GLOBALLY COUPLED CHAOTIC MAPS WITH BISTABLE BEHAVIOUR: LARGE DEVIATIONS

GERHARD KELLER

Abstract. For a system of globally coupled chaotic maps with bistable behaviour we relate the rate function for large deviations in the system size at finite time to dynamical properties of the self consistent Perron-Frobenius operator (SCPFO) that describes the system in the infinite size limit.

1. Introduction

The following system of globally coupled chaotic interval maps was studied in [BKZ09]: let $X = [-\frac{1}{2}, \frac{1}{2}]$ and, for $r \in (-2/3, 2/3)$, let

$$T_r(x) := f_r(x) \bmod \left(\mathbb{Z} + \frac{1}{2}\right) = \begin{cases} f_r(x) & \text{on } [-\frac{1}{2}, -\frac{r}{4}), \\ f_r(x) - 1 & \text{on } (-\frac{r}{4}, \frac{1}{2}], \end{cases} \tag{1}$$

where $f_r(x) = \frac{(r+4)x+r+1}{2rx+2}$ is linear fractional. Observe that $f_r(-\frac{1}{2}) = -\frac{1}{2}$, $f_r(-\frac{r}{4}) = \frac{1}{2}$ and $f_r(\frac{1}{2}) = \frac{3}{2}$. The T_r are thus smooth deformations of the doubling map T_0 on X, and they are certainly among the best understood smooth chaotic dynamical systems. They are semiconjugate to the full shift on two symbols, and they have a unique absolutely continuous invariant measure with an analytic density and the strongest possible exponential mixing properties.

The local units T_r are combined to a globally coupled map $\boldsymbol{T}_N : X^N \to X^N$ defined by

$$\boldsymbol{T}_N(\boldsymbol{x}) = (T_{r(\boldsymbol{x})}(x_1), \dots, \boldsymbol{T}_{r(\boldsymbol{x})}(x_N)) \quad \text{with} \quad r(\boldsymbol{x}) = G(\phi(\boldsymbol{x})) \tag{2}$$

where

$$\phi(\boldsymbol{x}) = N^{-1} \sum_i x_i \quad \text{and} \quad G(t) = A \cdot \tanh\left(\frac{B}{A} t\right) \text{ with } 0 < A \leqslant 0.4 \text{ and } 0 \leqslant B \leqslant 18. \tag{3}$$

The function G can indeed be a much more general sigmoidal function, see [BKZ09] for details.

The two faces of the dynamics of $\boldsymbol{T}_N$ are highlighted by the following theorems. The first one describes the long-time behaviour of the system at fixed system size N.

Theorem A (Theorem 1 of [BKZ09])**.** *For any $N \in \mathbb{N}$ and all parameters A, B as in (3), the map $\boldsymbol{T}_N : X^N \to X^N$ has a unique absolutely continuous invariant probability measure $\boldsymbol{\mu}_N$. Its density is strictly positive and real analytic. The systems $(\boldsymbol{T}_N, \boldsymbol{\mu}_N)$ are exponentially mixing in various strong senses, in particular do Hölder observables have exponentially decreasing correlations.*

Date: April 2, 2010.

This note could not have been written without the collaboration leading to the publication [BKZ09] and, in particular, without many discussions on large deviations with Jean-Baptiste Bardet. The work was supported by DFG grant Ke-514/7-1.

As $\boldsymbol{T}_N$ is equivariant under coordinate permutations, all $(X^N, \boldsymbol{T}_N)$ can be embedded into the dynamical system $(\mathsf{P}(X), \widetilde{T})$ where $\mathsf{P}(X)$ denotes the space of Borel probability measures on X and $\widetilde{T}(Q) := Q \circ T^{-1}_{G(\phi(Q))}$ with $\phi(Q) := \int_X x\, Q(dx)$. Indeed, using the empirical measures $\epsilon_N : X^N \to \mathsf{P}(X)$, $\epsilon_N(\boldsymbol{x}) = N^{-1} \sum_i \delta_{x_i}$, it is obvious that $\epsilon_N \circ \boldsymbol{T}_N = \widetilde{T} \circ \epsilon_N$. Of course, the dynamics of $\widetilde{T}$ are not easier to understand than those of the $\boldsymbol{T}_N$. But if $\widetilde{T}$ is restricted to the space of probabilities $Q = u\,\lambda$ absolutely continuous w.r.t Lebesgue measure λ on X, then $\frac{d(\widetilde{T}Q)}{d\lambda} = \widetilde{P}u$ where $\widetilde{P} : \mathcal{D} \to \mathcal{D}$ is the *self consistent Perron-Frobenius operator* of the system acting nonlinearly on the space $\mathcal{D}$ of probability densities on X by

$$\widetilde{P}u = P_{G(\phi(u))}u \quad \text{where } \phi(u) = \int_X x\, u(x)\, dx. \tag{4}$$

Here $P_r : \mathcal{D} \to \mathcal{D}$ is the usual (linear) Perron-Frobenius operator of T_r. It has a unique fixed point, namely the unique invariant density u_r of the map T_r. The monotone increasing map $H : (-\frac{2}{3}, \frac{2}{3}) \to (-\frac{2}{3}, \frac{2}{3})$, $H(r) = G(\phi(u_r))$ always has the fixed point $r = 0$. It has two more fixed points $-r_* < 0 < r_*$ if and only if $H'(0) = \frac{A}{6} > 1$ [BKZ09, Section 2.5].

The dynamics of $\widetilde{P}$ are summarised in the following theorem.

Theorem B (Theorem 2 and Remark 5 of [BKZ09])**.** *Consider $\widetilde{P} : \mathcal{D} \to \mathcal{D}$, $\mathcal{D}$ equipped with the metric inherited from L^1_λ. Then the following holds:*

1) If $A \leqslant 6$ (the stable regime*), then u_0 is the unique fixed point of $\widetilde{P}$, and it attracts all densities, that is,*

$$\lim_{n\to\infty} \widetilde{P}^n u = u_0 \qquad \textit{for all } u \in \mathcal{D}.$$

2) If $A > 6$ (the bistable regime*), then u_{-r_*}, u_0, u_{r_*} are the only fixed points of $\widetilde{P}$. Now u_0 is unstable, while u_{-r_*} and u_{r_*} are stable. More precisely:*
   - *a) $u_{\pm r_*}$ are stable fixed points for $\widetilde{P}$ in the sense that their respective basins of attraction are L^1_λ-open. They are indeed Lyapunov-stable.*
   - *b) If $u \in \mathcal{D}$ is not attracted by u_{-r_*} or u_{r_*}, then it is attracted by u_0.*
   - *c) u_0 is not stable. Indeed, u_0 can be L^1_λ-approximated by convex analytic densities from either basin. It is a hyperbolic fixed point of $\widetilde{P}$ in a sense made precise in Proposition 5 of* [BKZ09].

Remark 1. *a) In Proposition 3 it is shown that* $\mathrm{DA}(u_0)$ *is the boundary of* $\mathrm{DA}(u_-) \cup \mathrm{DA}(u_+)$. *I conjecture that even more is true, namely that* $\mathrm{DA}(u_0)$ *is the common boundary of* $\mathrm{DA}(u_-)$ *and* $\mathrm{DA}(u_+)$.
b) The formula for the codimension one stable eigenspace given in [BKZ09, Proposition 5] *is wrong. We correct it at the end of section 4.*

What are links between the finite-dimensional dynamics of the $\boldsymbol{T}_N$ and their infinite-dimensional idealization $\widetilde{P}$? Here is a first one: it is a rather general fact for mean field coupled piecewise expanding maps that all weak accumulation points of the sequence $(\boldsymbol{\mu}_N \circ \epsilon_N^{-1})_{N>0}$ of probability measures on $\mathsf{P}(X)$ are supported by the set $\mathsf{P}_\lambda := \{Q \in \mathsf{P}(X) : Q \ll \lambda\}$ [Kel00]. Together with the observation from Theorem B that, for each $u \in \mathcal{D}$, $\widetilde{P}^n u$ converges to one of u_{-r_*}, u_0 or u_{r_*}, this implies the next result.

