Virasoro and W-constraints for the q-KP hierarchy

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Abstract. Based on the Adler-Shiota-van Moerbeke (ASvM) formula, the Virasoro constraints and W-constraints for the p-reduced q-deformed Kadomtsev-Petviashvili (q-KP) hierarchy are established.

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INTRODUCTION

The origin of q-calculus (quantum calculus) [1, 2] traces back to the early 20th century. Many mathematicians have important works in the area of q-calculus and q-hypergeometric series. The q-deformation of classical nonlinear integrable system (also called q-deformed integrable system) started in 1990's by means of q-derivative ∂_q instead of usual derivative ∂ with respect to x in the classical system. As we know, the q-deformed integrable system reduces to a classical integrable system as q goes to 1.

Several q-deformed integrable systems have been presented, for example, q-deformation of the KdV hierarchy [3, 4, 5], q-Toda equation [6], q-Calogero-Moser equation [7]. Obviously, the q-deformed Kadomtsev-Petviashvili (q-KP) hierarchy is also a subject of intensive study in the literature from [8] to [13].

The additional symmetries, string equations and Virasoro constraints [14, 15, 16, 17, 18, 19] of the classical KP hierarchy are important since they are involved in the matrix models of the string theory [20]. For example, there are several new works [21, 22, 23, 24, 25] on this topic. It is quite interesting to study the analogous properties of q-deformed KP hierarchy by this expanding method. In [11], the additional symmetries of the q-KP hierarchy were provided. Recently, additional symmetries and the string equations associated with the q-KP hierarchy have already been reported in [11, 13]. The negative Virasoro constraint generators $\{L_{-n}, n \geq 1\}$ of the $2-$reduced q-KP hierarchy are also obtained in [13] by the similar method of [18].

Our main purpose of this article is to give the complete Virasoro constraint generators $\{L_n, n \geq -1\}$ and W-constraints $\{w_m, m \geq -2\}$ for the p-reduced q-KP hierarchy by the different process with negative part of Virasoro constraints given in [13]. The method of this paper is based on Adler-Shiota-van Moerbeke (ASvM) formula.

This paper is organized as follows. We give a brief description of q-calculus and q-KP hierarchy in Section 2 for reader's convenience. The main results are stated and proved in Section 3, which are the Virasoro constraints and W-constraints on the τ function for the p-reduced q-KP hierarchy. Section 4 is devoted to conclusions and discussions.

q-CALCULUS AND q-KP HIERARCHY

At the beginning of the this section, Let us recall some useful facts of q-calculus [2] in the following to make this paper be self-contained.

The Euler-Jackson q-difference ∂_q is defined by

$$\partial_q(f(x)) = \frac{f(qx)-f(x)}{x(q-1)}, \qquad q \neq 1 \tag{1}$$

and the q-shift operator is $\theta(f(x)) = f(qx)$. It is worth pointing out that θ does not commute with ∂_q, indeed, the relation $(\partial_q \theta^k(f)) = q^k \theta^k(\partial_q f), k \in \mathbb{Z}$ holds. The limit of $\partial_q(f(x))$ as q approaches 1 is the ordinary differentiation $\partial_x(f(x))$. We denote the formal inverse of ∂_q as ∂_q^{-1}. The following q-deformed Leibnitz rule holds

$$\partial_q^n \circ f = \sum_{k\geq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q^k f)\partial_q^{n-k}, \qquad n \in \mathbb{Z} \tag{2}$$

where the q-number $(n)_q = \frac{q^n-1}{q-1}$ and the q-binomial is introduced as

$$\binom{n}{0}_q = 1, \qquad \binom{n}{k}_q = \frac{(n)_q(n-1)_q\cdots(n-k+1)_q}{(1)_q(2)_q\cdots(k)_q}.$$

Let $(n)_q! = (n)_q(n-1)_q(n-2)_q\cdots(1)_q$, the q-exponent $e_q(x)$ is defined by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!} = \exp(\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k).$$

Similar to the general way of describing the classical KP hierarchy [14, 19], we will give a brief introduction of q-KP hierarchy and its additional symmetries based on [10, 11].

