

Separation of variables and explicit theta-function solution of the classical Steklov–Lyapunov systems: A geometric and algebraic geometric background. *

Yuri Fedorov

e-mail: Yuri.Fedorov@upc.edu

Department de Matemàtica Aplicada I,
Universitat Politècnica de Catalunya,
Barcelona, E-08028 Spain

and

Inna Basak

Department de Matemàtica Aplicada I,
Universitat Politècnica de Catalunya,
Barcelona, E-08028 Spain

e-mail: Inna.Basak@upc.edu

Abstract

The paper revises the explicit integration of the classical Steklov–Lyapunov systems via separation of variables, which was first made by F. Kötter in 1900, but was not well understood until recently. We give a geometric interpretation of the separating variables and then, applying the Weierstrass hyperelliptic root functions, obtain explicit theta-function solution to the problem. We also analyze the structure of its poles on the corresponding Abelian variety. This enables us to obtain a solution for an alternative set of phase variables of the systems that has a specific compact form.

1 Introduction

The motion of a rigid body in the ideal incompressible fluid is described by the classical Kirchhoff equations

$$\dot{K} = K \times \frac{\partial H}{\partial K} + p \times \frac{\partial H}{\partial p}, \quad \dot{p} = p \times \frac{\partial H}{\partial p},$$

where $K, p \in \mathbb{R}^3$ are the vectors of the impulsive momentum and the impulsive force, and $H = H(K, p)$ is the Hamiltonian, which is quadratic in K, p . Note that this system always possesses two trivial integrals (Casimir functions of the coalgebra $e^*(3)$) $\langle K, p \rangle, \langle p, p \rangle$ and the Hamiltonian itself is also a first integral.

Steklov [20] noticed that the classical Kirchhoff equations are integrable under certain conditions i.e., when the Hamiltonian has the form

$$H_1 = \frac{1}{2} \sum_{\alpha=1}^3 \left(b_\alpha K_\alpha^2 + 2\nu b_\beta b_\gamma K_\alpha p_\alpha + \nu^2 b_\alpha (b_\beta - b_\gamma)^2 p_\alpha^2 \right), \quad (\alpha, \beta, \gamma) = (1, 2, 3), \quad (1)$$

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b_1, b_2, b_3 and ν being arbitrary parameters. Under the Steklov condition, the equations possess fourth additional integral

$$H_2 = \frac{1}{2} \sum_{\alpha=1}^3 \left(K_\alpha^2 - 2\nu b_\alpha K_\alpha p_\alpha + \nu^2 (b_\beta - b_\gamma)^2 p_\alpha^2 \right). \quad (2)$$

Later Lyapunov [17] discovered an integrable case of the Kirchhoff equations whose Hamiltonian was a linear combination of the additional integral (2) and the two trivial integrals. Thus, the Steklov and Lyapunov integrable systems actually define different trajectories on the same invariant manifolds, two-dimensional tori. This fact was first noticed in [14].

In the sequel, without loss of generality, we assume $\nu = 1$ (this can always be made by an appropriate rescaling $p \rightarrow p/\nu$).

The Kirchhoff equations with the Hamiltonians (1), (2) were first solved explicitly by Kötter [16], who used the change of variables $(K, p) \rightarrow (z, p)$:

$$2z_\alpha = K_\alpha - (b_\beta + b_\gamma)p_\alpha, \quad \alpha = 1, 2, 3, \quad (\alpha, \beta, \gamma) = (1, 2, 3), \quad (3)$$

which transforms the Steklov–Lyapunov systems to the form

$$\dot{z} = z \times Bz - Bp \times Bz, \quad \dot{p} = p \times Bz, \quad B = \text{diag}(b_1, b_2, b_3) \quad (4)$$

and, respectively,

$$\dot{z} = p \times Bz, \quad \dot{p} = p \times (z - Bp). \quad (5)$$

Kötter implicitly showed that the above systems admit the following Lax representation with 3×3 skew-symmetric matrices and a spectral parameter

$$\begin{aligned} \dot{L}(s) &= [L(s), A(s)], \quad L(s), A(s) \in so(3), \quad s \in \mathbb{C}, \\ L(s)_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma} \left(\sqrt{s - b_\gamma} (z_\gamma + sp_\gamma) \right), \end{aligned} \quad (6)$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor. Equations (4) and (5) are generated by the operators

$$A(s)_{\alpha\beta} = \frac{\varepsilon_{\alpha\beta\gamma}}{s} \sqrt{(s - b_\alpha)(s - b_\beta)} b_\gamma z_\gamma, \quad \text{resp.} \quad A(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} \sqrt{(s - b_\alpha)(s - b_\beta)} p_\gamma. \quad (7)$$

The radicals in (6)–(7) are single-valued functions on the elliptic curve $\widehat{\mathcal{E}}$, the 4-sheeted unramified covering of the plane curve $\mathcal{E} = \{w^2 = (s - b_1)(s - b_2)(s - b_3)\}$. For this reason, the Lax representation has an elliptic spectral parameter.

Writing out the characteristic equation for $L(s)$, we arrive at the following family of quadratic integrals

$$\mathcal{F}(s) = \sum_{\gamma=1}^3 (s - b_\gamma) (z_\gamma + sp_\gamma)^2 \equiv J_1 s^3 + J_2 s^2 + 2sH_2 - 2H_1, \quad (8)$$

where

$$H_1 = \frac{1}{2} \langle z, Bz \rangle, \quad H_2 = \frac{1}{2} \langle z, z \rangle - \langle Bz, p \rangle, \quad J_2 = 2 \langle z, p \rangle - \langle Bp, p \rangle, \quad J_1 = \langle p, p \rangle. \quad (9)$$

It is seen that under the Kötter substitution (3) the functions J_1, J_2 transform into invariants of the coalgebra $e^*(3)$, whereas the integrals $H_1(z, p), H_2(z, p)$ (up to a linear combination of the invariants) become the Hamiltonians (1),(2).

An analog of the Lax pair (6) was later rediscovered in [5] and was used to obtain theta-function solution of the systems by using the method of Baker–Akhieser functions (see [4]). However, the resulting formulas appeared to be quite tedious, and it was not evident how to compare or identify them with the theta-function solution of Kötter.

Note that the latter was found in the classical manner, i.e., by a separation of variables and reduction of the equations of motion to quadratures, which have the form of the Abel–Jacobi map associated to a genus 2 hyperelliptic curve. The phase variables of the Kirchhoff equations have been expressed in terms of the separating variables in a quite symmetric but complicated way. Until recently, various attempts to check these expressions, as well as the reduction to quadratures made by Kötter, even using packages of modern computer algebra, were not successful. This even made some specialists to believe that the results of [16] are not reliable hence useless.

One of the first step in verification of Kötters’ calculations was made in [7], where the Steklov–Lyapunov systems on $e^*(3)$, as well as their higher-dimensional generalizations, have been considered as Poisson reductions of certain Hamiltonian systems in a bigger phase space. The latter systems were shown to possess 2×2 matrix Lax representations in a generalized Gaudin form with a rational spectral parameter. This fact easily allowed to find separating variables, which coincided with those suggested by Kötter, and, as a byproduct, prove their commutativity with respect to the Lie–Poisson bracket on $e^*(3)$. A similar approach to the separation of variables was made in [22].

The main aim of the present paper is to reconstruct the rest of the results of the paper [16]¹.

For our purposes we shall also use another set of phase variables which depend linearly on z, p . Namely, putting in (8) successively $s = b_1$, $s = b_2$, $s = b_3$ we obtain three independent quadratic integrals defining rank 3 quadrics in \mathbb{P}^6 :

$$\begin{aligned} (b_1 - b_2)(z_2 + b_1 p_2)^2 + (b_1 - b_3)(z_3 + b_1 p_3)^2 &= \mathcal{F}(b_1), \\ (b_2 - b_1)(z_1 + b_2 p_1)^2 + (b_2 - b_3)(z_3 + b_2 p_3)^2 &= \mathcal{F}(b_2), \\ (b_3 - b_1)(z_1 + b_3 p_1)^2 + (b_3 - b_2)(z_2 + b_3 p_2)^2 &= \mathcal{F}(b_3). \end{aligned} \tag{10}$$

Then it is natural to introduce new variables

$$\begin{aligned} v_1 &= \sqrt{(b_2 - b_3)(b_1 - b_2)} (z_2 + b_1 p_2), \\ v_2 &= \sqrt{(b_2 - b_3)(b_3 - b_1)} (z_3 + b_1 p_3), \\ v_3 &= \sqrt{(b_3 - b_1)(b_1 - b_2)} (z_1 + b_2 p_1), \\ v_4 &= \sqrt{(b_2 - b_3)(b_3 - b_1)} (z_3 + b_2 p_3), \\ v_5 &= \sqrt{(b_3 - b_1)(b_1 - b_2)} (z_1 + b_3 p_1), \\ v_6 &= \sqrt{(b_2 - b_3)(b_1 - b_2)} (z_2 + b_3 p_2), \end{aligned} \tag{11}$$

which, in particular, imply

$$\begin{aligned} p_1 &= \frac{v_3 - v_5}{\sqrt{\mathcal{S}}\sqrt{b_2 - b_3}}, \quad p_2 = \frac{v_1 - v_6}{\sqrt{\mathcal{S}}\sqrt{b_3 - b_1}}, \quad p_3 = \frac{v_2 - v_4}{\sqrt{\mathcal{S}}\sqrt{b_1 - b_2}}, \\ \mathcal{S} &= (b_1 - b_2)(b_2 - b_3)(b_3 - b_1). \end{aligned}$$

Then the integrals (10) and $(p, p) = J_1$ take the following compact form

$$\begin{aligned} v_1^2 - v_2^2 &= \psi(b_1) / (b_2 - b_3), \\ v_3^2 - v_4^2 &= \psi(b_2) / (b_3 - b_1), \\ v_5^2 - v_6^2 &= \psi(b_3) / (b_1 - b_2), \\ \frac{(v_3 - v_5)^2}{b_2 - b_3} + \frac{(v_1 - v_6)^2}{b_3 - b_1} + \frac{(v_2 - v_4)^2}{b_1 - b_2} &= J_1(b_1 - b_2)(b_2 - b_3)(b_3 - b_1). \end{aligned} \tag{12}$$

The Steklov–Lyapunov systems written in terms of v_1, \dots, v_6 , as well as the integrals (12), are quite similar to those describing the reduction of the integrable geodesic flow on the group $SO(4)$

¹Note that apart from the solutions of the Kirchhoff equations, Kötter also provided (although in an extremely brief form) the theta-solutions describing the motion of the group $E(3)$, that is, the components of the rotation matrix of the body and the trajectory of its center in space. We could not reconstruct these solutions.

with the diagonal metric Π to the algebra $so(4)$, which was considered in details in [1, 2]. In fact, as was shown by several authors (see e.g., [5]), there is a linear isomorphism connecting the above systems². We shall use this property and the results of [2] to obtain theta function expressions for the sums and differences of v_i , which have an especially simple form.