Theorem C (Theorem 3 of [BKZ09])**.** *The $\boldsymbol{T}_N$-invariant probability measures $\boldsymbol{\mu}_N$ on X^N correspond to $\widetilde{T}$-invariant probability measures $\boldsymbol{\mu}_N \circ \epsilon_N^{-1}$ on $\mathsf{P}(X)$. All weak accumulation* *points π of the latter sequence are $\widetilde{T}$-invariant probability measures on $\mathsf{P}(X)$ supported by the three fixed points $u_{-r_*}\lambda$, $u_0\lambda = \lambda$ and $u_{r_*}\lambda$.*

The proof of this theorem does not use the fact that $u_{\pm r_*}$ are stable while u_0 is unstable under $\widetilde{P}$. Taking this into account one would expect that all weak accumulation points π are supported just by u_{-r_*} and u_{r_*} so that, due to the sign flip symmetry of the system, the measures $\boldsymbol{\mu}_N \circ \epsilon_N^{-1}$ converge weakly to $\frac{1}{2}\delta_{u_{-r_*}\lambda} + \frac{1}{2}\delta_{u_{r_*}\lambda}$.

Theorem C (partially) describes the limit "$N \to \infty$" *after* the limit "time $\to \infty$" was taken. (Observe that $\boldsymbol{\mu}_N = \lim_{t\to\infty} \lambda \circ \boldsymbol{T}_N^{-t}$ for each N because of the mixing properties of the T_N.) In this note we describe another link between the dynamics of $\boldsymbol{T}_N$ and $\widetilde{P}$. It is based on rate functions for large deviations in the system size (that is when $N \to \infty$) at finite time t, and certain limits of these rate functions when $t \to \infty$. In section 2 we draw some consequences from Sanov's theorem and the contraction principle that apply to rather general mean field coupled dynamical systems and provide asymptotic expressions for related large deviations rates in Theorem 1. For the special example introduced above we evaluate the expressions from Theorem 1 in section 3. Corollary 1 relates these rates to the dynamics of $\widetilde{P}$ on $\mathcal{D}$, and Theorem 2 shows that, although $\widetilde{P}$ is bistable, the complicated boundaries of the domains of attraction preclude a simple expression for these rates.

2. Large deviations in mean field coupled dynamical systems

We begin with some general considerations on large deviations in mean field coupled dynamical systems. So we allow that (X, d) is a complete metric space with a "reference" Borel measure λ. Each $T_r : X \to X$ is assumed to be continuous λ-a.e., and the r-dependence should be continuous in the following sense:

$$\forall r\, \forall \epsilon > 0\, \exists \delta > 0 :\ \lambda(\mathrm{cl}\{x : \exists r' \in (r-\delta, r+\delta) \text{ s.th. } d(T_r(x), T_{r'}(x)) > \epsilon\}) < \epsilon \tag{5}$$

where $\mathrm{cl}(A)$ denotes the topological closure of a set A. This means that small changes in the parameter cannot cause large deformations of the graphs of the map. The particular model from the introduction obviously satisfies these assumptions.

We are interested in the large deviation behaviour of the systems $(X^N, \boldsymbol{T}_N)$ under initial distributions $Q_{\tilde{u}}^N = (\tilde{u}\,\lambda)^{\otimes N}$ on X^N, i.e. when the initial vector $\boldsymbol{x}$ has i.i.d. entries with density $\tilde{u}$ with respect to the measure λ on X. Each $\boldsymbol{x} \in X^N$ gives rise to an empirical measure $\varepsilon_{\boldsymbol{x}} := \frac{1}{N}\sum_{k=1}^N \delta_{x_k}$, and Sanov's theorem (see e.g. [DZ98, Theorem 6.2.10]) states that the empirical measures $\varepsilon_{\boldsymbol{x}}$ satisfy a large deviations principle (LDP) with good convex rate function in the space $\mathsf{P}(X)$ of all Borel probability measures on X (equipped with the topology of weak convergence) when $N \to \infty$. The rate function $I : \mathsf{P}(X) \to [0, +\infty]$ is given by

$$I(Q) = H(Q|\tilde{u}\lambda) = \begin{cases} \int u \cdot \log \frac{u}{\tilde{u}}\, d\lambda & \text{if } Q = u\lambda \ll \tilde{u}\lambda \\ +\infty & \text{otherwise} \end{cases} \tag{6}$$

Instead of $H(Q|\tilde{u}\lambda)$ we write $H(u|\tilde{u})$ if $Q = u\lambda$.

To capture the time evolution of the system consider the map $\Gamma : \mathsf{P}(X) \to \mathsf{P}(X)^{\mathbb{N}}$,

$$\Gamma(Q) = \left(Q, \widetilde{T}Q, \widetilde{T}^2 Q, \widetilde{T}^3 Q, \dots\right) \tag{7}$$

In order to study large deviations as $N \to \infty$ for the measures $Q_{\tilde{u}}^N \circ \Gamma^{-1}$ one needs to apply the contraction principle to Γ. To this end it suffices to choose a topology on $\mathsf{P}(X)^{\mathbb{N}}$ such that Γ is continuous at each Q with $I(Q) < \infty$, see [Puh91] or [Gar04] for details. The most obvious choice is the product topology. As $\widetilde{T}$ is continuous at all measures $Q \ll \lambda$ (see Lemma 2 in section 4), the same holds for each $\widetilde{T}^t$. Therefore, the measures $Q_{\tilde{u}}^N \circ \Gamma^{-1}$ on $\mathsf{P}(X)^{\mathbb{N}}$ satisfy a LDP with good rate function

$$I_\Gamma(\boldsymbol{\nu}) = \inf\{I(Q) : \Gamma(Q) = \boldsymbol{\nu}\} \tag{8}$$

Consider the following special instances of such an LDP: Fix a measurable subset $A \subseteq \mathsf{P}(X)$ and consider the sets

$$\boldsymbol{A}_{s,t} = \{\boldsymbol{\nu} = (\nu_k)_{k\in\mathbb{N}} \in \mathsf{P}(X)^{\mathbb{N}} : \nu_k \in A \, (s \leqslant k < t)\}$$

for $s < t \leqslant \infty$. Then, by the LDP,

$$\begin{aligned} -\mathring{\gamma}_{s,t} := -\inf\{I_\Gamma(\boldsymbol{\nu}) : \boldsymbol{\nu} \in \mathring{\boldsymbol{A}}_{s,t}\} &\leqslant \liminf_{N\to\infty} \frac{1}{N} \log Q_{\tilde{u}}^N\{\boldsymbol{x} \in X^N : \Gamma(\epsilon_{\boldsymbol{x}}) \in \boldsymbol{A}_{s,t}\} \\ &\leqslant \limsup_{N\to\infty} \frac{1}{N} \log Q_{\tilde{u}}^N\{\boldsymbol{x} \in X^N : \Gamma(\epsilon_{\boldsymbol{x}}) \in \boldsymbol{A}_{s,t}\} \\ &\leqslant -\bar{\gamma}_{s,t} := -\inf\{I_\Gamma(\boldsymbol{\nu}) : \boldsymbol{\nu} \in \bar{\boldsymbol{A}}_{s,t}\}. \end{aligned} \tag{9}$$

Here $\mathsf{P}(X)$ is equipped with the Borel-σ-algebra of the weak topology on $\mathsf{P}(X)$. It coincides with the Borel-σ-algebra of the τ-topology on $\mathsf{P}(X)$ which is stronger than the weak topology because it is generated by all bounded measurable test functions $\psi : X \to \mathbb{R}$. For details see e.g. [DZ98, section 6.2]. We note here that the restriction of the τ-topology to the space $\{Q \in \mathsf{P}(X) : Q \ll \lambda\}$ coincides with the restriction of the weak topology on L^1_λ to $\mathcal{D}$ (after identification of a density u with its measure $Q = u\,\lambda$).

From a dynamical point of view, it would be most interesting to study large deviations for the sets $\boldsymbol{A}_{s,\infty}$. But since each $(\boldsymbol{T}_N, \boldsymbol{\mu}_N)$ is mixing, $Q_{\tilde{u}}^N\{\boldsymbol{x} \in X^N : \Gamma(\epsilon_{\boldsymbol{x}}) \in \boldsymbol{A}_{s,\infty}\} = 0$ for all $s > 0$ except if the set $\{\boldsymbol{x} \in X^N : \epsilon_{\boldsymbol{x}} \in A\}$ has full λ^N-measure in X^N, so that this leads mostly to trivial results. Therefore the only meaningful way to look at the limit $t \to \infty$ is to study $\mathring{\gamma}_{s,t}$ and $\bar{\gamma}_{s,t}$ for large t. To this end we provide the following lemma.