The Lax operator L of q-KP hierarchy is given by

$$L = \partial_q + u_0 + u_{-1}\partial_q^{-1} + u_{-2}\partial_q^{-2} + \cdots. \tag{3}$$

where $u_i = u_i(x,t_1,t_2,t_3,\cdots,), i = 0,-1,-2,-3,\cdots$. The corresponding Lax equation of the q-KP hierarchy is defined as

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1,2,3,\cdots, \tag{4}$$

here the differential part $B_n = (L^n)_+ = \sum_{i=0}^{n} b_i \partial_q^i$ and the integral part $L_-^n = L^n - L_+^n$. L in eq.(3) can be generated by dressing operator $S = 1 + \sum_{k=1}^{\infty} s_k \partial_q^{-k}$ in the following way

$$L = S\partial_q S^{-1}. \tag{5}$$

Dressing operator S satisfies Sato equation

$$\frac{\partial S}{\partial t_n} = -(L^n)_- S, \quad n = 1,2,3,\cdots. \tag{6}$$

The q-wave function $w_q(x,t;z)$ and the q-adjoint function $w_q^*(x,t;z)$ of q-KP hierarchy are given by

$$w_q(x,t;z) = Se_q(xz)\exp(\sum_{i=1}^{\infty} t_i z^i), \qquad w_q^*(x,t;z) = (S^*)^{-1}|_{x/q} e_{1/q}(-xz)\exp(-\sum_{i=1}^{\infty} t_i z^i),$$

which satisfies following linear q-differential equations

$$Lw_q = zw_q, \quad L^*|_{x/q} w_q^* = zw_q^*,$$

here the notation $P|_{x/t} = \sum_i P_i(x/t)t^i \partial_q^i$ is used for a q-pseudo-differential operator of the form $P = \sum_i p_i(x)\partial_q^i$, and the conjugate operation "$*$" for P is defined by $P^* = \sum_i (\partial_q^*)^i p_i(x)$ with $\partial_q^* = -\partial_q \theta^{-1} = -\frac{1}{q}\partial_{\frac{1}{q}}$, $(\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta\partial_q^{-1}$, $(PQ)^* = Q^*P^*$ for any two q-PDOs.

Furthermore, $w_q(x,t;z)$ and $w_q^*(x,t;z)$ of q-KP hierarchy can be expressed by sole function $\tau_q(x;t)$ [10] as

$$w_q = \frac{\tau_q(x;t-[z^{-1}])}{\tau_q(x;t)} e_q(xz)e^{\xi(t,z)} = \frac{e_q(xz)e^{\xi(t,z)}e^{-\sum_{i=1}^{\infty}\frac{z^{-i}}{i}\partial_i}\tau_q}{\tau_q}, \tag{7}$$

$$w_q^* = \frac{\tau_q(x;t+[z^{-1}])}{\tau_q(x;t)} e_{1/q}(-xz)e^{-\xi(t,z)} = \frac{e_{1/q}(-xz)e^{-\xi(t,z)}e^{+\sum_{i=1}^{\infty}\frac{z^{-i}}{i}\partial_i}\tau_q}{\tau_q},$$

where $\xi(t,z) = \sum_{i=1}^{\infty} t_i z^i$ and $[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots\right)$. The operator $G(z)$ is introduced as $G(z)f(t) = f(t-[z^{-1}])$, then

$$w_q = \frac{G(z)\tau_q}{\tau_q} e_q(xz)e^{\xi(t,z)} \equiv \hat{w}_q e_q(xz)e^{\xi(t,z)}. \tag{8}$$

The following Lemma shows there exist an essential correspondence between q-KP hierarchy and KP hierarchy.