2 Separation of variables by F. Kötter.

The explicit solution of the Steklov–Lyapunov systems in the generic case was given by Kötter in the brief communication [16], where he presented the following scheme.

Let us fix the constants of motion in (9), then the invariant polynomial (8) can be written as

$$\mathcal{F}(s) = c_0(s - c_1)(s - c_2)(s - c_3), \quad c_0, c_1, c_2, c_3 = \text{const.} \quad (13)$$

Assume, without loss of generality, that $b_1 < b_2 < b_3$. Then one can show that for real z, p there are two possibilities:

- 1) c_1, c_2, c_3 are all real, then $b_1 \leq c_1 \leq c_2 \leq c_3 \leq b_3$;
- 2) c_1 is real and c_2, c_3 are complex conjugated, then $b_1 \leq c_1 \leq b_3$ and either $\rho = \Re c_2 = \Re c_3 < b_1$ or $\rho > b_3$.

Next, when no one of c_α coincides with b_1, b_2, b_3 , the level variety of the four first integrals of the problem (given by the coefficients at s^3, s^2, s, s^0) is a union of two-dimensional tori in $\mathbb{R}^6 = (z, p)$. We restrict ourselves to this generic situation, excluding the other cases, which correspond to periodic or asymptotic motions of the body.

Let λ_1, λ_2 be the roots of the equation

$$f(\lambda) = \sum_{i=1}^3 \frac{(z_j p_k - z_k p_j)^2}{\lambda - b_i} = 0, \quad (i, j, k) = (1, 2, 3), \quad (14)$$

where, when all c_α are real,

$$\lambda_1 \in [b_1, c_1], \quad \lambda_2 \in [c_3, b_3]. \quad (15)$$

Then for fixed c_0, c_1, c_2, c_3 the variables z, p can be expressed in terms of λ_1, λ_2 in such a way that for any $s \in \mathbb{C}$ the following relation holds (see formula (7) in [16])

$$z_i + s p_i = \sqrt{c_0} \frac{x_i \sum_{\alpha=1}^3 (s - c_\alpha) \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \left(\frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{(\lambda_1 - b_i)(\lambda_2 - c_\alpha)} - \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{(\lambda_2 - b_i)(\lambda_1 - c_\alpha)} \right)}{(\lambda_1 - \lambda_2) \sum_{\alpha=1}^3 \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)}}, \quad (16)$$

where

$$\Phi(\lambda) = (\lambda - b_1)(\lambda - b_2)(\lambda - b_3), \quad \psi(\lambda) = (\lambda - c_1)(\lambda - c_2)(\lambda - c_3), \quad (17)$$

$$x_i = \frac{\sqrt{(\lambda_1 - b_i)(\lambda_2 - b_i)}}{\sqrt{(b_i - b_j)(b_i - b_k)}}, \quad (18)$$

$$(i, j, k) = (1, 2, 3), \quad (\alpha, \beta, \gamma) = (1, 2, 3).$$

Setting in the above expression $s \rightarrow \infty$ and $s = 0$, one obtains the corresponding formulas for p_i, z_i .

Note that for real z_i, p_i , in the case (1) (all c_α are real), in view of the condition (15) all the expressions under the radicals in (16) are non-negative. In the rest of the cases the roots can be

²On the other hand, one of the Steklov–Lyapunov systems on $e^*(3)$ can also be regarded as a limit of the system on $so(4)$.

complex. For any $\alpha = 1, 2, 3$, the branches of $\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}$ in the numerator and the denominator of (16) must be the same.

Next, the evolution of λ_1, λ_2 is described by the quadratures

$$\begin{aligned} \frac{d\lambda_1}{\sqrt{R(\lambda_1)}} + \frac{d\lambda_2}{\sqrt{R(\lambda_2)}} &= \delta_1 dt, \\ \frac{\lambda_1 d\lambda_1}{\sqrt{R(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{R(\lambda_2)}} &= \delta_2 dt, \\ R(\lambda) &= -\Phi(\lambda)\psi(\lambda) \end{aligned} \tag{19}$$

with certain constants δ_1, δ_2 depending on the choice of the Hamiltonian only. In other words, in the variables λ_1, λ_2 the systems separate.

Note that the paper [16] does not describe explicitly how to find δ_1, δ_2 . They were calculated in [7], [22].

The above quadratures rewritten in the integral form

$$\begin{aligned} \int_{\lambda_0}^{\lambda_1} \frac{d\lambda}{2\sqrt{R(\lambda)}} + \int_{\lambda_0}^{\lambda_2} \frac{d\lambda}{2\sqrt{R(\lambda)}} &= u_1, \\ \int_{\lambda_0}^{\lambda_1} \frac{\lambda d\lambda}{2\sqrt{R(\lambda)}} + \int_{\lambda_0}^{\lambda_2} \frac{\lambda d\lambda}{2\sqrt{R(\lambda)}} &= u_2, \\ u_1 &= \delta_1 t + u_{10}, \quad u_2 = \delta_2 t + u_{20}, \end{aligned} \tag{20}$$

$$\tag{21}$$

which represent the Abel–Jacobi map associated to the genus 2 hyperelliptic curve $\mu^2 = -\Phi(\lambda)\psi(\lambda)$. Inverting the map (20) and substituting symmetric functions of $\lambda_1, \lambda_2, \mu_1, \mu_2$ into (16), one finally finds z, p as functions of time.

Everyone who had read paper [16] might be surprised by how Kötter managed to invent the intricate substitution $(z, p) \rightarrow (\lambda_1, \lambda_2, c_0, c_1, c_2, c_3)$ and to represent the result in the symmetric form (16). Unfortunately, the author of [16] gave no explanations of his computations. Nevertheless, it is clear that behind the striking formulas there must be a certain geometric idea, which we try to reconstruct in the next section.

3 A geometric background of Kötter’s solution.

Let $(x_1 : x_2 : x_3)$ be homogeneous coordinates in \mathbb{P}^2 defined up to multiplication by the same non-zero factor. Consider a line l in $\mathbb{P}^2 = (x_1 : x_2 : x_3)$ defined by equation

$$y_1 x_1 + y_2 x_2 + y_3 x_3 = 0.$$

Following Plücker (see e.g., [13]), the coefficients y_1, y_2, y_3 can be regarded as homogeneous coordinates of a point in the dual projective space $(\mathbb{P}^2)^*$. Now let l_1, l_2 be two intersecting lines in \mathbb{P}^2 with the Plücker coordinates $(y_1^{(1)}, y_2^{(1)}, y_3^{(1)}), (y_1^{(2)}, y_2^{(2)}, y_3^{(2)})$.

Then, for any constants $\lambda, \mu \in \mathbb{C}$ not vanishing simultaneously, the linear combination $\lambda y_\alpha^{(1)} + \mu y_\alpha^{(2)}$ are also Plücker coordinates of a line $l_{\lambda, \mu} \subset \mathbb{P}^2$. Hence, we arrive at an important geometric object, a *pencil of lines* in \mathbb{P}^2 , i.e., a one-parameter family $l_{\lambda, \mu}$. It is remarkable that all the lines of a pencil intersect at the same point $\mathbf{P} \in \mathbb{P}^2$ called *the focus* of the pencil.

Theorem 1. ([13]) *Let $l_{\lambda, \mu}$ be a pencil of lines in \mathbb{P}^2 defined by the Plücker coordinates $\lambda y_\alpha^{(1)} + \mu y_\alpha^{(2)}$, $(\lambda : \mu) \in \mathbb{P}$. Then the homogeneous coordinates of the focus are*

$$\mathbf{P} = \left(y_2^{(1)} y_3^{(2)} - y_3^{(1)} y_2^{(2)} : y_1^{(1)} y_3^{(2)} - y_3^{(1)} y_1^{(2)} : y_1^{(1)} y_2^{(2)} - y_2^{(1)} y_1^{(2)} \right).$$

Next, consider the family of confocal quadrics in \mathbb{P}^2

$$Q(s) = \left\{ \frac{x_1^2}{s-b_1} + \frac{x_2^2}{s-b_2} + \frac{x_3^2}{s-b_3} = 0 \right\} \quad (22)$$

and a fixed point $P = (X_1 : X_2 : X_3)$. Then one defines the spheroconical coordinates λ_1, λ_2 of this point (with respect to $Q(s)$) as the roots of the equation

$$\frac{X_1^2}{\lambda-b_1} + \frac{X_2^2}{\lambda-b_2} + \frac{X_3^2}{\lambda-b_3} = 0.$$

Now, going back to the Steklov–Lyapunov systems, we make the following observation.