Proposition 1. *Fix $s > 0$.*

a) $\mathring{\gamma}_{s,t} = \inf\{I(Q) : \widetilde{T}^k Q \in \mathring{A} \, (s \leqslant k < t)\}$ *and* $\bar{\gamma}_{s,t} = \inf\{I(Q) : \widetilde{T}^k Q \in \bar{A} \, (s \leqslant k < t)\}$ *for all* $s < t \leqslant \infty$.
b) $\mathring{\gamma}_{s,t} \leqslant \mathring{\gamma}_{s,\infty}$ *and* $\bar{\gamma}_{s,t} \leqslant \bar{\gamma}_{s,\infty}$ *for all* $t > s$.
c) $\lim_{t\to\infty} \bar{\gamma}_{s,t} = \sup_{t>s} \bar{\gamma}_{s,t} = \bar{\gamma}_{s,\infty}$.

Only the last statement is not obvious. Its *proof* is deferred to section 4.

In order to get rid of the fixed initial time s we define

$$\mathring{\gamma}(A, \tilde{u}) := \inf_s \mathring{\gamma}_{s,\infty} = \lim_{s\to\infty} \mathring{\gamma}_{s,\infty} \quad \text{and} \quad \bar{\gamma}(A, \tilde{u}) := \inf_s \bar{\gamma}_{s\infty} = \lim_{s\to\infty} \bar{\gamma}_{s\infty}. \tag{10}$$

The notation recalls the dependence of all γ's on A and $\tilde{u}$. The following corollary characterises $\overset{\circ/-}{\gamma}(A; \tilde{u})$ in terms of asymptotic dynamical properties of the dynamical system $(\mathcal{D}, \widetilde{P})$.

Theorem 1. *For $A \subset \mathsf{P}(X)$ let $A' = \{u \in \mathcal{D} : u\lambda \in A\}$. Then the asymptotic values for the rates in the LDP from* (9) *satisfy*

$$\overset{\circ/-}{\gamma}(A, \tilde{u}) = \inf\left\{H(u|\tilde{u}) : \exists s > 0 \text{ such that } \widetilde{P}^k u \in \overset{\circ/-}{A'} \text{ for all } k \geqslant s\right\} \tag{11}$$

where $\overset{\circ/-}{A'}$ *denotes the interior/closure of* A' *is w.r.t. the weak topology on* L^1_λ.

Proof. If one replaces the condition $\widetilde{P}^k u \in \overset{\circ/-}{A'}$ in (11) by $\widetilde{T}^k(u\lambda) \in \overset{\circ/-}{A}$, the identity (11) is an immediate consequence of Proposition 1. So we have to show the equivalence of these two conditions. Recall that the bijection $u \leftrightarrow u\lambda$ is a homeomorphism between $\mathcal{D}$ (equipped with the weak L^1_λ-topology) and $\mathsf{P}_\lambda := \{Q \in \mathsf{P}(X) : Q \ll \lambda\}$ (equipped with the τ-topology). Therefore, $\widetilde{P}^k u \in \overline{A'}$ is equivalent to $\widetilde{T}^k(u\lambda) \in \overline{A \cap \mathsf{P}_\lambda}^{\mathsf{P}_\lambda}$ (the closure in the relative τ-topology on P_λ), which coincides with $\bar{A} \cap \mathsf{P}_\lambda$. As $\widetilde{T}^k(u\lambda) \in \mathsf{P}_\lambda$ for every $u \in \mathcal{D}$, this shows that $\widetilde{P}^k u \in \overline{A'}$ is equivalent to $\widetilde{T}^k(u\lambda) \in \bar{A}$. The argument for the interior is similar. □

In general, the infimum in (11) is difficult to evaluate. In the next section we have a closer look at this problem for the special system introduced in section 1.

3. Large deviations for a bistable system with chaotic units

Recall from Theorem B that, for the example introduced in section 1, the dynamical system $(\mathcal{D}, \widetilde{P})$ either has the unique globally stable fixed point u_0, or it has the stable fixed points $u_\pm := u_{\pm r_*}$ plus the "hyperbolic" fixed point u_0. In the latter case the basins of $u_\pm$ are open in the L^1_λ-norm topology on $\mathcal{D}$.

We specialise to sets $A_\alpha = \{Q \in \mathsf{P}(X) : \phi(Q) > \alpha\}$. It is open in the weak (and, a fortiori, in the τ-topology), and its closure is $\bar{A}_\alpha = \{Q \in \mathsf{P}(X) : \phi(Q) \geqslant \alpha\}$.

Theorem B from the introduction together with Theorem 1 implies the following identity.

Corollary 1. *For each* $\alpha < \phi(u_+)$,

$$\gamma(\tilde{u}) := \inf\{H(u|\tilde{u}) : u \in \mathrm{DA}(u_+)\} = \overset{\circ}{\gamma}(A_\alpha, \tilde{u}) = \bar{\gamma}(A_\alpha, \tilde{u}) \tag{12}$$

where $\mathrm{DA}(u_+)$ *is the domain of attraction of the stable fixed point* u_+ *of* $\widetilde{P} : \mathcal{D} \to \mathcal{D}$. *Note that this quantity is independent of* α.

The determination of $\gamma(\tilde{u})$ is difficult. What can be said immediately is:

I) If $\tilde{u}$ belongs to $\mathrm{DA}(u_+)$, then $\gamma(\tilde{u}) = 0$.

II) If $\tilde{u}$ belongs to $\mathrm{DA}(u_-)$, then $\gamma(\tilde{u})$ is strictly positive, because $\mathrm{DA}(u_-)$ is L^1_λ - open. There is no explicit lower bound on $\gamma(\tilde{u})$ that I can give in case $\tilde{u} \in \mathrm{DA}(u_-)$, but the following holds: As $\mathrm{DA}(u_-)$ is open in L^1_λ, there is $\delta > 0$ such that $B_\delta(\tilde{u}) \subset \mathrm{DA}(u_-)$, and as $H(u|\tilde{u}) \geqslant \|u - \tilde{u}\|_1^2$ by Pinsker's inequality, this implies immediately that $\gamma(\tilde{u}) \geqslant \delta^2$. As $\tilde{u} \in \mathrm{DA}(u_-)$, the density $\hat{\tilde{u}} : x \mapsto \tilde{u}(-x)$ belongs to $\mathrm{DA}(u_+)$, and hence

$$\gamma(\tilde{u}) \leqslant H(\hat{\tilde{u}}|\tilde{u}) \tag{13}$$

Denote the "multiplicative" symmetrisation of $\tilde{u}$ by

$$u_* = Z_*^{-1}\,(\tilde{u}\hat{\tilde{u}})^{1/2} \quad \text{where } Z_* = \int (\tilde{u}\hat{\tilde{u}})^{1/2}\, d\lambda \leqslant 1. \tag{14}$$

Estimate (13) can be rewritten now

$$\gamma(\tilde{u}) \leqslant H(\hat{\tilde{u}}|\tilde{u}) = -2\log Z_* + H(\tilde{u}|u_*).$$

A much better upper bound for $\gamma(\tilde{u})$ than this one is available.

Proposition 2. *If* $\tilde{u}$ *belongs to* $\mathrm{DA}(u_-)$, *then* $0 < \gamma(\tilde{u}) \leqslant H(u_*|\tilde{u}) = -\log Z_* < -\int \log \tilde{u}(x)dx$.