Lemma 1. [10] Let $L_1 = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots$, where $\partial = \partial/\partial x$, be a solution of the classical KP hierarchy and τ be its tau function. Then $\tau_q(x,t) = \tau(t+[x]_q)$ is a tau function of the q-KP hierarchy associated with Lax operator L in eq.(3), where

$$[x]_q = \big(x, \frac{(1-q)^2}{2(1-q^2)}x^2, \frac{(1-q)^3}{3(1-q^3)}x^3, \cdots, \frac{(1-q)^i}{i(1-q^i)}x^i, \cdots\big).$$

Define Γ_q and Orlov-Shulman's M operator [11] for q-KP hierarchy as $M = S\Gamma_q S^{-1}$ and $\Gamma_q = \sum_{i=1}^{\infty}\left(it_i + \frac{(1-q)^i}{(1-q^i)}x^i\right)\partial_q^{i-1}$. The the additional flows for each pair $\{m,n\}$ are difined as follows

$$\frac{\partial S}{\partial t^*_{m,n}} = -(M^m L^n)_- S, \tag{9}$$

or equivalently

$$\frac{\partial L}{\partial t^*_{m,n}} = -[(M^m L^n)_-, L], \qquad \frac{\partial M}{\partial t^*_{m,n}} = -[(M^m L^n)_-, M]. \tag{10}$$

The additional flows $\partial^*_{mn} = \dfrac{\partial}{\partial t^*_{m,n}}$ commute with the hierarchy $\partial_k = \dfrac{\partial}{\partial t_k}$, i.e. $[\partial^*_{mn}, \partial_k] = 0$ but do not commute with each other, so they are additional symmetries [12]. $(M^m L^n)_-$ serves as the generator of the additional symmetries along the trajectory parametrized by $t^*_{m,n}$.

Theorem 1.[13] If an operator L does not depend on the parameters t_n and the additional variables $t^*_{1,-n+1}$, then L^n is a purely differential operator, and the string equations of the q-KP hierarchy are given by

$$[L^n, \frac{1}{n}(ML^{-n+1})_+] = 1,\ n = 2,3,4,\cdots \tag{11}$$

VIRASORO AND W-CONSTRAINTS FOR THE q-KP HIERARCHY

In this section, we mainly study the Virasoro constraints and W-constraints on τ-function of the p-reduced q-KP hierarchy. To this end, two useful vertex operators $X_q(\mu,\lambda)$ and $Y_q(\mu,\lambda)$ would be introduced.

The vertex operator $X_q(\mu,\lambda)$ is defined in [11] as

$$X_q(\mu,\lambda) = e_q(x\mu)e_q^{-1}(x\lambda)exp(\sum_{i=1}^{\infty} t_i(\mu^i - \lambda^i))exp(-\sum_{i=1}^{\infty}\frac{\mu^{-i}-\lambda^{-i}}{i}\partial_i). \tag{12}$$

We can also denote the vertex operator $X_q(\mu,\lambda)$ by

$$X_q(\mu,\lambda) =: exp(\alpha(\lambda) - \alpha(\mu)) : \tag{13}$$

where the symbol :: means that we keep t_i be always left side of ∂_j, and $\alpha(\lambda) = \sum \alpha_n \cdot \frac{\lambda^{-n}}{n}$, here $\alpha_0 = 0, \alpha_n = \partial_n = \dfrac{\partial}{\partial t_n}$ for $n > o$, $\alpha_n = |n|t_{|n|} + \frac{(1-q)^{|n|}}{1-q^{|n|}}x^{|n|}$ for $n < o$.

The following lemma is given without proof.

Lemma 2. Taylor expansion of the $X_q(\mu,\lambda)$ on μ at the point of λ is

$$X_q(\mu,\lambda) = \sum_{m=0}^{\infty}\frac{(\mu-\lambda)^m}{m!}\sum_{n=-\infty}^{\infty}\lambda^{-m-n}W_n^{(m)},$$

here $\sum_{n=-\infty}^{\infty}\lambda^{-m-n}W_n^{(m)} = \partial_\mu^m X_q(\mu,\lambda)|_{\mu=\lambda}$.

The first items of $W_n^{(m)}$ are

$$W_n^{(o)} = \delta_{n,0},$$
$$W_n^{(1)} = \alpha_n,$$
$$W_n^{(2)} = (-n-1)\alpha_n + \sum_{i+j=n} :\alpha_i\alpha_j:$$
$$W_n^{(3)} = (n+1)(n+1)\alpha_n + \sum_{i+j+k=n} :\alpha_i\alpha_j\alpha_k: -\frac{3}{2}(n+2)\sum_{i+j=n} :\alpha_i\alpha_j:$$