Proposition 2. *The separating variables λ_1, λ_2 defined by formula (14) are spheroconical coordinates of the focus \mathbf{P} of the pencil of lines in \mathbb{P}^2 with the Plücker coordinates $z + sp = (z_1 + sp_1 : z_2 + sp_2 : z_3 + sp_3)$, $s \in \mathbb{P}$ with respect to the family of quadrics (22).*

Proof. According to Theorem 1, the homogeneous coordinates of the focus \mathbf{P} are

$$(z_2p_3 - z_3p_2 : z_3p_1 - z_1p_3 : z_1p_2 - z_2p_1), \quad (23)$$

hence, the spheroconical coordinates of \mathbf{P} with respect to the family (22) are precisely the roots of the equation (14), i.e., λ_1, λ_2 . \square

Note also the following property: for $\alpha = 1, 2, 3$, the line ℓ_α with the Plücker coordinates $z + c_\alpha p$ is tangent to the quadric $Q_\alpha = Q(c_\alpha)$. Indeed, setting in the right hand side of (8) $s = c_\alpha$, we obtain

$$\sum_{i=1}^3 (c_\alpha - b_i)(z_i + c_\alpha p_i)^2 = 0,$$

which represents the condition of tangency of the line ℓ_α and the quadric Q_α .

As a result, the following configuration holds: *the three lines ℓ_1, ℓ_2, ℓ_3 in \mathbb{P}^2 intersect at the same point \mathbf{P} and are tangent to the quadrics Q_1, Q_2, Q_3 respectively.* An example of such a configuration is shown in Fig. 1.

It follows that a solution $z(t), p(t)$ defines a trajectory of the focus \mathbf{P} on \mathbb{P}^2 or on $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$, and it natural to suppose that the Steklov–Lyapunov systems define dynamical systems on the sphere. Indeed, some of these systems were studied in [22] and were shown to be related to a generalization of the classical Neumann system with an additional quartic potential.

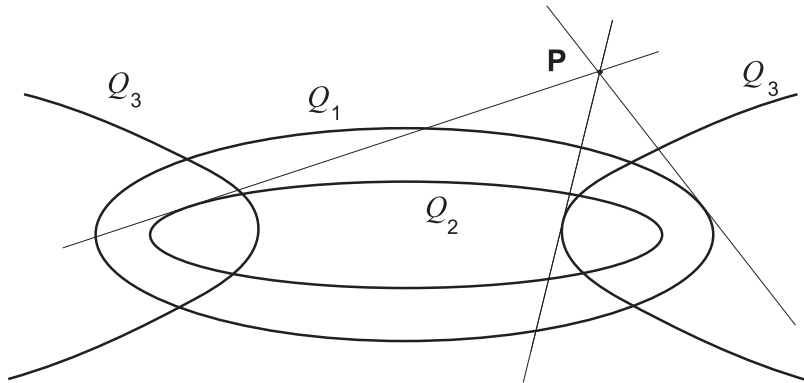


Figure 1: A configuration of tangent lines in $\mathbb{R}^2 = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3} \right)$ for the case $b_1 < c_1 < b_2 < c_2 < c_3 < b_3$, when the quadrics Q_α are two ellipses and a hyperbola.

In the sequel our main goal will be to recover the variables z and p as functions of the spherical coordinates of the focus \mathbf{P} , that is, to reconstruct the Kötter formula (16). Obviously, the solution is not unique: due to square roots in (18), each pair (λ_1, λ_2) , $\lambda_k \neq b_1, b_2, b_3$ gives 4 points on \mathbb{P}^2 , and for each point \mathbf{P} that does not lie on any of the quadrics $Q(c_\alpha)$, $2^3 = 8$ different configurations of tangent lines ℓ_1, ℓ_2, ℓ_3 are possible (Fig. 1 shows just one of them). Thus, under the above generality conditions, a pair (λ_1, λ_2) gives 32 different tangent configurations.

Reconstruction of z, p in terms of the separating variables. Let $(\mathbb{P}^2)^* = (G_1 : G_2 : G_3)$ be the dual space to $\mathbb{P}^2 = (x_1 : x_2 : x_3)$, (G_i being the Plücker coordinates of lines in \mathbb{P}^2). It is convenient to regard G_i also as Cartesian coordinates in the space $(\mathbb{C}^3)^* = (G_1, G_2, G_3)$. The pencil $\sigma(\mathbf{P})$ of lines in \mathbb{P}^2 with the focus (23) is represented by a line in $(\mathbb{P}^2)^*$ or by plane

$$\pi = \{(z_2 p_3 - z_3 p_2)G_1 + (z_3 p_1 - z_1 p_3)G_2 + (z_1 p_2 - z_2 p_1)G_3 = 0\} \subset (\mathbb{C}^3)^*.$$

Consider the line $\bar{\sigma}(\mathbf{P}) = \{z + sp | s \in \mathbb{R}\} \subset (\mathbb{C}^3)^*$. Obviously, $\{z + sp\} \subset \pi$. Now let us use the condition for the three lines ℓ_1, ℓ_2, ℓ_3 defined by the points $z + c_1 p$, $z + c_2 p$, $z + c_3 p$ in $(\mathbb{P}^2)^*$ to be tangent to the quadrics $Q(c_1)$, $Q(c_2)$, $Q(c_3)$ respectively. Let $\mathbf{V}_\alpha = (V_{\alpha 1}, V_{\alpha 2}, V_{\alpha 3}) \subset \pi$, $\alpha = 1, 2, 3$ be some vectors in $(\mathbb{C}^3)^*$ representing these points, so that $\ell_\alpha = \{V_{\alpha 1}x_1 + V_{\alpha 2}x_2 + V_{\alpha 3}x_3 = 0\}$. Then we have

$$z + c_1 p - \mu_1 \mathbf{V}_1 = 0, \quad z + c_2 p - \mu_2 \mathbf{V}_2 = 0, \quad z + c_3 p - \mu_3 \mathbf{V}_3 = 0 \quad (24)$$

for some indefinite factors μ_α . This system is equivalent to a homogeneous system of 9 scalar equations for 9 variables $z_\alpha, p_\alpha, \mu_\alpha$, $\alpha = 1, 2, 3$. Thus the variables can be found up to multiplication by a common factor. Eliminating z, p from (24), we obtain the following homogeneous system for μ_1, μ_2, μ_3

$$(c_2 - c_3)V_{\alpha 1}\mu_1 + (c_3 - c_1)V_{\alpha 2}\mu_2 + (c_1 - c_2)V_{\alpha 3}\mu_3 = 0, \quad \alpha = 1, 2, 3,$$

which has a nontrivial solution, since $\det \|V_{\alpha i}\| = 0$ (the vectors \mathbf{V}_α lie in the same hyperplane π). It follows, for example, that

$$\mu_1 = \mu \Sigma_1 / (c_2 - c_3), \quad \mu_2 = \mu \Sigma_2 / (c_3 - c_1), \quad \mu_3 = \mu \Sigma_3 / (c_1 - c_2), \quad (25)$$

$$\Sigma_1 = V_{22}V_{33} - V_{32}V_{23}, \quad \Sigma_2 = V_{32}V_{13} - V_{33}V_{12}, \quad \Sigma_3 = V_{12}V_{23} - V_{13}V_{22}, \quad (26)$$

$\mu \neq 0$ being an arbitrary factor. Substituting these expressions into (24) and using the obvious identity

$$\Sigma_1 \mathbf{V}_1 + \Sigma_2 \mathbf{V}_2 + \Sigma_3 \mathbf{V}_3 = 0,$$

after transformations we find

$$p = \frac{\mu}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} (c_1 \Sigma_1 \mathbf{V}_1 + c_2 \Sigma_2 \mathbf{V}_2 + c_3 \Sigma_3 \mathbf{V}_3), \quad (27)$$

$$z = \frac{\mu}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} (c_2 c_3 \Sigma_1 \mathbf{V}_1 + c_1 c_3 \Sigma_2 \mathbf{V}_2 + c_1 c_2 \Sigma_3 \mathbf{V}_3). \quad (28)$$

As a result,

$$z + sp = \frac{\mu}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} \sum_{\alpha=1}^3 (c_\alpha s + c_\beta c_\gamma) \Sigma_\alpha \mathbf{V}_\alpha. \quad (29)$$

Now we express the components of \mathbf{V}_α in terms of λ_1, λ_2 . Up to an arbitrary nonzero factor, they can be found from the system of equations

$$V_{\alpha 1}x_1 + V_{\alpha 2}x_2 + V_{\alpha 3}x_3 = 0, \quad \sum_{i=1}^3 (c_\alpha - b_i)V_{\alpha i}^2 = 0, \quad \alpha = 1, 2, 3, \quad (30)$$

which represent the conditions that the line ℓ_α passes through the focus $\mathbf{P} = (x_1 : x_2 : x_3)$ and touches the quadric $Q(c_\alpha)$.

In the sequel we apply the normalization $x_1^2 + x_2^2 + x_3^2 = 1$, which gives rise to expressions (18).

For $\mathbf{P} \notin Q(c_\alpha)$, this system possesses two different solutions, and for $\mathbf{P} \in Q(c_\alpha)$ a single one (the line touches $Q(c_\alpha)$ at the point \mathbf{P}). In the latter case we can just put

$$V_{\alpha i} = x_i / (c_\alpha - b_i). \quad (31)$$

Next, it is obvious that under reflection $(x_1 : x_2 : x_3) \rightarrow (-x_1 : x_2 : x_3)$, a solution $(V_{\alpha 1} : V_{\alpha 2} : V_{\alpha 3})$ transforms to $(-V_{\alpha 1} : V_{\alpha 2} : V_{\alpha 3})$ (similarly, for the two other reflections). Let us seek solutions of equations (30) in the form of symmetric functions of the complex coordinates λ_1, λ_2 such that

- 1) for $\lambda_1 = c_\alpha$ or $\lambda_2 = c_\alpha$ (i.e., when $\mathbf{P} \in Q(c_\alpha)$) there is a unique solution proportional to (31);
- 2) if λ_1 or λ_2 circles around the point $\lambda = c_\alpha$ on the complex plane λ , the two solutions transform into each other;
- 3) for $\lambda_1 = b_i$ or $\lambda_2 = b_i$ (i.e., when $x_i = 0$), $V_{\alpha i}$ does not vanishes.