Proof. We will see in Proposition 3 that each symmetric density $u \in \mathcal{D}$ can be L^1_λ-approximated by densities v from $\mathrm{DA}(u_+)$ in such a way that the infimum of the entropies $H(v|\tilde{u})$ of the approximating densities is less or equal to $H(u|\tilde{u})$. Therefore,

$$\gamma(\tilde{u}) \leqslant \inf \{H(u|\tilde{u}) : u \in \mathcal{D} \,\text{symmetric}\}.$$

We evaluate this infimum: for symmetric densities u,

$$H(u|\tilde{u}) = \frac{1}{2}\int_X u \log \frac{u}{\tilde{u}}\, d\lambda + \frac{1}{2}\int_X \hat{u} \log \frac{\hat{u}}{\hat{\tilde{u}}}\, d\lambda = \int u \log \frac{u}{u_*} d\lambda - \log Z_*.$$

This quantity is minimised by $u = u_*$ so that $\gamma \leqslant H(u_*|\tilde{u}) = -\log Z_* < -\int \log u_* d\lambda = \int \log \tilde{u} d\lambda$ by Jensen's inequality. □

Proposition 3. *Each $u \in \mathrm{DA}(u_0)$ belongs to the L^1_λ - closure of $\mathrm{DA}(u_-)$ or of $\mathrm{DA}(u_+)$ in $\mathcal{D}$ or to both. If u is symmetric, it belongs to both closures.*

In any case $H(u|\tilde{u}) \geqslant \liminf_{\delta\to 0} H(v_\delta|\tilde{u})$ where the liminf is taken along a family of densities $v_\delta \in \mathrm{DA}(u_+)$ that approximates u.

The *proof* is deferred to section 4.

Because of the strong symmetry properties of our system one might conjecture that the estimate in Proposition 2 is optimal. Our main result, Theorem 2 below, shows that this is not the case. We start with some

Lemma 1. *Suppose that $\inf \tilde{u} > 0$ and let $u_* = Z_*^{-1}(\tilde{u}\hat{\tilde{u}})^{1/2}$ as before. Let g be a bounded function on X with $\int g\, d\lambda = 0$. Then, in the limit $\|g\|_\infty \to 0$,*

$$H(u_* + g|\tilde{u}) = H(u_*|\tilde{u}) - \int \log \tilde{u} \cdot g_{\mathrm{as}}\, d\lambda + \mathcal{O}\left(\left\|\frac{g^2}{u_*}\right\|_\infty\right)$$

where $g_{\mathrm{as}}(x) = \frac{1}{2}(g(x) - g(-x))$ is the antisymmetric part of g.

The *proof* is a direct calculation.

Remark 2. *Consider the case of Gibbs (=exponential) densities $\tilde{u} = Z_\beta^{-1} e^{\beta x}$ for $\beta \in \mathbb{R}$. So we assume that the initial configuration $(x_1, \dots, x_N)$ is chosen "as randomly as possible" under the sole constraint $\mathbb{E}[x_i] = \phi(\tilde{u}) = \frac{1}{2}\coth(\beta) - \frac{1}{\beta}$, a quantity that ranges from $-\frac{1}{2}$ to 0 when β is varied in $(-\infty, 0)$. As $\tilde{u}$ is decreasing for such β and as $\widetilde{P}$ maps decreasing densities to decreasing ones, $\widetilde{P}^n\tilde{u}$ is decreasing for all n, and so $\tilde{u}$ belongs to $\mathrm{DA}(u_-) \cup \mathrm{DA}(u_0)$.*

In this case $u_* = 1$, $Z_* = Z_\beta^{-1}$ so that $\gamma(\tilde{u}) \leqslant \log Z_\beta = \frac{\beta}{2} + \log \frac{1-e^{-\beta}}{\beta}$, see Proposition 2. Furthermore, $\frac{d}{dt}H(u_* + tg|\tilde{u})_{|t=0} = -\beta\phi(g_{\mathrm{as}}) = -\beta\phi(g)$ by Lemma 1, so that the entropy can be decreased only if the field of g is negative (recall that $\beta < 0$). This is the reason that exponential densities $\tilde{u}$ need a separate proof in the next theorem.

Theorem 2. *Suppose $\inf \tilde{u} > 0$ and let $u_* = Z_*^{-1}(\tilde{u}\hat{\tilde{u}})^{1/2}$ be as before. Then there are densities $u \in \mathrm{DA}(u_+)$ such that $H(u|\tilde{u}) < H(u_*|\tilde{u})$.*

As a consequence, $\gamma(\tilde{u}) < H(u_*|\tilde{u}) = -\log Z_*$ with strict inequality.

Remark 3. *I conjecture that Proposition 3 can be strengthened to show that $\mathrm{DA}(u_0)$ is the common boundary of $\mathrm{DA}(u_+)$ and $\mathrm{DA}(u_-)$ and that $\gamma(\tilde{u}) = \inf\{H(u|\tilde{u}) : u \in \mathrm{DA}(u_0)\}$.*

4. Proofs

Lemma 2. *Under assumption* (5)*, if each* T_r *is continuous* λ*-a.e., then the map* $\widetilde{T} : \mathsf{P}(X) \to \mathsf{P}(X)$ *is continuous at all measures* $Q \ll \lambda$.

Proof. The proof is similar to that of [BKZ09, Lemma 1]. Let $Q, Q_n \in \mathsf{P}(X)$ and suppose that $Q_n \rightharpoonup Q$ weakly. It suffices to prove that $\int \psi\, d(\widetilde{T}Q_n) \to \int \psi\, d(\widetilde{T}Q)$ for any Lipschitz continuous $\psi : X \to \mathbb{R}$. Define $r_n = r(Q_n) := G(\phi(\int x\, Q_n(dx)))$ and $r = r(Q)$ analogously.

Fix $\varepsilon > 0$ and denote $U_\varepsilon = \mathrm{cl}\{x \in X : \exists n \geqslant n_\varepsilon \text{ s.th. } d(T_r(x), T_{r_n}(x)) > \varepsilon\}$ where the n_ε can be chosen such that $\lambda(U_\varepsilon) < \varepsilon$ by assumption (5). The weak convergence of Q_n to Q implies that $\limsup_{n\to+\infty} Q_n(U_\varepsilon) \leqslant Q(U_\varepsilon)$. Then, for $n \geqslant n_\varepsilon$,

$$\begin{aligned}
&\Big|\int_X \psi\, d(\widetilde{T}Q) - \int_X \psi\, d(\widetilde{T}Q_n)\Big| = \Big|\int_X \psi\circ T_r\, dQ - \int_X \psi\circ T_{r_n}\, dQ_n\Big| \\
\leqslant &\Big|\int_X \psi\circ T_r\, d(Q-Q_n)\Big| + \Big|\int_{U_\varepsilon^c} (\psi\circ T_r - \psi\circ T_{r_n})\, dQ_n\Big| + \Big|\int_{U_\varepsilon} (\psi\circ T_r - \psi\circ T_{r_n})\, dQ_n\Big| \\
\leqslant &\Big|\int_X \psi\circ T_r\, d(Q-Q_n)\Big| + \mathrm{Lip}(\psi)\,\varepsilon + 2\|\psi\|_\infty\, Q_n(U_\varepsilon).
\end{aligned}$$

Since $Q \ll \lambda$ and the map T_r is continuous λ-a.e., the first term converges to zero as $n \to \infty$ and the third term is asymptotically bounded by $\limsup_{n\to\infty} 2\|\psi\|_\infty\, Q_n(U_\varepsilon) \leqslant 2\|\psi\|_\infty\, Q(U_\varepsilon)$ which tends to zero when $\varepsilon \to 0$. □

Proof of Proposition 1. a) For finite t and also for $t = \infty$ this is a simple consequence of the definitions and the properties of the product topology.

b) and c) It follows directly from the definitions of $\bar{\gamma}_{s,t}$ and $\mathring{\gamma}_{s,t}$ that both quantities are increasing in t. This implies b) and shows that the limit and the supremum in c) coincide.

