

Comment on: "New exact solutions of the (3+1)-dimensional Burgers System" [Phys. Lett. A 373 (2009) 181]

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Abstract

We demonstrate that all exact solutions of the Riccati equation by Dai and Wang [C.-Q. Dai, Y.-Y. Wang, Phys. Lett. A 373 (2009) 181–187] are not new and cannot be new because the general solution of this equation was obtained more than one century ago. Moreover we show that some "new solutions" by Dai and Wang of the Riccati equation do not satisfy this equation. We also illustrate that the authors did not obtain any new solutions for solution of the (3+1)-dimensional Burgers system.

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1 On "new" solutions of the Riccati equation by Dai and Wang

Dai and Wang [1] looked for exact solutions of the Riccati equation

$$\frac{d\varphi(\xi)}{d\xi} = l_0 + \varphi^2(\xi), \quad (1)$$

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where l_0 is constant. In their letter these authors say: "we firstly use the the Exp—function method [2] to seek new exact solutions of the Riccati equation (1)". Firstly Dai and Wang are wrong here because they are not first who applied the Exp-function method to find "new solutions" of the Riccati equation. Zhang was first in [3] who used the Exp - function method to find "new generalized solitary solutions of Riccati equation". Criticism of the paper by Zhang [3] was given in our recent work [4].

Secondly Dai and Wang claim: "we obtain some new exact solutions of the Riccati equation". The Riccati equation (1) has been studied during several centuries, therefore statement by Dai and Wang may cause a sensation. Unfortunately there is no sensation because this statement is wrong as well.

Let us show that some "solutions" by Dai and Wang are not new and some of them do not satisfy the Riccati equation. "Introducing a complex variable" $\eta = k\xi + \xi_0$ Dai and Wang rewrite equation (1) in the form

$$k\varphi' - l_0 - \varphi^2 = 0, \quad (2)$$

where k is a constant, ξ_0 is an arbitrary constant. It is well known (see any textbook on differential equations) that this equation has the general solution in the form

$$\varphi(\eta) = -\sqrt{-l_0} \tanh\left(\frac{\sqrt{-l_0}}{k}\eta\right), \quad \eta = k\xi + \xi_0. \quad (3)$$

This solution depends on one arbitrary constant ξ_0 . In the limit $k \rightarrow \mp 0$ this formula degenerates to the constant solution

$$\varphi(\eta) = \pm\sqrt{-l_0}. \quad (4)$$

There is no any other solution of equation (2) besides (3) and (4).

Using the Exp-function method Dai and Wang found *five exact solutions* of the Riccati equation (2). These solutions are as follows:

$$\varphi_1 = \frac{-\sqrt{-l_0}b_1e^\eta + a_{-1}e^{-\eta}}{b_1e^\eta + \frac{a_{-1}}{\sqrt{-l_0}}e^{-\eta}}, \quad \eta = \sqrt{-l_0}\xi + \xi_0, \quad (5)$$

$$\varphi_2 = \frac{-i\sqrt{l_0}b_1e^\eta + a_{-1}e^{-\eta}}{b_1e^\eta - i\frac{a_{-1}}{\sqrt{l_0}}e^{-\eta}}, \quad \eta = i\sqrt{l_0}\xi + \xi_0, \quad (6)$$

$$\varphi_3 = \frac{-\sqrt{-l_0}b_1e^\eta + a_0 - \sqrt{-l_0}b_{-1}e^{-\eta}}{\frac{a_0^2 + l_0b_0^2}{4l_0b_{-1}}e^\eta + b_0 + b_{-1}e^{-\eta}}, \quad \eta = 2\sqrt{-l_0}\xi + \xi_0, \quad (7)$$

$$\varphi_4 = \frac{\sqrt{-l_0}b_2e^{2\eta} + a_1e^\eta + a_0 - \frac{\sqrt{-l_0}}{l_0}b_{-1}e^{-\eta}}{b_2e^{2\eta} + b_1e^\eta + \frac{a_1^2+l_0b_1^2+2\sqrt{-l_0}a_0b_2}{2l_0b_2} + \frac{(a_1+\sqrt{-l_0}b_1)(4a_0b_2-\sqrt{-l_0}a_1^2-l_0\sqrt{-l_0}b_1^2)}{8l_0^2b_2^2}e^{-\eta}},$$