Using the Jacobi identities

$$\sum_{i=1}^n \frac{a_i^k}{\prod (a_i - a_j)} = \begin{cases} 0, & k < n-1 \\ 1, & k = n-1 \\ \sum_{i=1}^n a_i, & k = n, \end{cases} \quad (32)$$

one can check that the following expressions satisfy equations (30) and the above three conditions

$$V_{\alpha i} = x_i \left(\frac{\sqrt{\Phi(\lambda_1)(\lambda_2 - c_\alpha)}}{\lambda_1 - b_i} + \frac{\sqrt{\Phi(\lambda_2)(\lambda_1 - c_\alpha)}}{\lambda_2 - b_i} \right), \quad x_i = \frac{\sqrt{(\lambda_1 - b_i)(\lambda_2 - b_i)}}{\sqrt{(b_i - b_j)(b_i - b_k)}}. \quad (33)$$

Then, using again the identities (32), we have

$$\langle \mathbf{V}_\alpha, \mathbf{V}_\beta \rangle \equiv (\lambda_2 - \lambda_1) \left(\sqrt{(\lambda_2 - c_\alpha)(\lambda_2 - c_\beta)} - \sqrt{(\lambda_1 - c_\alpha)(\lambda_1 - c_\beta)} \right). \quad (34)$$

and, in particular, $\langle \mathbf{V}_\alpha, \mathbf{V}_\alpha \rangle = (\lambda_1 - \lambda_2)^2$ for $\alpha = 1, 2, 3$.

Next, substituting (33) into (26) and applying the symbolic multiplication rule $\sqrt{ab}\sqrt{ac} = a\sqrt{bc}$, we find the factors Σ_α in form

$$\Sigma_\alpha = (\lambda_1 - \lambda_2)x_1 \left(\sqrt{-(\lambda_1 - c_\gamma)(\lambda_2 - c_\beta)} - \sqrt{-(\lambda_1 - c_\beta)(\lambda_2 - c_\gamma)} \right), \quad (35)$$

$$(\alpha, \beta, \gamma) = (1, 2, 3).$$

Further, putting (33), (35) into (29), we obtain

$$\begin{aligned} z_i + sp_i &= \frac{\mu(\lambda_1 - \lambda_2)x_1}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} x_i \cdot \sum_{\alpha=1}^3 (c_\alpha s + c_\beta c_\gamma) \\ &\cdot \left[\frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{\lambda_1 - b_i} \left(\sqrt{\frac{\lambda_1 - c_\gamma}{\lambda_2 - c_\gamma}} - \sqrt{\frac{\lambda_1 - c_\beta}{\lambda_2 - c_\beta}} \right) + \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{\lambda_2 - b_i} \left(\sqrt{\frac{\lambda_2 - c_\gamma}{\lambda_1 - c_\gamma}} - \sqrt{\frac{\lambda_1 - c_\beta}{\lambda_2 - c_\beta}} \right) \right] \\ &\equiv \mu(\lambda_1 - \lambda_2)x_1 x_i \sum_{\alpha=1}^3 (s - c_\alpha) \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \left(\frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{(\lambda_1 - b_i)(\lambda_2 - c_\alpha)} - \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{(\lambda_2 - b_i)(\lambda_1 - c_\alpha)} \right), \end{aligned} \quad (36)$$

which, up to multiplication by a common factor, coincides with the numerator in Kötter's formula (16).

To determine the factor μ in (29) and in (36), we apply the condition $\langle p, p \rangle = c_0$ which follows from (13). Then, from (27) we get

$$\frac{c_0}{\mu^2} = \frac{|c_1 \Sigma_1 \mathbf{V}_1 + c_2 \Sigma_2 \mathbf{V}_2 + c_3 \Sigma_3 \mathbf{V}_3|^2}{(c_1 - c_2)^2 (c_2 - c_3)^2 (c_3 - c_1)^2}. \quad (37)$$

Using the expressions (34), we obtain

$$\begin{aligned}
\left| \sum_{\alpha=1}^3 c_{\alpha} \Sigma_{\alpha} \mathbf{V}_{\alpha} \right|^2 &\equiv \sum_{\alpha=1}^3 [c_{\alpha}^2 \Sigma_{\alpha}^2 \langle \mathbf{V}_{\alpha}, \mathbf{V}_{\alpha} \rangle + 2c_{\beta} c_{\gamma} \Sigma_{\beta} \Sigma_{\gamma} \langle \mathbf{V}_{\beta}, \mathbf{V}_{\gamma} \rangle] \\
&= (\lambda_1 - \lambda_2)^3 x_1^2 \sum_{\alpha=1}^3 \left[c_{\alpha}^2 (\lambda_1 - \lambda_2) \left(\sqrt{-(\lambda_1 - c_{\gamma})(\lambda_2 - c_{\beta})} - \sqrt{-(\lambda_1 - c_{\beta})(\lambda_2 - c_{\gamma})} \right)^2 \right. \\
&\quad + 2c_{\beta} c_{\gamma} \left(\sqrt{-(\lambda_1 - c_{\gamma})(\lambda_2 - c_{\beta})} - \sqrt{-(\lambda_1 - c_{\beta})(\lambda_2 - c_{\gamma})} \right) \\
&\quad \cdot \left(\sqrt{-(\lambda_1 - c_{\alpha})(\lambda_2 - c_{\gamma})} - \sqrt{-(\lambda_1 - c_{\gamma})(\lambda_2 - c_{\alpha})} \right) \\
&\quad \left. \cdot \left(\sqrt{-(\lambda_2 - c_{\beta})(\lambda_2 - c_{\gamma})} - \sqrt{-(\lambda_1 - c_{\beta})(\lambda_1 - c_{\gamma})} \right) \right].
\end{aligned}$$

Simplifying the above expression and again using symbolic multiplication of square roots, one can verify that it is a full square:

$$\left| \sum_{\alpha=1}^3 c_{\alpha} \Sigma_{\alpha} \mathbf{V}_{\alpha} \right|^2 = x_1^2 (\lambda_1 - \lambda_2)^4 \left(\sum_{\alpha=1}^3 (c_{\beta} - c_{\gamma}) \sqrt{-(\lambda_1 - c_{\alpha})(\lambda_2 - c_{\alpha})} \right)^2.$$

Hence, from (37) we find

$$\frac{\sqrt{c_0}}{\mu} = x_1 (\lambda_1 - \lambda_2)^2 \sum_{\alpha=1}^3 \frac{\sqrt{-(\lambda_1 - c_{\alpha})(\lambda_2 - c_{\alpha})}}{(c_{\alpha} - c_{\beta})(c_{\alpha} - c_{\gamma})}.$$

Combining this with (36), we finally arrive at (16).

Thus, we derived the remarkable Kötter formula by making use of the geometric interpretation of the variables λ_1, λ_2 . We also note that the expressions (16) are symmetric in λ_1, λ_2 .

Remark 1. As noticed above, a disordered generic pair (λ_1, λ_2) gives 32 different configurations of tangent lines to the quadrics $Q(c_1), Q(c_2), Q(c_3)$. Since the common factor μ in (29) is defined up to sign flip, we conclude that, according to the formula (16), *to each generic pair (λ_1, λ_2) there correspond 64 different points (z, p) on the invariant manifold (a union of 2-dimensional tori) defined by the constants c_0, c_1, c_2, c_3 .* This ambiguity corresponds to different signs of the square roots in the Kötter formula.

In the next section we shall use the expressions (16) and the quadratures (20) to find explicit theta-functional solutions for the Steklov–Lyapunov systems.

4 Explicit theta-function solution of the Steklov-Lyapunov systems

In order to give explicit theta-functions solution, we first recall some basic formulas describing inversion of the quadratures (19). We shall mainly follow the description given in [3, 4, 10]. Consider an even order hyperelliptic Riemann surface of genus g represented in the standard form

$$\Gamma = \{ \mu^2 = (\lambda - E_1) \cdots (\lambda - E_{2g+2}) \} \in \mathbb{C}^2(\lambda, \mu).$$

In the sequel we shall regard Γ as its complex compactification obtained by gluing two infinite points ∞_-, ∞_+ , where the coordinate λ equals infinity.

Consider also differential 1-form (differential) $\omega = \phi(\tau) d\tau$ on Γ , where τ is a local parameter at a point $P \in \Gamma$. A differential ω is called holomorphic if $\phi(\tau)$ is a holomorphic function for any point

P . We choose the canonical basis of cycles $\mathbf{a}_1, \dots, \mathbf{a}_g, \mathbf{b}_1, \dots, \mathbf{b}_g$ on the surface Γ such that their intersections are of the form:

$$\mathbf{a}_i \circ \mathbf{a}_j = \mathbf{b}_i \circ \mathbf{b}_j = 0, \quad \mathbf{a}_i \circ \mathbf{b}_j = \delta_{ij}, \quad i, j = 1, \dots, g,$$

where $\gamma_1 \circ \gamma_2$ denotes the intersection index of the cycles γ_1, γ_2 .