It remains to show that $\sup_{t>s} \bar{\gamma}_{s,t} \geqslant \bar{\gamma}_{s,\infty}$. We may assume that $\gamma_s := \sup_{t>s} \gamma_{s,t} < \infty$, and consider a sequence of densities $(u_t)_{t>s}$ such that $\widetilde{T}^k(u_t\lambda) \in \bar{A}$ for $k = s, \dots, t$ and $\lim_{t\to\infty} H(u_t|\tilde{u}) = \gamma_s$. Let $m = \tilde{u}\lambda$ and $h_t = u_t/\tilde{u}$. Then the the densities h_t are uniformly integrable w.r.t. m because they have uniformly bounded entropy [Kal01, Exercise 4.6], and hence $(h_t)_{t>s}$ is precompact in the weak topology on L^1_m [Kal01, Lemma 4.13]. Let h be any weak limit point of the h_t. We show in Lemma 3 below that, for all weakly compact subsets $\mathcal{D}_c \subset \mathcal{D} \subset L^1_m$, the maps $\mathcal{D}_c \to \mathbb{R}$, $f \mapsto \frac{d}{dm}\widetilde{T}^k(fm)$ $(k \geqslant 0)$ are continuous w.r.t. the weak topology on L^1_m. A fortiori, each measure $\widetilde{T}^k(hm)$ $(k \geqslant s)$ with h as above is a limit point of the the measures $\widetilde{T}^k(u_t\lambda) = \widetilde{T}^k(h_t m)$ in the weak topology on $\mathsf{P}(X)$, and therefore $\widetilde{T}^k(hm) \in \bar{A}$.

By part a) of the lemma it remains to show that $H(h\tilde{u}|\tilde{u}) \leqslant \gamma_s$. Let, for $0 < \eta < 1$, $h^\eta = (1-\eta)(h \wedge \eta^{-1}) + \tilde{\eta}$ where $\tilde{\eta}$ is chosen such that h^η is a probability density w.r.t. m. Observe that $\tilde{\eta} \geqslant \eta$ and that $\tilde{\eta} \to 0$ when $\eta \to 0$. As

$$h(x)\ln h^\eta(x) \geqslant \begin{cases} h(x)\ln h(x) \geqslant -\frac{1}{e} & \text{if } h(x) \leqslant 1 \\ 0 & \text{if } h(x) > 1 \end{cases},$$

these functions have an integrable minorant, and we can apply Fatou's lemma:

$$H(h\tilde{u}|\tilde{u}) = \int h\ln h\, dm = \int \liminf_{\eta\to 0} h\ln h^\eta\, dm \leqslant \liminf_{\eta\to 0} \int h \ln h^\eta\, dm.$$

As, for each fixed η, h^η is bounded away from 0 and $+\infty$, the weak convergence of the h_t leads to

$$\begin{aligned} H(h\tilde{u}|\tilde{u}) &\leqslant \liminf_{\eta\to 0} \lim_{t\to\infty} \int h_t \ln h^\eta \, dm = \liminf_{\eta\to 0} \lim_{t\to\infty} \int u_t \ln h^\eta \, d\lambda \\ &= \liminf_{\eta\to 0} \lim_{t\to\infty} \left(\int u_t \ln h_t \, d\lambda - \int u_t \ln \frac{h_t}{h^\eta} \, d\lambda \right) \\ &= \liminf_{\eta\to 0} \lim_{t\to\infty} \left(H(u_t|\tilde{u}) - H(u_t|\tilde{u}h^\eta) \right) \\ &\leqslant \lim_{t\to\infty} H(u_t|\tilde{u}) \\ &= \gamma_{\alpha,s} \end{aligned}$$

□

Lemma 3. *Let $m = \tilde{u}\lambda$ be as in the proof of Proposition 1. For each $n \geqslant 0$ the map $\varphi_n : \mathcal{D}_c \to \mathbb{R}$, $f \mapsto \phi(\widetilde{P}^n(f\tilde{u}))$, is continuous w.r.t. the weak topology on L^1_m for all weakly compact subsets $\mathcal{D}_c \subset \mathcal{D} \subset L^1_m$.*

Proof. We prove this by induction on n. For $n = 0$, $f \mapsto \varphi_0(f) = \phi(f\tilde{u}) = \int x f(x)\, m(dx)$ is obviously continuous. Suppose the statement holds for $k = 0, \dots, n-1$. Then $f \mapsto r_{k+1}(f) := G(\varphi_k(f))$ is continuous for all such k. Suppose now that $f_t \in \mathcal{D}_c$ and $f_t \to f$ weakly in L^1_m. Denote $T_{\boldsymbol{r}(f)} := T_{r_n(f)} \circ \ldots \circ T_{r_1(f)}$. Then there are $\eta_t \to 0$ and measurable subsets $A_t \subset X$ such that

$$|T_{\boldsymbol{r}(f_t)}(x) - T_{\boldsymbol{r}(f)}(x)| < \eta_t \text{ for all } x \in X \setminus A_t \text{ and } m(A_t) < \eta_t.$$

Hence,

$$\begin{aligned} |\phi(\widetilde{P}^n(f_t\tilde{u})) - \phi(\widetilde{P}^n(f\tilde{u}))| &= \left| \int T_{\boldsymbol{r}(f_t)}(x) \cdot f_t(x)\, m(dx) - \int T_{\boldsymbol{r}(f)}(x) \cdot f(x)\, m(dx) \right| \\ &\leqslant \left| \int T_{\boldsymbol{r}(f)}(x) \cdot (f_t(x) - f(x))\, m(dx) \right| + \eta_t + \int_{A_t} f_t(x)\, m(dx) \end{aligned}$$

The first term in this sum converges to zero as $t \to \infty$ because the f_t converge to f weakly in L^1_m, the second term converges to zero by definition, and the third one because $\mathcal{D}_c$ is weakly compact in L^1_m so that the f_t are uniformly m-integrable and $m(A_t) < \eta_t \to 0$. □

For the remaining proofs we need to collect some notation and facts from [BKZ09]. In that paper essential use was made of a subclass $\mathcal{D}' \subset \mathcal{D}$ of probability densities on $X = [-\frac{1}{2}, \frac{1}{2}]$ that are (generalised) convex combinations of densities $w_y(x) = \frac{1-y^2/4}{(1-xy)^2}$, $y \in Y := [-\frac{2}{3}, \frac{2}{3}]$. Recall that $v \in \mathcal{D}'$ is a generalised convex combination of the w_y if there is a Borel probability μ on Y (the *representing measure*) such that $v(x) = \int_Y w_y(x)\, \mu(dy)$.

The following aspect of these densities will be used in the sequel:

(15) Each fractional linear map on $\mathbb{R}$ whose asymptotic value does not belong to $[-\frac{3}{2}, \frac{3}{2}]$ can be written in the form $f(x) = \frac{x+b}{yx+d}$ with $y \in Y$. Its inverse is $f^{-1}(z) = \frac{dz-b}{1-yz}$, whose derivative $(f^{-1})'(z) = (d-by)(1-\frac{y^2}{4})^{-1} w_y(z)$ is a multiple of w_y. Therefore, if an interval J is mapped diffeomorphically onto X by f, then the normalised Lebesgue measure on J is transformed into the measure $w_y\lambda$ on X. In particular, the density of this transformed measure belongs to $\mathcal{D}'$.

As each map T_r has two increasing linear fractional branches, each mapping a subinterval of X onto X, a density w_y is transformed under T_r into a convex combination $P_r w_y$ of two densities of this type. More generally, convex combinations of densities w_y are transformed to convex combinations of w_y's which means that $P_r(\mathcal{D}')\subseteq\mathcal{D}'$. The action of P_r on $\mathcal{D}'$ induces an action $\mathcal{L}_r^*:\mathsf{P}(Y)\to\mathsf{P}(Y)$ on their representing measures such that $P_r u=\int_Y w_y\,\mathcal{L}_r^*\mu(dy)$ for $u=\int_Y w_y\,\mu(dy)$. Analogously, there is a nonlinear (self consistent) operator $\widetilde{\mathcal{L}}^*$ on $\mathsf{P}(Y)$ such that $\widetilde{P}u=\int_Y w_y\,\widetilde{\mathcal{L}}^*\mu(dy)$ [BKZ09, Section 4.1].

$\mathcal{L}_r^*$ can be described in terms of two contractions $\sigma_r,\tau_r:Y\to Y$, namely $\mathcal{L}_r^*\delta_y=p_r\,\delta_{\sigma_r(y)}+(1-p_r)\,\delta_{\tau_r(y)}$. Both contractions are increasing on Y, and they satisfy $\sigma_r\leqslant\tau_r$. The maps σ_r and τ_r correspond to the mass transport by the left and right branch of T_r, respectively.

