutions to Eq.(1) are not listed here.

With the aid of Maple, we have checked the solutions $q_j(x,t)(j = 1, \cdots, 5)$ by putting them back into Eq.(1).

Acknowledgments

References

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