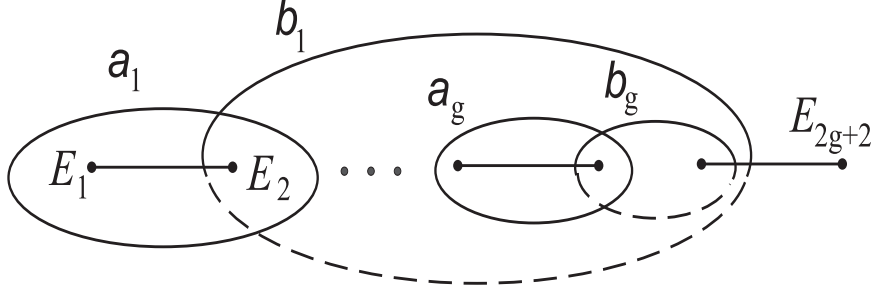


Figure 2:

An example of a canonical basis of cycles on Γ is shown on Figure 2. The parts of the cycles on the lower sheet are shown by dashed lines.

Next, let $\bar{\omega}_1, \dots, \bar{\omega}_g$ be the conjugated basis of *normalized* holomorphic differentials on Γ such that

$$\oint_{\mathbf{a}_j} \bar{\omega}_i = 2\pi j \delta_{ij}, \quad j = \sqrt{-1}.$$

The $g \times g$ matrix of b -periods $B_{ij} = \oint_{\mathbf{b}_j} \bar{\omega}_i$ is symmetric and has a negative definite real part. Consider the period lattice $\Lambda^0 = \{2\pi j \mathbb{Z}^g + B \mathbb{Z}^g\}$ of rank $2g$ in $\mathbb{C}^g = (z_1, \dots, z_g)$. The complex torus $\text{Jac}(G) = \mathbb{C}^g / \Lambda^0$ is called the Jacobi variety (*Jacobian*) of the curve G .

Now consider a generic divisor of points $P_1 = (\lambda_1, \mu_1), \dots, P_g = (\lambda_g, \mu_g)$ on it, and the Abel–Jacobi mapping with a basepoint P_0

$$\int_{P_0}^{P_1} \bar{\omega} + \dots + \int_{P_0}^{P_g} \bar{\omega} = z, \quad (38)$$

$$\bar{\omega} = (\bar{\omega}_1, \dots, \bar{\omega}_g)^T, \quad z = (z_1, \dots, z_g)^T \in \mathbb{C}^g.$$

Under the mapping, functions on $S^g \Gamma$, i.e., symmetric functions of the points P_1, \dots, P_g are $2g$ -fold periodic functions of the complex variables z_1, \dots, z_g with the above period lattice Λ^0 (Abelian functions).

Explicit expressions of such functions can be obtained by means of theta-functions on the universal covering $\mathbb{C}^g = (z_1, \dots, z_g)$ of the complex torus. Recall that customary Riemann’s theta-function $\theta(z|B)$ associated with the Riemann matrix B is defined by the series³

$$\theta(z|B) = \sum_{M \in \mathbb{Z}^g} \exp(\langle BM, M \rangle + \langle M, z \rangle), \quad (39)$$

$$\langle M, z \rangle = \sum_{i=1}^g M_i z_i, \quad \langle BM, M \rangle = \sum_{i,j=1}^g B_{ij} M_i M_j.$$

Equation $\theta(z|B) = 0$ defines a codimension one subvariety $\Theta \in \text{Jac}(\Gamma)$ (for $g > 2$ with singularities) called *theta-divisor*.

³The expression for $\theta(z)$ we use here is different from that chosen in a series of books on theta-functions by multiplication of z by a constant factor.

We shall also use theta-functions with characteristics $\alpha = (\alpha_1, \dots, \alpha_g)$, $\beta = (\beta_1, \dots, \beta_g)$, $\alpha_j, \beta_j \in \mathbb{R}$, which are obtained from $\theta(z|B)$ by shifting the argument z and multiplying by an exponent⁴:

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z) \equiv \theta \begin{bmatrix} \alpha_1 \cdots \alpha_g \\ \beta_1 \cdots \beta_g \end{bmatrix} (z) = \exp\{\langle B\alpha, \alpha \rangle / 2 + \langle z + 2\pi_j \beta, \alpha \rangle\} \theta(z + 2\pi_j \beta + B\alpha).$$

Then for a pair of characteristics one has the following useful relations

$$\theta \begin{bmatrix} \alpha + \alpha' \\ \beta + \beta' \end{bmatrix} (z) = \exp\{(B\alpha', \alpha') / 2 + (z + 2\pi_j \beta + 2\pi_j \beta', \alpha')\} \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z + 2\pi_j \beta' + B\alpha'). \quad (40)$$

All these functions possess the quadiperiodic property

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z + 2\pi_j K + BM) = \exp(2\pi_j \epsilon) \exp\{-\langle BM, M \rangle / 2 - \langle M, z \rangle\} \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z), \quad (41)$$

$$\epsilon = \langle \alpha, K \rangle - \langle \beta, M \rangle,$$

An important particular case is represented by theta-functions with half-integer characteristics

$$\Delta = \begin{pmatrix} \Delta' \\ \Delta'' \end{pmatrix}, \quad \eta_i = \begin{pmatrix} \eta'_i \\ \eta''_i \end{pmatrix}, \quad \text{and} \quad \eta_{ij} = \eta_i + \eta_j \pmod{\mathbb{Z}^{2g} / \mathbb{Z}^{2g}}, \quad \Delta', \Delta'', \eta'_i, \eta''_i \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g$$

such that

$$2\pi_j \eta''_i + B\eta'_i = \int_{E_{2g+2}}^{E_i} \bar{\omega} \pmod{\Lambda}, \quad (42)$$

$$2\pi_j \Delta'' + B\Delta' = \mathcal{K} \pmod{\Lambda},$$

$\mathcal{K} \in \mathbb{C}^g$ being the vector of the Riemann constants and E_i briefly denotes the branch point $(E_i, 0)$ on Γ .

The half-integer characteristic $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is odd (even) if $\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z)$ is odd (respectively, even).

For the case $g = 2$ and for the chosen canonical basis of cycles $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ on Γ the above characteristics Δ, η_i are

$$\Delta = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}, \quad (43)$$

$$\eta_3 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \eta_4 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad \eta_5 = \begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix},$$

and, by convention, η_6 is the zero theta-characteristic. Note also the property

$$\eta_1 + \eta_3 + \eta_5 = \eta_2 + \eta_4 = \Delta \pmod{\mathbb{Z}^{2g} / \mathbb{Z}^{2g}}. \quad (44)$$

The six functions $\theta[\Delta + \eta_i](z)$, $i = 1, \dots, 6$ are odd, that is, $\theta[\Delta + \eta_i](0) = 0$, whereas the other 10 functions $\theta[\Delta + \eta_{ij}](z)$, $i, j \neq 6$ are even. In the case $g = 2$ no one of the latter functions vanishes at zero.

The root functions. To obtain theta-functions solution for many problems linearized on Jacobians of hyperelliptic curves, one can apply some remarkable relations between roots of certain functions on symmetric products of such curves and quotients of theta-functions with half-integer characteristics, which are historically referred to as *root functions*. For the case of odd order hyperelliptic curves such functions were obtained by Weierstrass and Rosenheim [23, 15], see also [3, 4].

For our purposes it is sufficient to quote only several root functions for the particular case $g = 2$ and the even-order hyperelliptic curve

$$\Gamma = \{\mu^2 = R(\lambda)\}, \quad R(\lambda) = (\lambda - E_1) \cdots (\lambda - E_6).$$

Let us introduce the polinomial $U(\lambda, s) = (s - \lambda_1)(s - \lambda_2)$.

⁴Here and below we omit B in the theta-functional notation.

Proposition 3. *Under the Abel–Jacobi mapping (38) with $g = 2$ and the basepoint $P_0 = E_6$ the following relations hold*

$$U(\lambda, E_i) \equiv (\lambda_1 - E_i)(\lambda_2 - E_i) = \kappa_i \frac{\theta^2[\Delta + \eta_i](z)}{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}, \quad (45)$$

$$q = \int_{\infty_-}^{\infty_+} \bar{\omega} = 2 \int_{E_6}^{\infty_+} \bar{\omega}, \quad \kappa_i = \text{const}, \quad i = 1, \dots, 6,$$

$$\begin{aligned} \frac{1}{\lambda_1 - \lambda_2} & \left(\frac{\sqrt{R(\lambda_1)}}{(E_i - \lambda_1)(E_j - \lambda_1)(E_s - \lambda_1)} - \frac{\sqrt{R(\lambda_2)}}{(E_i - \lambda_2)(E_j - \lambda_2)(E_s - \lambda_2)} \right) \\ & = \kappa_{ijs} \frac{\theta[\Delta + \eta_i + \eta_j + \eta_s](z)\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}{\theta[\Delta + \eta_i](z)\theta[\Delta + \eta_j](z)\theta[\Delta + \eta_s](z)}, \end{aligned} \quad (46)$$

$$\begin{aligned} \frac{\sqrt{U(\lambda, E_i)}\sqrt{U(\lambda, E_j)}}{\lambda_1 - \lambda_2} & \left(\frac{\sqrt{R(\lambda_1)}}{(E_i - \lambda_1)(E_j - \lambda_1)(E_s - \lambda_1)} - \frac{\sqrt{R(\lambda_2)}}{(E_i - \lambda_2)(E_j - \lambda_2)(E_s - \lambda_2)} \right) \\ & = \kappa'_{ijs} \frac{\theta[\Delta + \eta_i + \eta_j + \eta_s](z)}{\theta[\Delta + \eta_s](z)}, \end{aligned} \quad (47)$$

$$\kappa_{ijs}, \kappa'_{ijs} = \text{const}, \quad i, j, s = 1, \dots, 6, \quad i \neq j \neq s \neq i,$$

where, as above, η_6 is the zero theta-characteristic and ∞_+, ∞_- are the infinite points of the compactified curve Γ . The constant factors $\kappa_i, \kappa_{ijs}, \kappa'_{ijs}$ depend on the moduli of Γ only.

Note that various expressions of symmetric functions of the λ, μ -coordinates on an even hyperelliptic curve were obtained in [11] on the basis of the Klein–Weierstrass realization of Abelian functions outlined in [3] and [8].