Let $R:=[-\frac{2}{5},\frac{2}{5}]$. For $\boldsymbol{r}=(r_1,\dots,r_n)\in R^n$ denote by $T_{\boldsymbol{r}}^{(n)}$ and $P_{\boldsymbol{r}}^{(n)}$ the composition $T_{r_n}\circ\cdots\circ T_{r_1}$ and its corresponding Perron Frobenius operator, respectively. For each $u\in\mathcal{D}$, $\widetilde{P}^n u=P_{\boldsymbol{r}}^{(n)}u$ for a suitable $\boldsymbol{r}\in R^n$ that is determined by the density u. Analogously, for $u\in\mathcal{D}'$ with representing measure μ, $\widetilde{\mathcal{L}}^{*n}\mu=\mathcal{L}_{\boldsymbol{r}}^{*(n)}\mu$. We note the following fact:

(16) $\mathcal{L}^{*(n)}\delta_y$ is a convex combination of 2^n point masses in points $\kappa_n\circ\cdots\circ\kappa_1(y)$ where $\kappa_i\in\{\sigma_{r_i},\tau_{r_i}\}$ for all i. Each of these contributions represents the transport of w_y by the corresponding branch of $T_{\boldsymbol{r}}$. In view of the monotonicity properties of the σ_r and τ_r, $\sigma_{\boldsymbol{r}}(y)\leqslant\kappa_n\circ\cdots\circ\kappa_1(y)\leqslant\tau_{\boldsymbol{r}}(y)$ for all these points.

For later use we note some further estimates. They follow easily from the explicit formulas $w_y(x)=\frac{1-y^2/4}{(1-xy)^2}$ and $\sigma_r(y)=\frac{2(y+r)}{(r+1)y+r+4}$ [BKZ09, Section 4.1] using $\sigma_r^{-1}(r)=\frac{r}{1-r}$.

$$\begin{aligned}\|w_{\sigma_r^{-1}(r)}-1\|_\infty&\leqslant\mathrm{const}\cdot|\sigma_r^{-1}(r)|\leqslant\mathrm{const}\cdot|r|\\ |\phi(w_y)|&\leqslant\mathrm{const}\cdot|y|\\ |G^{-1}(r)|&\leqslant\mathrm{const}\cdot|r|\end{aligned}\tag{17}$$

We will also use the fact that the $\mathcal{L}_r^*$ and also $\widetilde{\mathcal{L}}^*$ respect the stochastic order $\prec$ on $\mathsf{P}(Y)$ [BKZ09, Section 4.3]. Finally we introduce one further notation: $\chi_r(x)=-1$ if $x<-\frac{r}{4}$ and $\chi_r(x)=1$ if $x\geqslant-\frac{r}{4}$. Recall from (1) that $-\frac{r}{4}$ is the discontinuity of T_r.

Proof of Proposition 3. Let $u\in\mathrm{DA}(u_0)$ and $\varepsilon>0$. We will show that there is $v\in\mathrm{DA}(u_-)\cup\mathrm{DA}(u_+)$ such that $\|u-v\|_1<3\varepsilon$. First there is a Lipschitz-continuous $v_1\in\mathcal{D}$ such that $\|u-v_1\|_1<\varepsilon$. Fix n such that $\|v_1-\mathbb{E}[v_1|\mathcal{Z}]\|<\varepsilon$ for each partition $\mathcal{Z}$ of X into subintervals of length at most $(\frac{3}{4})^n$. As $\inf T_r'\geqslant\frac{4}{3}$ for each $r\in R$, each composition $T_{\boldsymbol{r}}^{(n)}$ with $\boldsymbol{r}=(r_1,\dots,r_n)\in R^n$ gives rise to a decomposition $\mathcal{Z}_{\boldsymbol{r}}$ of X into 2^n monotonicity intervals of at most this size. Hence, letting $v_{\boldsymbol{r}}:=\mathbb{E}[v_1|\mathcal{Z}_{\boldsymbol{r}}]$, we have $\|v_1-v_{\boldsymbol{r}}\|_1<\varepsilon$ for sufficiently large n. Observe that, typically, $\widetilde{P}^n v_{\boldsymbol{r}}\neq P_{\boldsymbol{r}}^{(n)}v_{\boldsymbol{r}}$, because $G(\phi(\widetilde{P}^i v_{\boldsymbol{r}}))$ does not coincide with r_{i+1} for all $i=0,\dots,n-1$. But the map associating with $\boldsymbol{r}\in R^n$ the n-tupel of parameters $(G(\phi(v_{\boldsymbol{r}})),G(\phi(\widetilde{P}v_{\boldsymbol{r}})),\dots,G(\phi(\widetilde{P}^{n-1}v_{\boldsymbol{r}})))\in R^n$ is continuous so that it has at least one fixed point according to Brouwer's theorem. From now on, $\boldsymbol{r}$ denotes such a fixed point. Then $\tilde{v}:=\widetilde{P}^n v_{\boldsymbol{r}}=P_{\boldsymbol{r}}^{(n)}v_{\boldsymbol{r}}$ is a convex combination of the 2^n densities $w_{\kappa_n\circ\cdots\circ\kappa_1(0)}$, see (16). So $\tilde{v}$ belongs to $\mathcal{D}'$, see (15). If $\tilde{v}\in\mathrm{DA}(u_-)\cup\mathrm{DA}(u_+)$, then also $v_{\boldsymbol{r}}\in\mathrm{DA}(u_-)\cup\mathrm{DA}(u_+)$ and as $\|v_{\boldsymbol{r}}-u\|_1<2\varepsilon$, we are done. Otherwise, $v_{\boldsymbol{r}}\in\mathrm{DA}(u_0)$, and we will approximate it by some $v_\delta\in\mathrm{DA}(u_-)\cup\mathrm{DA}(u_+)$.

To this end fix functions $g_j\in L_m^1$ such that $\phi(g_j)=1$ and $P_{r_{j+1}}g_j=0$ for $j=0,\dots,n-1$. Possible choices are $g_j=\beta_j\cdot\chi_{r_{j+1}}\cdot T_{r_{j+1}}'$ where β_j is a suitable normalising constant. Finally let $q\in\mathcal{D}$ be constant on the rightmost (or leftmost) monotonicity interval of $T_{\boldsymbol{r}}^{(n)}$ and 0 elsewhere. Then define

$$v_\delta := (1-\delta)v_{\boldsymbol{r}} + \delta\left(q + \sum_{j=0}^{n-1}\gamma_j \frac{g_j}{P_{\boldsymbol{r}}^{(j)}1}\circ T_{\boldsymbol{r}}^{(j)}\right)$$

whereIt follows that $\delta > 0$ is small and where $\gamma_0, \dots, \gamma_{n-1}$ are determined such that $\phi(P_{\boldsymbol{r}}^{(k)}v_\delta) = \phi(P_{\boldsymbol{r}}^{(k)}v_{\boldsymbol{r}})$ for $k = 0, \dots, n-1$. This can be done recursively for $k = n-1, \dots, 0$ because

$$\begin{aligned}
&\phi(P_{\boldsymbol{r}}^{(k)}v_\delta) - \phi(P_{\boldsymbol{r}}^{(k)}v_{\boldsymbol{r}}) \\
=&\delta\left(\phi(P_{\boldsymbol{r}}^{(k)}(q - v_{\boldsymbol{r}})) + \sum_{j=0}^{k-1}\gamma_j\phi(P_{r_k}\circ\ldots\circ P_{r_{j+1}}g_j)\right. \\
&\left.\quad + \gamma_k\phi(g_j) + \sum_{j=k+1}^{n-1}\gamma_j\phi\left(P_{\boldsymbol{r}}^{(k)}1\cdot\left(\frac{g_j}{P_{\boldsymbol{r}}^{(j)}1}\circ T_{r_j}\circ\ldots\circ T_{r_{k+1}}\right)\right)\right) \\
=&\delta\left(\phi(P_{\boldsymbol{r}}^{(k)}(q - v_{\boldsymbol{r}})) + \gamma_k + \sum_{j=k+1}^{n-1}\gamma_j\phi\left(P_{\boldsymbol{r}}^{(k)}1\cdot\left(\frac{g_j}{P_{\boldsymbol{r}}^{(j)}1}\circ T_{r_j}\circ\ldots\circ T_{r_{k+1}}\right)\right)\right)
\end{aligned}$$

for all these k. Note that the choice of the γ_j does not depend on δ. This implies that $\|v_\delta - v_{\boldsymbol{r}}\|_1 \leqslant \delta C_n$ for some constant C_n independent of δ and therefore $\|v_\delta - u\|_1 < 3\varepsilon$ when $\delta < C_n^{-1}\varepsilon$. Note also that $v_\delta \in \mathcal{D}$ for sufficiently small $\delta > 0$ because $\int \frac{g_j}{P_{\boldsymbol{r}}^{(j)}1}\circ T_{\boldsymbol{r}}^{(j)}(x)dx = \int g_j(x)dx = \int P_{r_{j+1}}g_j(x)dx = 0$, and $\inf_x v_{\boldsymbol{r}}(x) > 0$ since $v_{\boldsymbol{r}} \in \mathcal{D}'$.