$$\eta = 2\sqrt{-l_0}\xi + \xi_0, \quad (8)$$

$$\varphi_5 = \frac{\sqrt{-l_0}b_2e^{2\eta} - \sqrt{-l_0}b_1e^\eta - \frac{a_{-1}b_2}{b_1} + a_{-1}e^{-\eta}}{b_2 \exp(2\eta) + b_1e^\eta + \frac{\sqrt{-l_0}a_{-1}b_2}{l_0b_1} + \frac{\sqrt{-l_0}}{l_0}e^{-\eta}}, \quad \eta = -2\sqrt{-l_0}\xi + \xi_0. \quad (9)$$

Solutions (5)–(9) correspond to formulae (17), (20), (23), (27) and (28) in the work [1].

Note that the Riccati equation (2) is the first-order differential equation and all solutions of this equation can have *only one arbitrary constant* [5–8]. Therefore formulae (5)–(9) can contain only one arbitrary constant. But Dai and Wang believe that there are more arbitrary constants in each expression (5)–(9). Namely, they claim that there are: two arbitrary constants a_{-1} and b_1 in φ_1 and φ_2 , three arbitrary constants a_0 , b_0 , and b_{-1} in φ_3 , four arbitrary constants a_0 , a_1 , b_1 , and b_2 in φ_4 , three arbitrary constants a_{-1} , b_1 , and b_2 in φ_5 .

Let us demonstrate that this is not the case because it is not possible never. Rewriting expression (5) we have

$$\varphi_1 = \frac{-\sqrt{-l_0}b_1e^\eta + a_{-1}e^{-\eta}}{b_1e^\eta + \frac{a_{-1}}{\sqrt{-l_0}}e^{-\eta}} = -\sqrt{l_0} \frac{1 - \frac{a_{-1}}{\sqrt{-l_0}b_1}e^{2\eta}}{1 + \frac{a_{-1}}{\sqrt{-l_0}b_1}e^{2\eta}} =$$

$$= -\sqrt{-l_0} \tanh \left(\eta - \frac{1}{2} \log \frac{a_{-1}}{\sqrt{-l_0}b_1} \right). \quad (10)$$

Due to $\eta = \sqrt{-l_0}\xi + \xi_0$ we obtain that there is only one arbitrary constant $\xi_0 - \frac{1}{2} \log \frac{a_{-1}}{\sqrt{-l_0}b_1}$ in the argument of tanh. Therefore solution φ_1 by Dai and Wang coincides with known solution (3).

We can see that solution φ_2 by Dai and Wang is equal to φ_1 because $\sqrt{-l_0} = i\sqrt{l_0}$. Therefore φ_2 coincides with solution (3).

Substituting φ_3 in Eq.(2) we do not get zero. Therefore expression φ_3 does not satisfy Eq.(2) and this expression is not a solution of the Riccati equation (2) if a_0 , b_0 , b_{-1} are arbitrary constants. Function φ_3 can be solution of the Riccati equation (2) only if we assume additional constraints

$$a_0^2 + b_0^2l_0 = 0, \quad b_{-1} \left(b_0l_0 - a_0\sqrt{-l_0} \right) = 0 \quad (11)$$

However in this case expression φ_3 is the trivial solution (4).

Substituting φ_4 in equation (2) we do not obtain zero as well. So function φ_4 is not a solution of the Riccati equation (2) if a_0 , a_1 , b_1 , and b_2 are arbitrary constants. Expression φ_4 is a solution of the Riccati equation (2) only if we take additional constraints into account

$$a_1 = -\sqrt{-l_0}b_1, \quad b_1b_2 = 0, \quad a_0b_2 = 0, \quad (12)$$

or

$$a_1 = \sqrt{-l_0}b_1, \quad a_0b_2 = 0, \quad a_0b_1 = 0. \quad (13)$$

In these cases expression φ_4 is the trivial solution (4) again.