Sketch of proof of Proposition 3. The left and right hand sides of (45) are meromorphic functions on $\text{Jac}(\Gamma)$, which have the same zeros and poles with the same multiplicity. This implies that their quotient is an analytic function on a compact complex manifold without poles and therefore a constant.

The root functions (46), (47) can be deduced from the corresponding root functions for the case of odd-order hyperelliptic curve, by making a fractionally-linear transformation of λ that sends the Weierstrass point E_{2g+2} on Γ to infinity. \square

The constants $\kappa_i, \kappa_{ijs}, \kappa'_{ijs}$ can be calculated explicitly in terms of the coordinates E_1, \dots, E_6 and theta-constants by equating λ_1, λ_2 to certain E_i and the argument z to the corresponding half-period in $\text{Jac}(\Gamma)$ (see, e.g., [3]).

Explicit solution. Now we are able to write explicit solution for the Steklov–Lyapunov systems by comparing the root functions (45), (47) with the Kötter expression (16).

Namely, let $\Gamma = \{\mu^2 = \Phi(\lambda)\varphi(\lambda)\}$ where the polynomials ϕ and φ are defined in (17) and identify (without ordering) the sets

$$\{E_1, E_2, E_3, E_4, E_5, E_6\} = \{b_1, b_2, b_3, c_1, c_2, c_3\}.$$

By $\eta_{b_i}, \eta_{c_\alpha}$ we denote the half-integer characteristics corresponding to the branch points $(b_i, 0), (c_\alpha, 0)$ respectively, according to formula (42).

Theorem 4. *For fixed constants of motion c_1, c_2, c_3 the variables z, p can be expressed in terms of theta-functions of the curve Γ in a such a way that for any $s \in \mathbb{C}$*

$$z_i + sp_i = \frac{\sum_{\alpha=1}^3 k_{i\alpha} (s - c_\alpha) \theta[\Delta + \eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}](z)}{\sum_{\alpha=1}^3 k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z)}, \quad (\alpha, \beta, \gamma) = (1, 2, 3), \quad (48)$$

where $k_{i\alpha}, k_{0\alpha}$ are certain constants depending on the moduli of Γ only, and the components of the argument z are linear functions of t :

$$z_1 = C_{11}\delta_1 t + C_{12}\delta_2 t + z_{10}, \quad z_2 = C_{21}\delta_1 t + C_{22}\delta_2 t + z_{10}, \quad z_{10}, z_{20} = \text{const}, \quad C = A^{-1} \quad (49)$$

A being is the matrix of \mathbf{a} -periods of the differentials $d\lambda/\mu, \lambda d\lambda/\mu$ on Γ .

Thus, we have recovered the theta-function solution of the systems obtained by Kötter in [16]. The proof is given in the end of the section.

Remark 2. In view of the definition of theta-function with characteristics, under the argument shift $z \rightarrow z - \mathcal{K}$ the special characteristic Δ is killed and the solutions (48) are simplified to

$$z_i + sp_i = \frac{\sum_{\alpha=1}^3 \bar{k}_{i\alpha} (s - c_\alpha) \theta[\eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}](z)}{\sum_{\alpha=1}^3 \bar{k}_{0\alpha} \theta[\eta_{c_\alpha}](z)}, \quad (\alpha, \beta, \gamma) = (1, 2, 3), \quad (50)$$

where the constants $\bar{k}_{i\alpha}, \bar{k}_{0\alpha}$ coincide with $k_{i\alpha}, k_{0\alpha}$ in (48) up to multiplication by a quartic root of unity. In each concrete case of position of b_i, c_α , one can also simplify the sums of characteristics in the numerator of (50) by using the relations (44).

Remark 3. In view of the quasi-periodic property (41), when the complex argument z changes by a period vector in $\text{Jac}(\Gamma)$, the theta-functions in (48), (50) are multiplied by generally different factors. Hence, the variables z_i, p_i cannot be single valued on the Jacobian variety Γ , and a simple accounting shows that they are meromorphic on $\widetilde{\text{Jac}}(\Gamma)$, the 16-fold unramified covering of it, obtained by doubling all the four period vectors in $\text{Jac}(\Gamma)$. This implies that $\widetilde{\text{Jac}}(\Gamma)$ is also a principally polarized Abelian variety isomorphic to $\text{Jac}(\Gamma)$. As follows from the structure of (48), all z_i, p_i have a common set of simple poles (the pole divisor), which we denote $\mathcal{D} \subset \widetilde{\text{Jac}}(\Gamma)$.

The degree of the covering $\widetilde{\text{Jac}}(\Gamma) \rightarrow \text{Jac}(\Gamma)$ can also be found in another way: According to Remark 1, each generic pair (λ_1, λ_2) corresponds to 64 different points (z, p) on the invariant manifold $\widetilde{\text{Jac}}(\Gamma)$. On the other hand, the same pair gives rise to 4 different points in $\text{Jac}(\Gamma)$ defined by the divisors $\{(\lambda_1, \pm\sqrt{R_6(\lambda_1)}), (\lambda_2, \pm\sqrt{R_6(\lambda_2)})\}$. Hence a generic point of $\text{Jac}(\Gamma)$ corresponds to $64/4=16$ points in $\widetilde{\text{Jac}}(\Gamma)$.

Proof of Theorem 4. The summands in the numerator of the Kötter solution (16), when divided by $\lambda_1 - \lambda_2$, can be written as

$$\begin{aligned} & \frac{s - c_\alpha}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{\lambda_1 - \lambda_2} \cdot \left(\frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{(\lambda_1 - b_i)(\lambda_2 - c_\alpha)} - \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{(\lambda_2 - b_i)(\lambda_1 - c_\alpha)} \right) \\ &= \frac{s - c_\alpha}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \frac{\sqrt{-(\lambda_1 - c_\beta)(\lambda_2 - c_\beta)} \sqrt{-(\lambda_1 - c_\gamma)(\lambda_2 - c_\gamma)}}{\lambda_1 - \lambda_2} \\ & \quad \times \left(\frac{\mu_1}{(\lambda_1 - b_i)(\lambda_1 - c_\beta)(\lambda_1 - c_\gamma)} - \frac{\mu_2}{(\lambda_2 - b_i)(\lambda_2 - c_\beta)(\lambda_2 - c_\gamma)} \right), \\ & \quad \mu_1 = \sqrt{\Phi(\lambda_1)\psi(\lambda_1)}, \quad \mu_2 = \sqrt{\Phi(\lambda_2)\psi(\lambda_2)}. \end{aligned}$$

The right hand sides have the form of the root function (47). Hence, up to a constant factor, they are equal to

$$(s - c_\alpha) \frac{\theta[\Delta + \eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}](z)}{\theta[\Delta + \eta_{b_i}](z)}.$$

Next, in view of (45), we obtain

$$\begin{aligned} x_i &= \varkappa_i \frac{\theta[\Delta + \eta_{b_i}](z)}{\sqrt{\theta[\Delta](z - q/2) \theta[\Delta](z + q/2)}}, \\ \sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)} &= \varkappa_\alpha \frac{\theta[\Delta + \eta_{c_\alpha}](z)}{\sqrt{\theta[\Delta](z - q/2) \theta[\Delta](z + q/2)}}, \end{aligned} \quad (51)$$

$$\varkappa_i, \varkappa_\alpha = \text{const},$$

$$\sum_{\alpha=1}^3 \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} = \sum_{\alpha=1}^3 \frac{k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z)}{\sqrt{\theta[\Delta](z - q/2) \theta[\Delta](z + q/2)}}. \quad (52)$$

Combining the above expressions, we rewrite the right hand side of (16) in the form

$$\frac{\theta[\Delta + \eta_{b_i}](z)}{\sqrt{c_0} \sqrt{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}} \frac{\sum_{\alpha=1}^3 \frac{k_{i\alpha} (s - c_\alpha) \theta[\Delta + \eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}](z)}{\theta[\Delta + \eta_{b_i}](z)}}{\sum_{\alpha=1}^3 \frac{k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z)}{\sqrt{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}}},$$

which, after simplifications, gives (48).

Expressions (49) follow from the relation $(\bar{\omega}_1, \bar{\omega}_2)^T = C(d\lambda/\mu, \lambda d\lambda/\mu)$, where, as above, $\bar{\omega}_j$ are the normalized holomorphic differentials on Γ , which implies $(\bar{\omega}_1, \bar{\omega}_2)^T = C(u_1, u_2)^T$, where u_1, u_2 are the right hand sides of the quadratures (21). \square

5 The divisor of poles and the alternative form of the theta-function solution.

The nice form of the Kötter solution (48) itself tells us a little about the structure of zeros and poles of z_i, p_i on the 2-dimensional Abelian variety $\widetilde{\text{Jac}}(\Gamma)$. It is possible however to give a quite detailed description of the set of common poles of these variables, called the divisor of poles \mathcal{D} . Obviously, $\mathcal{D} = \{\sum_{\alpha=1}^3 k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z) = 0\} \subset \widetilde{\text{Jac}}(\Gamma)$.

Namely, for each $\alpha = 1, 2, 3$, the zeros of $\theta[\Delta + \eta_{c_\alpha}](z)$ in $\text{Jac}(\Gamma)$ form a translate Θ_α of the theta-divisor Θ by the half-period $2\pi j\eta''_{c_\alpha} + 2B\eta'_{c_\alpha}$. Each translate passes via six half-periods, and $\Theta_1, \Theta_2, \Theta_3$ have a unique common intersection in the origin (neutral point) $\mathcal{O} \in \text{Jac}(\Gamma)$. This is depicted in Fig. 3 (a), where Θ_α are shown as circles and the half-periods in $\text{Jac}(\Gamma)$ as black dots. Hence, at $z = \mathcal{O}$ the denominator of (48) vanishes. Then, under the covering $\pi : \widetilde{\text{Jac}}(\Gamma) \rightarrow \text{Jac}(\Gamma)$, the preimage of \mathcal{O} consists of all the 16 half-periods in $\widetilde{\text{Jac}}(\Gamma)$, which therefore belong to the divisor \mathcal{D} .