Therefore, $\widetilde{P}^n v_\delta = P_{\boldsymbol{r}}^{(n)}v_\delta = (1-\delta)P_{\boldsymbol{r}}^{(n)}v_{\boldsymbol{r}} + \delta P_{\boldsymbol{r}}^{(n)}q$ belongs to $\mathcal{D}'$ and, denoting the representing measures of $P_{\boldsymbol{r}}^{(n)}v_\delta$, $P_{\boldsymbol{r}}^{(n)}v_{\boldsymbol{r}}$ and $P_{\boldsymbol{r}}^{(n)}q$ by μ_δ, $\mu_{\boldsymbol{r}}$ and μ_q, respectively, it follows that $\mu_\delta = (1-\delta)\mu_{\boldsymbol{r}} + \delta\mu_q$. Now the choice of q as being concentrated on the rightmost (or leftmost) monotonicity interval of $T_{\boldsymbol{r}}^{(n)}$ implies that $\mu_q \succ \mu_{\boldsymbol{r}}$ (or $\mu_q \prec \mu_{\boldsymbol{r}}$), compare (16). Therefore, $\mu_\delta \succ \mu_{\boldsymbol{r}}$ (or $\mu_\delta \prec \mu_{\boldsymbol{r}}$).

It follows that at most one of the two densities $\widetilde{P}^n v_\delta = P_{\boldsymbol{r}}^{(n)}v_\delta$ and $\tilde{v} = P_{\boldsymbol{r}}^{(n)}v_{\boldsymbol{r}} = \widetilde{P}^n v_{\boldsymbol{r}}$ can belong to $\mathrm{DA}(u_0)$, see [BKZ09, Lemma 13]. As we assumed here that $v_{\boldsymbol{r}} \in \mathrm{DA}(u_0)$, we conclude that $\widetilde{P}^n v_\delta \in \mathrm{DA}(u_-) \cup \mathrm{DA}(u_+)$ and hence also $v_\delta \in \mathrm{DA}(u_-) \cup \mathrm{DA}(u_+)$.

It remains to check that the entropies $H(v_\delta|\tilde{u})$ approach $H(u|\tilde{u})$ when first $\delta \to 0$ and then $n \to \infty$ and $\varepsilon \to 0$. As it is possible to perform the first approximation of u by a Lipschitz density v in such a way that $|H(v|\tilde{u}) - H(u|\tilde{u})| < \varepsilon$, this follows immediately, because all other densities occurring in the proof are bounded in a way that L^1_λ - convergence implies convergence of the relative entropies. $\square$

Proof of Theorem 2.
Exponential densities: As indicated in Remark 2, we begin with the special case of exponential densities $\tilde{u}(x) = Z_\beta^{-1}e^{\beta x}$, $\beta < 0$. Recall that $u_* = 1$ in this case.

As $H(u_* + g|\tilde{u}) = H(u_*|\tilde{u}) - \beta\phi(g_{\mathrm{as}}) + \mathcal{O}(\|g\|_\infty^2) = H(u_*|\tilde{u}) - \beta\phi(g) + \mathcal{O}(\|g\|_\infty^2)$ for functions g with $\int g(x)\,dx = 0$, we will construct densities

$$u = u_* + g\ , \quad g = (w_{\sigma_r^{-1}(r)} - 1 + t_r g_r) \quad (r > 0)$$

with suitable functions g_r such that $P_r g_r = 0$ and $\phi(g_r) = 1$. This ensures on the one hand that

$$\phi(g) = \phi(w_{\sigma_r^{-1}(r)} - 1 + t_r g_r) = \phi(w_{\sigma_r^{-1}(r)}) + t_r,$$

so that, choosing $t_r = G^{-1}(r) - \phi(w_{\sigma_r^{-1}(r)})$, we can make sure that $\phi(g) = G^{-1}(r) > 0$ and hence $H(u|\tilde{u}) < H(u_*|\tilde{u})$ for sufficiently small $r > 0$. To control the $\mathcal{O}(||g||_\infty^2)$ - term one has to use the estimates in (17). On the other hand, such a choice guarantees that $\widetilde{P}u = P_{G(\phi(u))}u = P_{G(\phi(g))}u = P_r u = P_r w_{\sigma_r^{-1}(r)} = p_r(\sigma_r^{-1}(r)) \cdot w_{\sigma_r(\sigma_r^{-1}(r))} + (1 - p_r(\sigma_r^{-1}(r))) \cdot w_{\tau_r(\sigma_r^{-1}(r))} = (\frac{1}{2} - \frac{r}{2-r}) \cdot w_r + (\frac{1}{2} + \frac{r}{2-r}) \cdot w_{\tau_r(\sigma_r^{-1}(r))}$ where we used the explicit formula for $p_r(y)$ from [BKZ09, Eq. (4.4)]. Therefore, $\widetilde{P}u$ belongs to $\mathcal{D}'$ with representing measure $(\frac{1}{2} - \frac{r}{2-r})\delta_r + (\frac{1}{2} + \frac{r}{2-r})\delta_{\tau_r(\sigma_r^{-1}(r))} \succ \delta_r \succ \delta_0$ (observe that $\tau_r(\sigma_r^{-1}(r)) > \sigma_r(\sigma_r^{-1}(r)) = r$). Hence $\widetilde{P}u \in \mathrm{DA}(u_+)$ by [BKZ09, Proposition 2].

It remains to construct the functions g_r. Possible choices are $g_r = (2 + O(r))\, \chi_r\, T_r'$ with $O(r)$ chosen such that $\phi(g_r) = 1$.

The case of general $\tilde{u}$: For each antisymmetric, bounded $g : X \to \mathbb{R}$ such that $P_0 g = 0$, we have

$$0 = \int P_0 g(x) dx = \int g(x) T_0(x) dx = \int g(x)(2x - \frac{1}{2}\chi_0(x)) dx = 2\phi(g) - \int_0^{1/2} g(x) dx,$$

so that $\phi(g) = \frac{1}{2}\int_0^{1/2} g(x)dx$. Motivated by Lemma 1 we fix any such g with

$$\phi(g) = 0 \quad \text{and} \quad \int \log \tilde{u}(x) \cdot g(x) dx > 0. \tag{18}$$

As the linear space of all bounded, antisymmetric g with $P_0 g = 0$ is infinite-dimensional, such a function g will always exist except if $\tilde{u}$ is an exponential density. (For example, $g(x) = \frac{1}{3}\chi_0(x) - \mathrm{sign}(\sin(6\pi x))$ will do in case $\tilde{u} = u_r$ for $r < 0$, as a numerical check shows. Indeed, $\int \log u_r(x) \cdot g(x) dx > 0$ for all $r < 0$ in this case.) Observe that, for sufficiently small $t > 0$,

$$\Delta_t := H(u_*|\tilde{u}) - H(u_* + tg|\tilde{u}) > 0 \tag{19}$$

because of Lemma 1 and (18).