There is Adler-Shiota-van Moerbeke (ASvM) formula [11] for q-KP hierarchy as

$$X_q(\mu,\lambda)w_q(x,t;z) = (\lambda-\mu)Y_q(\mu,\lambda)w_q(x,t;z), \tag{14}$$

where the operator $Y_q(\mu,\lambda)$ is the generators of additional symmetry of q-KP hierarchy as

$$Y_q(\mu,\lambda) = \sum_{m=0}^{\infty}\frac{(\mu-\lambda)^m}{m!}\sum_{n=-\infty}^{\infty}\lambda^{-m-n-1}(M^mL^{m+n})_-. \tag{15}$$

ASvM formula is equivalent to the following equation

$$\partial^*_{m,n+m}\tau_q = \frac{W_n^{(m+1)}(\tau_q)}{m+1}. \tag{16}$$

The following theorem holds by virtue of the ASvM formula.

Theorem 2.

$$\left(\frac{W_n^{(m+1)}}{m+1} - c\right)\tau_q = 0,\ m = 0,1,2,3\cdots. \tag{17}$$

Proof. Consider the condition $\partial^*_{m,n+m}\hat{w}_q = 0$, from eq.(8), and denote $\tilde{\tau}_q = G(z)\tau_q$,

$$\partial^*_{m,n+m}\hat{w}_q = \partial^*_{m,n+m}\frac{\tilde{\tau}_q}{\tau_q} = \frac{\tilde{\tau}_q}{\tau_q}\left(\frac{\partial^*_{m,n+m}\tilde{\tau}_q}{\tilde{\tau}_q} - \frac{\partial^*_{m,n+m}\tau_q}{\tau_q}\right) = \hat{w}_q(G(z)-1)\frac{\partial^*_{m,n+m}\tau_q}{\tau_q} = 0.$$

The operator $G(z)$ has the property, which is $(G(z)-1)f(t) = 0$ implies $f(t)$ is a constant, from this we can get

$$\frac{\partial^*_{m,n+m}\tau_q}{\tau_q} = c \tag{18}$$

where c is constant. Combining eq.(16) with eq.(18) finishes the proof. □

Now we consider the p-reduced q-KP hierarchy, by setting $(L^p)_- = 0$, i.e. $L^p = (L^p)_+$. From Lax equation of q-KP hierarchy, the p-reduced condition means that L is independent on t_{jp} as $\partial_{jp}L = 0, j = 1,2,3,\cdots$ and τ_q is independent on t_{jp} as $\partial_{jp}\tau_q = 0, j = 1,2,3,\cdots$.

Based on theorem 2, the Virasoro constraints and W-constraints for the p-reduced q-KP hierarchy will be obtained. Let $n = kp$ in theorem 2 and denote

$$\tilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q^i)}x^i,\ i = 1,2,3,\cdots. \tag{19}$$

First of all, for $m=0$, eq.(17) in theorem 2 becomes

$$(W^{(1)}_{kp}-c)\tau_q=0. \tag{20}$$

Let $c=0$, we have that $\alpha_{kp}\tau_q=\frac{\partial\tau_q}{\partial t_{kp}}=0$, it is just the condition $L^p=(L^p)_+$ for p-reduced q-KP hierarchy.

For $m=1$, it is

$$(\frac{W^{(2)}_{kp}}{2}-c)\tau_q=0 \tag{21}$$

Theorem 3. The Virasoro constraints imposed on the tau function τ_q of the p-reduced q-KP hierarchy are

$$L_k\tau_q=0,\ \ k=-1,0,1,2,3,\cdots,$$

here

$$L_{-1}=\frac{1}{p}\sum_{\substack{n=p+1\\ n\neq 0(\mathrm{mod}p)}}^{\infty} n\tilde{t}_n\frac{\partial}{\partial\tilde{t}_{n-p}}+\frac{1}{2p}\sum_{i+j=p} ij\tilde{t}_i\tilde{t}_j,$$

$$L_0=\frac{1}{p}\sum_{\substack{n=1\\ n\neq 0(\mathrm{mod}p)}}^{\infty} n\tilde{t}_n\frac{\partial}{\partial\tilde{t}_n}+(\frac{p}{24}-\frac{1}{24p}),$$