Expression (9) can be presented as the following

$$\begin{aligned} \varphi_5 &= \frac{\sqrt{-l_0}b_2e^{2\eta} - \sqrt{-l_0}b_1e^\eta - \frac{a_{-1}b_2}{b_1} + a_{-1}e^{-\eta}}{b_2e^{2\eta} + b_1e^\eta + \frac{\sqrt{-l_0}a_{-1}b_2}{l_0b_1} + \frac{\sqrt{-l_0}}{l_0}a_{-1}e^{-\eta}} = \\ &= \frac{\sqrt{-l_0}(b_2e^\eta - b_1) \left(e^\eta - \frac{a_{-1}}{b_1\sqrt{-l_0}}e^{-\eta} \right)}{(b_2e^\eta + b_1) \left(e^\eta - \frac{a_{-1}}{b_1\sqrt{-l_0}}e^{-\eta} \right)} = \\ &= \sqrt{-l_0} \frac{(b_2e^\eta - b_1)}{(b_2e^\eta + b_1)} = \sqrt{-l_0} \tanh \left(\frac{\eta}{2} + \frac{1}{2} \log \frac{b_2}{b_1} \right). \end{aligned} \quad (14)$$

Due to $\eta = -2\sqrt{-l_0}\xi + \xi_0$ we obtain that there is only one arbitrary constant $\frac{\xi_0}{2} + \frac{1}{2} \log \frac{b_2}{b_1}$ in argument of tanh. Therefore solution φ_5 by Dai and Wang coincides with known solution (3).

Thus we have proved that Dai and Wang did not find new solutions of the Riccati equation (2). Moreover, functions φ_3 and φ_4 in general case are not solutions of the Riccati equation (2). What is more we state that nobody can not obtain new exact solutions of the Riccati equation (2). The statement by Dai and Wang in [1] on the Riccati equation is wrong.

2 On "new" solutions of the (3+1)-dimensional Burgers system by Dai and Wang

Dai and Wang have considered the (3+1)-dimensional Burgers system in [1] as well

$$u_t - 2uu_y - 2vu_x - 2wu_z - u_{xx} - u_{yy} - u_{zz} = 0, \quad (15)$$

$$u_x - v_y = 0, \quad u_z - w_y = 0.$$

Authors [1] claim: "based on the Riccati equation and its new exact solutions, we find new and more general exact solutions with two arbitrary functions of the (3+1)-dimensional Burgers system". In this section we show that this statement is wrong. All solutions of the (3+1)-dimensional Burgers system by Dai and Wang are not new.

Authors [1] have obtained two formal solutions of the system of equations (15)

$$u = -h\varphi(\xi), \quad v = \frac{p_t - p_{xx} - p_{zz}}{2p_x} - p_x\varphi(\xi), \quad w = -p_z\varphi(\xi) \quad (16)$$

and

$$\begin{aligned} u &= -\frac{1}{2}h\varphi(\xi) + \frac{1}{2}h\sqrt{l_0 + \varphi^2(\xi)}, \\ v &= \frac{p_t - p_{xx} - p_{zz}}{2p_x} - \frac{1}{2}p_x\varphi(\xi) + \frac{1}{2}p_x\sqrt{l_0 + \varphi^2(\xi)}, \\ w &= -\frac{1}{2}p_z\varphi(\xi) + \frac{1}{2}p_z\sqrt{l_0 + \varphi^2(\xi)}. \end{aligned} \quad (17)$$

Here $\varphi(\xi)$ is solution of the Riccati equation (1). In formulae (16)–(17) Dai and Wang take $\xi = p(x, z, t) + hy$, where $p(x, z, t)$ is an arbitrary function, h is an arbitrary constant.

Substituting expressions (5)–(9) to the formulae (16)–(17) Dai and Wang obtained ten "new solutions" of the Burgers system (15). We proved in previous section that expressions (5)–(9) by Dai and Wang are equivalent to two solutions (3) and (4) of the Riccati equation. Therefore Dai and Wang can get only four solutions of the Burgers system (15). Let us show that only two distinct solutions of (15) can be obtained in such a way.