Note that translations in $\text{Jac}(\Gamma)$ by a complete period \mathcal{V} correspond to translation in $\widetilde{\text{Jac}}(\Gamma)$ by the half-period $\mathcal{V}/2$.

Now assume, as above, that $b_1 < b_2 < b_3$ and that $(b_3, 0) = E_6 \in \Gamma$ is the basepoint of the Abel map (38) with $g = 2$. A further information about \mathcal{D} is given by

Proposition 5. *The divisor $\mathcal{D} \subset \widetilde{\text{Jac}}(\Gamma)$ is invariant under translations by the half-periods generated by*

$$\mathcal{V}_1/2 = 2\pi j\eta''_{b_1} + 2B\eta'_{b_1}, \quad \mathcal{V}_2/2 = 2\pi j\eta''_{b_2} + 2B\eta'_{b_2}, \quad \begin{pmatrix} \eta'_{b_i} \\ \eta''_{b_i} \end{pmatrix} = \eta_{b_i}. \quad (53)$$

Proof. Choose a generic point $q \in \mathcal{D}$ and let z^* be its projection onto $\text{Jac}(\Gamma)$, which gives

$$f(z^*) = \sum_{\alpha=1}^3 k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z^*) = 0.$$

In view of the quasi-periodic property (41) and the half-integer characteristics (43), under the translations $z^* \rightarrow z^* + M\mathcal{V}_1 + N\mathcal{V}_2$, $M, N \in \mathbb{Z}$ all the functions $\theta[\Delta + \eta_{c_\alpha}](z^*)$ are multiplied by the same factor and therefore $f(z^* + M\mathcal{V}_1 + N\mathcal{V}_2) = 0$. Hence the points $z^*/2 + M\mathcal{V}_1/2 + N\mathcal{V}_2/2$ in $\widetilde{\text{Jac}}(\Gamma)$ also belong to \mathcal{D} .

One can also show that this does not hold for the translations by the other half-periods. \square

Theorem 6. *The denominator of the solution (48) admits the factorization*

$$\sum_{\alpha=1}^3 k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z) = \exp(\chi z + \zeta) \cdot \theta[\Delta](z/2) \theta[\Delta + \eta_{b_1}](z/2) \theta[\Delta + \eta_{b_2}](z/2) \theta[\Delta + \eta_{b_1} + \eta_{b_2}](z/2) \quad (54)$$

with certain constants χ, ζ .

The proof of the theorem is based on the fourth Riemann identity (see, e.g., [3, 10]) and the theta-formulas of Frobenius and Thomae (see, e.g., [21, 18]). Technically, it is quite tedious and for this reason we move it into Appendix.

Now note that each of the 4 sets

$$\begin{aligned} \mathcal{D}_0 &= \{\theta[\Delta](z/2|B) = 0\}, & \mathcal{D}_1 &= \{\theta[\Delta + \eta_{b_1}](z/2|B) = 0\}, \\ \mathcal{D}_2 &= \{\theta[\Delta + \eta_{b_2}](z/2|B) = 0\}, & \mathcal{D}_3 &= \{\theta[\Delta + \eta_{b_1} + \eta_{b_2}](z/2|B) = 0\} \end{aligned}$$

describe a translate of the theta-divisor, the genus 2 curve Γ embedded into $\widetilde{\text{Jac}}(\Gamma)$. Then, Theorem 6 says that the pole divisor \mathcal{D} is a union of these translates, which are obtained from each other by shifts by the half-periods $\mathcal{V}_1/2, \mathcal{V}_2/2$, and $\mathcal{V}_3/2 = -\mathcal{V}_1/2 - \mathcal{V}_2/2$. The union passes through all the 16 half-periods in $\widetilde{\text{Jac}}(\Gamma)$. The action of the translations by $\mathcal{V}_1/2, \mathcal{V}_2/2, \mathcal{V}_3/2$ in $\widetilde{\text{Jac}}(\Gamma)$ on the components $(\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ gives respectively

$$(\mathcal{D}_1, \mathcal{D}_0, \mathcal{D}_3, \mathcal{D}_2), \quad (\mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_0, \mathcal{D}_1), \quad (\mathcal{D}_3, \mathcal{D}_2, \mathcal{D}_1, \mathcal{D}_0). \quad (55)$$

All these properties are in complete correspondence with our previous observations about the divisor \mathcal{D} .

Also, as was shown in [2] by applying the Kovalevskaya–Painlevé analysis, the pole divisor with the same structure appears in the integrable flow on the algebra $so(4)$ with the diagonal metric Π , already mentioned in Introduction. This result of [2] about \mathcal{D} equally holds for the Steklov–Lyapunov systems due to a linear isomorphism between them and the integrable flow on $so(4)$.

The intersection pattern for \mathcal{D} is shown in Fig. 3 (b), which we borrowed from [2]. Here the circles represent the translates \mathcal{D}_j and the 16 black dots depict the half-periods. Under the projection π all the above half-periods are mapped onto $\mathcal{O} \in \text{Jac}(\Gamma)$.

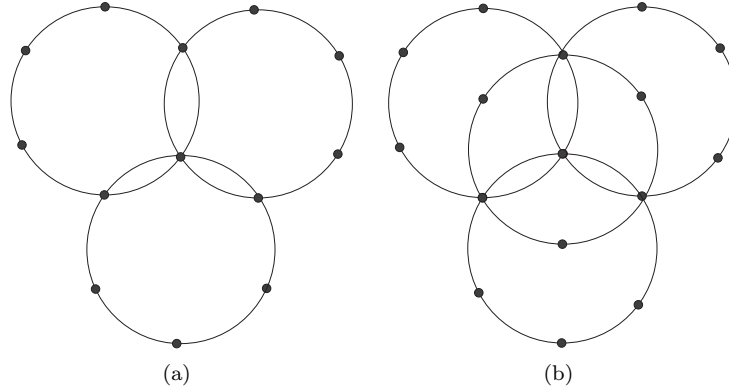


Figure 3: (a) Configuration of the translates Θ_α in $\text{Jac}(\Gamma)$. (b) The 4 translates of Γ in $\widetilde{\text{Jac}}(\Gamma)$ forming the pole divisor \mathcal{D} .

Solutions for the variables v_k . Let us choose the origin of $\widetilde{\text{Jac}}(\Gamma)$ at one of the four triple intersections of \mathcal{D}_j and denote for brevity the four theta-functions in (54) as $\theta_0(z/2), \theta_1(z/2), \theta_2(z/2), \theta_3(z/2)$.

Now we show that theta-function solutions for the new phase variables v_1, \dots, v_6 introduced in (11) have a rather specific and compact form. Namely, as follows from expressions (48) and (11), the functions $v_1 + v_2$ and $v_1 - v_2$ may have only *simple* poles *at most* along the components of the divisor \mathcal{D} . On the other hand, the form of the integrals (12) imply the following remarkable property: the poles (the zeros) of $v_1 + v_2$ are the zeros (resp. the poles) of $v_1 - v_2$. Since both functions are meromorphic on $\widetilde{\text{Jac}}(\Gamma)$, none of them can have simple poles along only one component \mathcal{D}_j . This necessarily implies that $v_1 + v_2$ has poles along two certain components $\mathcal{D}_{j_1}, \mathcal{D}_{j_2}$ and zeros along the other two components $\mathcal{D}_{j_3}, \mathcal{D}_{j_4}$, and vice versa for $v_1 - v_2$.

The same observations hold for the pairs $(v_3 + v_4, v_3 - v_4)$ and $(v_5 + v_6, v_5 - v_6)$. Note also that functions from different pairs cannot have the same poles, since in that case they would also have the same zeros and their quotient would be constant, which is not true.

Now let us fix the origin of $\widehat{\text{Jac}}(\Gamma)$ at one specific triple intersection of \mathcal{D}_j such that the 3 functions $v_1 + v_2, v_3 + v_4, v_5 + v_6$ have a common pole along the component \mathcal{D}_0 . In this case the following proposition holds.

Proposition 7. *The theta-function solutions for the phase variables v_k have the form*

$$\begin{aligned} v_1 + v_2 &= \chi_1 \frac{\theta_1(z/2)\theta_2(z/2)}{\theta_0(z/2)\theta_3(z/2)}, & v_1 - v_2 &= \chi_2 \frac{\theta_0(z/2)\theta_3(z/2)}{\theta_1(z/2)\theta_2(z/2)}, \\ v_3 + v_4 &= \chi_3 \frac{\theta_2(z/2)\theta_3(z/2)}{\theta_0(z/2)\theta_1(z/2)}, & v_3 - v_4 &= \chi_4 \frac{\theta_0(z/2)\theta_1(z/2)}{\theta_2(z/2)\theta_3(z/2)}, \\ v_5 + v_6 &= \chi_5 \frac{\theta_1(z/2)\theta_3(z/2)}{\theta_0(z/2)\theta_2(z/2)}, & v_5 - v_6 &= \chi_6 \frac{\theta_0(z/2)\theta_2(z/2)}{\theta_1(z/2)\theta_3(z/2)}, \end{aligned} \quad (56)$$

$$\chi_1, \chi_3, \chi_5 = \text{const}, \quad \chi_2 = \frac{\psi(b_1)}{(b_2 - b_3)\chi_1}, \quad \chi_4 = \frac{\psi(b_2)}{(b_3 - b_1)\chi_3}, \quad \chi_6 = \frac{\psi(b_3)}{(b_1 - b_2)\chi_5}, \quad (57)$$

where $z = (z_1, z_2)$ depend on t according to (49).