$$L_k=\frac{1}{p}\sum_{\substack{n=1\\ n\neq 0(\mathrm{mod}p)}}^{\infty} n\tilde{t}_n\frac{\partial}{\partial\tilde{t}_{n+kp}}+\frac{1}{2p}\sum_{\substack{i+j=kp\\ i,j\neq 0(\mathrm{mod}p)}} ij\tilde{t}_i\tilde{t}_j,\quad k\geq 1,$$

and L_n satisfy Virasoro algebra commutation relations

$$[L_n,L_m]=(n-m)L_{(n+m)},\ m,n=-1,0,1,2,3,\cdots. \tag{22}$$

Proof. Following the results in eq.(20) and eq.(21), we have

$$(\frac{W^{(2)}_{kp}}{2}-c)\tau_q=(\frac{1}{2}\sum_{i+j=kp}:\alpha_i\alpha_j:-c)\tau_q=0. \tag{23}$$

Define $L_k=\frac{W^{(2)}_{kp}}{p}$, let $c=\frac{p}{24}-\frac{1}{24p}$ in L_0, otherwise $c=0$. The p-reduced condition $n\neq 0(\mathrm{mod}p)$ can be naturally added without destroying the algebra structure, because of $\tilde{t}_{mp}$ is presented together with $\frac{\partial}{\partial\tilde{t}_{mp+kp}}$.

By a straightforward and tedious calculation, the Virasoro commutation relations

$$[L_n,L_m]=(n-m)L_{(n+m)},\ m,n=-1,0,1,2,3,\cdots$$

can be verified. $\square$

For $m=2$, it is

$$(\frac{W_{kp}^{(3)}}{3}-c)\tau_q=(\frac{1}{3}\sum_{i+j+h=kp}:\alpha_i\alpha_j\alpha_h:-c)\tau_q=0. \tag{24}$$

Theorem 4. Let

$$w_m=\sum_{\substack{i+j+h=mp\\ i,j,h\neq 0(\mathrm{mod}p)}}:\alpha_i\alpha_j\alpha_h:,\ m\geq -2,$$

the W-constraints on the tau function τ_q of the p-reduced q-KP hierarchy are

$$w_m\tau_q=0, m\geq -2,$$

and they satisfy following algebra commutation relations

$$[L_n,w_m]=(2n-m)w_{n+m}, n\geq -1, m\geq -2.$$

For $m\geq 3$, using the similar technique in theorem 3 and 4, we can deduce the higher order algebraic constrains on the tau function τ_q of the p-reduced q-KP hierarchy.

Remark 1. As we know, the q-deformed KP hierarchy reduces to the classical KP hierarchy when $q\to 1$ and $u_0=0$. The parameters $(\tilde{t}_1,\tilde{t}_2,\cdots,\tilde{t}_i,\cdots)$ tend to $(t_1+x,t_2,\cdots,t_i,\cdots)$ as $q\to 1$. One can further identify t_1+x with x in the classical KP hierarchy, i.e. $t_1+x\to x$. The deformation as q goes to 1 of Virasoro constraints and W-constraints for the p-reduced q-KP hierarchy are identical with the results of the classical KP hierarchy given by L.A.Dickey [16] and S.Panda, S.Roy [18].

CONCLUSIONS AND DISCUSSIONS

To summarize, we have derived the Virasoro constraints and W-constraints of the p-reduced q-KP hierarchy in theorem 3 and 4 respectively. The results of this paper show obviously that the Virasoro constraint generators $\{L_n, n\geq -1\}$ and W-constraints $\{w_m, m\geq -2\}$ for the p-reduced q-KP hierarchy are different with the form of the KP hierarchy. Furthermore, we also would like to point out the following interesting relation between the q-KP hierarchy and the KP hierarchy

$$L_n=\hat{L}_n|_{t_i\to\tilde{t}_i=t_i+\frac{(1-q)^i}{i(1-q^i)}x^i}$$

and it seems to demonstrate that q-deformation is a non-uniform transformation for coordinates $t_i\to\tilde{t}_i$, which is consistent with results on τ function [10] and the q-soliton [12] of the q-KP hierarchy. Here $\hat{L}_n$ [16, 18] are Virasoro generators of the KP hierarchy.

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