Taking solution (3) of the Riccati equation (1) in the form

$$\varphi(\xi) = -\sqrt{-l_0} \tanh\left(\sqrt{-l_0}(p + hy) + \xi_0\right), \quad \xi = p + hy, \quad (18)$$

and substituting (18) into the formal solution (16) we have

$$\begin{aligned} u &= h\sqrt{-l_0} \tanh\left(\sqrt{-l_0}(p + hy) + \xi_0\right), \\ v &= \frac{p_t - p_{xx} - p_{zz}}{2p_x} + p_x\sqrt{-l_0} \tanh\left(\sqrt{-l_0}(p + hy) + \xi_0\right), \\ w &= p_z\sqrt{-l_0} \tanh\left(\sqrt{-l_0}(p + hy) + \xi_0\right). \end{aligned} \quad (19)$$

Without loss of generality we can introduce new arbitrary function $q(x, z, t) = \sqrt{-l_0}p(x, z, t) + \xi_0$ and new arbitrary constant $r = \sqrt{-l_0}h$. Then expressions

(19) take the form

$$u = r \tanh(q + ry), \quad v = \frac{q_t - q_{xx} - q_{zz}}{2q_x} + q_x \tanh(q + ry), \quad (20)$$

$$w = q_z \tanh(q + ry),$$

Now let us substitute the function (18) in the formal solution (17). After some transformations we have

$$u = \frac{1}{2}h\sqrt{-l_0} \tanh\left(\frac{\sqrt{-l_0}}{2}(p + hy) + \frac{\xi_0}{2} - \frac{i\pi}{4}\right),$$

$$v = \frac{p_t - p_{xx} - p_{zz}}{2p_x} + \frac{1}{2}p_x\sqrt{-l_0} \tanh\left(\frac{\sqrt{-l_0}}{2}(p + hy) + \frac{\xi_0}{2} - \frac{i\pi}{4}\right), \quad (21)$$

$$w = \frac{1}{2}p_z\sqrt{-l_0} \tanh\left(\frac{\sqrt{-l_0}}{2}(p + hy) + \frac{\xi_0}{2} - \frac{i\pi}{4}\right).$$

Introducing new arbitrary function $q(x, z, t) = \frac{\sqrt{-l_0}}{2}p(x, z, t) + \frac{\xi_0}{2} - \frac{i\pi}{4}$ and new arbitrary constant $r = \frac{\sqrt{-l_0}h}{2}$ we can rewrite expressions (21) in the form (20).

Therefore solutions (19) and (21) are the same. However solution (20) of the Burgers system (15) is not new. This solution was found by Li et al in work [9].

Taking constant solution $\varphi(\xi) = \pm\sqrt{-l_0}$ of the Riccati equation (1) and substituting it in (16) we have

$$u = \pm h\sqrt{-l_0}, \quad v = \frac{p_t - p_{xx} - p_{zz}}{2p_x} \pm p_x\sqrt{-l_0}, \quad w = \pm p_z\sqrt{-l_0}. \quad (22)$$

Substituting the same function in (17) we obtain

$$u = \pm \frac{h}{2}\sqrt{-l_0}, \quad v = \frac{p_t - p_{xx} - p_{zz}}{2p_x} \pm \frac{p_x}{2}\sqrt{-l_0}, \quad w = \pm \frac{p_z}{2}\sqrt{-l_0}. \quad (23)$$

We can write solutions (22)–(23) in more general form

$$u = r, \quad v = q, \quad w = s, \quad (24)$$

where $q = q(x, z, t)$ and $s = s(x, z, t)$ are arbitrary functions, r is an arbitrary constant. Therefore solutions (22)–(23) are the same. Solution (24) of the system (15) is obvious and can be obtained without any calculations.

Thus we see that Dai and Wang get only two distinct solutions (20) and (24) of the Burgers system (15). Solution (20) is not new, solution (24) is the

trivial solution. So the statement by Dai and Wang cited in the beginning of this section is wrong. Authors [1] did not obtain new results for the Riccati equation (2) and for the Burgers system (15). Some statements and some results by Dai and Wang are wrong.

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