We proceed by approximating $u_* + tg$ by a density that is mapped under finitely many iterations of $\widetilde{P}$ to a density from $\mathcal{D}'$. Let $\mathcal{Z}_n$ be the partition of X into monotonicity intervals of T_0^n and $u_*^{(n)} = \mathbb{E}[u_*|\mathcal{Z}_n]$. As u_* is symmetric and as all intervals of $\mathcal{Z}_n$ have length 2^{-n}, also $u_*^{(n)}$ is symmetric. Choose $n = n(t)$ such that

$$|H(u_* + tg|\tilde{u}) - H(u_*^{(n)} + tg|\tilde{u})| = \left|\int g \log \frac{u_*}{u_*^{(n)}}\, d\lambda\right| + \mathcal{O}\left(\left\|\frac{g^2}{u_*}\right\|_\infty\right) + \mathcal{O}\left(\left\|\frac{g^2}{u_*^{(n)}}\right\|_\infty\right) < \frac{\Delta_t}{4} \tag{20}$$

in the limit $\|g\|_\infty \to 0$. (Observe that $\inf u_*^{(n)} \geqslant \inf u_* > 0$.) Let $\beta > 0$ and

$$v_{t,\beta} := u_*^{(n)} + t\left(g + \beta\left(\sum_{j=0}^{n-1} 4\gamma_j \chi_0 \circ T_0^j + 4(\chi_r - r/2) \circ T_0^n\right)\right)$$

where $r = r(\beta, t)$ is the unique (positive !) solution of $G(t\beta(1 - r^2/4)) = r$, and $\gamma_0, \dots, \gamma_{n-1}$ are determined such that

$$\phi(P_0^k v_{t,\beta}) = \begin{cases} 0 & \text{for } k = 0, \dots, n-1 \\ t\beta(1 - r^2/4) & \text{for } k = n. \end{cases}$$

This can be done recursively as in the proof of Proposition 3: as all $P_0^k u_*^{(n)}$ are symmetric, as $P_0\chi_0 = 0$, $P_0 g = 0$ and $\phi(g) = 0$, and as $\phi(P_0^k(\chi_0 \circ T_0^j)) = \int P_0^{j-k}\,\mathrm{id}\cdot\chi_0\,d\lambda = 2^{-(j-k)}\,\phi(\chi_0) = \frac{1}{4}2^{-(j-k)}$ for $j \geqslant k$,

$$\begin{aligned} \phi(v_{t,\beta}) &= t\beta\left(\sum_{j=0}^{n-1}\gamma_j 2^{-j} + 2^{-n}(1 - r^2/4)\right) \\ \phi(P_0^k v_{t,\beta}) &= t\beta\sum_{j=0}^{k-1}4\gamma_j\phi(P_0^{k-j}\chi_0) + t\beta\sum_{j=k}^{n-1}4\gamma_j\phi(\chi_0 \circ T_0^{j-k}) + t\beta 2^{-(n-k)}(1 - r^2/4) \\ &= t\beta\sum_{j=k}^{n-1}\gamma_j 2^{-(j-k)} + t\beta 2^{-(n-k)}(1 - r^2/4)\,\text{for } k \geqslant 1. \end{aligned}$$

Observe that the γ_j can be chosen independently of t and β. Observe also that $\int \chi_r(x)dx = r/2$ so that $v_{t,\beta}$ is indeed a probability density for small enough t and β. In view of (19) and (20),

$$H(v_{t,\beta}|\tilde{u}) = H(u_*|\tilde{u}) - \Delta_t + \{H(v_{t,\beta}|\tilde{u}) - H(u_*^{(n)} + tg|\tilde{u})\}.$$

The term in curly brackets is bounded by $\frac{\Delta_t}{4}$ provided $\|t\beta(\sum_{j=0}^{n-1}4\gamma_j\chi_0 \circ T_0^j + 4(\chi_r - r/2) \circ T_0^n)\|_\infty$ is small enough. But this quantity is bounded by $t\beta 4(\sum_{j=0}^{n}|\gamma_j| + 1)$, so it can be controlled by choosing β close to 0. Therefore, for all sufficiently small $t > 0$ and $n = n(t)$ as above, there is $\beta_t > 0$ such that

$$H(v_{t,\beta}|\tilde{u}) < H(u_*|\tilde{u}) - \frac{\Delta_t}{2} \quad \text{for all } |\beta| \leqslant \beta_t. \tag{21}$$

As a final step we show that $v_{t,\beta} \in \mathrm{DA}(u_+)$ if $0 < \beta \leqslant \beta_t$. As $G(\phi(P_0^k v_{t,\beta})) = G(0) = 0$ for $k = 0, \dots, n-1$, we have

$$\begin{aligned} \widetilde{P}^n v_{t,\beta} &= P_0^n v_{t,\beta} \\ &= P_0^n u_*^{(n)} + tP_0^n g + t\beta\left(\sum_{j=0}^{n-1}4\gamma_j P_0^{n-j}\chi_0 + 4(\chi_r - r/2)\right) \\ &= P_0^n u_*^{(n)} + t\beta 4(\chi_r - r/2) \\ &= 1 + t\beta 4(\chi_r - r/2). \end{aligned}$$

Here we used that $P_0^n u_*^{(n)} = 1$, $P_0 g = 0$ and $P_0\chi_0 = 0$. Therefore, by the above choice of $r = r(t, \beta)$,

$$G(\phi(\widetilde{P}^n v_{t,\beta})) = G(\phi(1 + t\beta 4(\chi_r - r/2))) = G(\phi(t\beta 4\chi_r)) = G(t\beta(1 - r^2/4)) = r$$

so that

$$\begin{aligned} \widetilde{P}^{n+1}v_{t,\beta} &= P_r(1 + t\beta 4(\chi_r - r/2)) \\ &= (1 + t\beta(-4 - 2r))P_r 1_{[-\frac{1}{2}, -\frac{r}{4}]} + (1 + t\beta(4 - 2r))P_r 1_{[-\frac{r}{4}, \frac{1}{2}]} \end{aligned}$$

Indeed, this is a convex combination of $(\frac{1}{2}-\frac{r}{4})^{-1}P_r 1_{[-\frac{1}{2},-\frac{r}{4}]}$ and $(\frac{1}{2}+\frac{r}{4})^{-1}P_r 1_{[-\frac{r}{4},\frac{1}{2}]}$, both densities in $\mathcal{D}'$. The representing measure of the first one is supported by the point $\sigma_r(0)$, that of the second one by $\tau_r(0)$. As $r>0$, these points are to the right hand side of 0 so that the supporting measure of $\widetilde{P}^{n+1}v_{t,\beta}$ is $\succ \delta_0$. It follows from Lemma 13 in [BKZ09] that $\widetilde{P}^{n+1}v_{t,\beta} \in \mathrm{DA}(u_+)$ so that also $v_{t,\beta} \in \mathrm{DA}(u_+)$.

□

Hyperbolicity of the fixed point $h_0 = 1$ - a correction

Proposition 5 of [BKZ09] states that $D\widetilde{P}_{|\mathcal{D}\cap\mathrm{BV}(X)}$ has $h_0=1$ as a hyperbolic fixed point. The proof gives (correctly) the unstable eigendirection $[x]$ with eigenvalue $\lambda = \frac{1}{2}+\frac{B}{12}$, but the codimension one stable eigenspace given there is wrong. Instead it is

$$E_s := \left\{ g \in \mathrm{BV}(X) : \int g(x)dx = 0 \,\text{and}\, \phi\left((\lambda - P_0)^{-1}(g)\right) = 0 \right\}.$$

Indeed, for $g \in E_s$ we have $\phi(g) = -\sum_{k=1}^{\infty} \phi\left((\lambda^{-1}P_0)^k(g)\right)$, from which it follows easily that

$$\left(D\widetilde{P}_{|h_0}\right)^n(g) = P_0^n g - B\cdot\phi\left((\lambda-P_0)^{-1}(P_0^n g)\right) = \mathcal{O}(2^{-n}).$$

References

[BKZ09] J.-B. Bardet, G. Keller, and R. Zweimüller. Stochastically stable globally coupled maps with bistable thermodynamic limit. *Commun. Math. Phys.*, 292:237–270, 2009.

[DZ98] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998.

[Gar04] J. Garcia. An extension of the contraction principle. *Journal of Theoretical Probability*, 17:403–434, 2004.

[Kal01] O. Kallenberg. *Foundations of Modern Probability*. Springer, 2nd edition, 2001.

[Kel00] G. Keller. An ergodic theoretic approach to mean field coupled maps. In *Fractal geometry and stochastics, II (Greifswald/Koserow, 1998)*, volume 46 of *Progress in Probability*, pages 183–208. Birkhäuser, 2000.

[Puh91] A. Puhalskii. On functional principle of large deviations. *New Trends Proba. Statist.*, 1:198–218, 1991.