Proof. First, note that the functions (56) have the same structure of zeros and poles, as prescribed above. Next, as follows from the Kötter formula (16) and theta-solutions (48), the translations by the period vectors $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_1 + \mathcal{V}_2$ in $\text{Jac}(\Gamma)$ generate the involutions

$$\begin{aligned} \sigma_1 &: (z_1, p_1, z_2, p_2, z_3, p_3) \mapsto (z_1, p_1, -z_2, -p_2, z_3, p_3), \\ \sigma_2 &: (z_1, p_1, z_2, p_2, z_3, p_3) \mapsto (-z_1, -p_1, -z_2, -p_2, z_3, p_3), \\ \sigma_3 &= \sigma_2 \circ \sigma_1 : (z_1, p_1, z_2, p_2, z_3, p_3) \mapsto (-z_1, -p_1, z_2, p_2, z_3, p_3), \end{aligned}$$

which, in view of (11), gives rise to the transformations

$$\begin{aligned} \sigma_1 &: v_2 + v_1 \longleftrightarrow v_2 - v_1, & v_4 \pm v_3 &\longleftrightarrow v_4 \pm v_3, & v_5 + v_6 &\longleftrightarrow v_5 - v_6, \\ \sigma_2 &: v_2 + v_1 \longleftrightarrow v_2 - v_1, & v_4 + v_3 &\longleftrightarrow v_4 - v_3, & v_5 \pm v_6 &\longleftrightarrow -(v_5 \pm v_6) \\ \sigma_3 &: v_2 \pm v_1 \longleftrightarrow v_2 \pm v_1, & v_4 + v_3 &\longleftrightarrow v_4 - v_3, & v_6 + v_5 &\longleftrightarrow v_6 - v_5. \end{aligned}$$

Now observe that the relations (56) are invariant under the action of σ_i on the left-hand sides and the corresponding transformation of $\theta_0(z/2), \dots, \theta_3(z/2)$ under the action (55). Moreover, one can check that the left- and right hand sides of (56) are multiplied by the same factors under the shift of z by any period vector of $\text{Jac}(\Gamma)$. This proves (56).

The relations (57) between the constants χ_i follow from the first 3 integrals in (12). \square

The constants χ_1, χ_2, χ_3 can be calculated explicitly in terms of b_i, c_α and theta-constants of Γ .

As follows from the solutions (56), the product $(v_1 + v_2)(v_3 + v_4)$ and the other two similar products have double poles along \mathcal{D}_0 only:

$$\begin{aligned} (v_1 + v_2)(v_3 + v_4) &= g_2 \frac{\theta_2^2(z/2)}{\theta_0^2(z/2)}, \\ (v_3 + v_4)(v_5 + v_6) &= g_3 \frac{\theta_3^2(z/2)}{\theta_0^2(z/2)}, & (v_1 + v_2)(v_5 + v_6) &= g_1 \frac{\theta_1^2(z/2)}{\theta_0^2(z/2)}, \\ g_1, g_2, g_3 &= \text{const}. \end{aligned}$$

Analogous of some of these expressions were obtained in paper [9] in relation with separation of variables for the integrable system on $so(4)$ with the diagonal metric II. Due to the linear isomorphism between this system and the Steklov–Lyapunov systems, the separating variables presented in [9] can also be regarded as new separating variables for (4), (5).

6 Conclusive Remarks

In given paper we gave a justification of the separation of variables and the theta-function solution of the Steklov–Lyapunov systems obtained by F. Kötter [16]. Using the results of [1, 2], we also presented such a solution for an alternative set of variables, which have a simpler form.

On the other hand, there exist several nontrivial integrable generalizations of the systems: the first of them was discovered by V. Rubanovsky [19] and describes a motion of a gyrostat in an ideal fluid under the action of the Archimedes torque, which arises when the barycenter of the gyrostat does not coincide with its volume center. In this generalization the Hamiltonian of the Kirchhoff equations, apart from quadratic terms, contains linear (gyroscopic) terms in K, p . Under the change of variables (3), the gyroscopic generalizations of the systems (4), (5) take the form

$$\dot{z} = z \times (Bz - g) - Bp \times (Bz - g), \quad \dot{p} = p \times (Bz - g)$$

and, respectively,

$$\dot{z} = p \times (Bz - g), \quad \dot{p} = p \times (z - Bp),$$

where $g = (g_1, g_2, g_3)^T$ is an arbitrary constant vector related to the angular momentum of the rotor inside the gyrostat.

Following [12], these systems admit the following generalizations of Kötter's Lax pair with an elliptic spectral parameter

$$\begin{aligned} \dot{L}(s) &= [L(s), A(s)], \quad L(s), A(s) \in so(3), \quad s \in \mathbb{C}, \\ L(s)_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma} \left(\sqrt{s - b_\gamma} (z_\gamma + sp_\gamma) + g_\gamma / \sqrt{s - b_\gamma} \right), \\ A(s)_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma} \frac{1}{s} \sqrt{(s - b_\alpha)(s - b_\beta)} (b_\gamma z_\gamma - g_\gamma), \quad \text{resp.} \quad A(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} \sqrt{(s - b_\alpha)(s - b_\beta)} p_\gamma, \end{aligned}$$

which provides a sufficient set of constants of motion and makes possible to obtain theta-function solutions. Like in the case of the Steklov–Lyapunov systems, generic invariant manifolds of the Rubanovsky systems are two-dimensional tori, which can be extended to affine parts of Abelian varieties. However, as we plan to show in a forthcoming publication, an explicit integration of the latter systems appears to be more complicated, and the Abelian varieties are not Jacobians of genus 2 hyperelliptic curves, but Prym subvarieties.

The problem of separation of variables for the Rubanovsky systems is still unsolved.

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Appendix. Proof of Theorem 6.

The proof is based on the fourth Riemann identity (see, e.g., [3, 10])

$$\theta(y_1) \theta(y_2) \theta(y_3) \theta(y_4) = \frac{1}{4} \sum \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (w_1) \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (w_2) \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (w_3) \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (w_4), \quad (58)$$

where the summation is over all the half-period characteristics $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and the arguments $y_j, w_j \in \mathbb{C}^g$ (in our case $g = 2$) are related as follows

$$(w_1 \ w_2 \ w_3 \ w_4) = (y_1 \ y_2 \ y_3 \ y_4) T, \quad T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Up to multiplication by a simple exponent of z , the theta-product in (54) can be written as

$$\theta(z'/2) \theta(z'/2 + \mathcal{V}_1) \theta(z'/2 + \mathcal{V}_2) \theta(z'/2 + \mathcal{V}_1 + \mathcal{V}_2), \quad (59)$$

where $z' = z + 2\mathcal{K}$, i.e., the translation by the complete period in $\text{Jac}(\Gamma)$, and $\mathcal{V}_{1,2}$ are the periods defined by (53). In view of the identity (58), the product (59) gives the following sum of 16 theta-products:

$$\frac{1}{4} \sum_{2(\alpha,\beta) \in (\mathbb{Z}_2)^4} \theta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] \left(z' + \frac{\mathcal{V}_1 + \mathcal{V}_2}{2} \right) \theta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] \left(\frac{\mathcal{V}_1}{2} \right) \theta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] \left(\frac{\mathcal{V}_2}{2} \right) \theta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] (0).$$

(Note that in each product the variable z' enters only once.)

Next, in view of the property (40), this sum can be written as product of an exponent of z and the sum

$$\frac{1}{4} \sum_{2(\alpha,\beta) \in (\mathbb{Z}_2)^4} \theta \left[\begin{matrix} \alpha + \alpha' \\ \beta + \beta' \end{matrix} \right] (z') \theta \left[\begin{matrix} \alpha + \alpha' \\ \beta + \beta' \end{matrix} \right] \left(-\frac{\mathcal{V}_1}{2} \right) \theta \left[\begin{matrix} \alpha + \alpha' \\ \beta + \beta' \end{matrix} \right] \left(-\frac{\mathcal{V}_2}{2} \right) \theta \left[\begin{matrix} \alpha + \alpha' \\ \beta + \beta' \end{matrix} \right] \left(-\frac{\mathcal{V}_1 + \mathcal{V}_2}{2} \right),$$

$$2\pi j \beta' + B\alpha' = (\mathcal{V}_1 + \mathcal{V}_2)/2,$$

which, under the corresponding re-indexation, reads

$$\begin{aligned} & \frac{1}{4} \sum_{2(\alpha,\beta) \in (\mathbb{Z}_2)^4} \epsilon_{\alpha,\beta} \theta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] (z') \theta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] \left(\frac{\mathcal{V}_1}{2} \right) \theta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] \left(\frac{\mathcal{V}_2}{2} \right) \theta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] \left(\frac{\mathcal{V}_1 + \mathcal{V}_2}{2} \right) \\ &= -\frac{1}{4} \sum_{i=1}^6 \theta[\Delta + \eta_i](z') \theta[\Delta + \eta_i] \left(\frac{\mathcal{V}_1}{2} \right) \theta[\Delta + \eta_i] \left(\frac{\mathcal{V}_2}{2} \right) \theta[\Delta + \eta_i] \left(\frac{\mathcal{V}_1 + \mathcal{V}_2}{2} \right) \\ &+ \frac{1}{4} \sum_{1 \leq i < j \leq 5} \theta[\Delta + \eta_{ij}](z') \theta[\Delta + \eta_{ij}] \left(\frac{\mathcal{V}_1}{2} \right) \theta[\Delta + \eta_{ij}] \left(\frac{\mathcal{V}_2}{2} \right) \theta[\Delta + \eta_{ij}] \left(\frac{\mathcal{V}_1 + \mathcal{V}_2}{2} \right), \quad (60) \end{aligned}$$

where $\epsilon_{\alpha,\beta} = -1$ if $\theta[\alpha\beta](z)$ is odd and $+1$ otherwise, and, as above, $\eta_{ij} = \eta_j + \eta_j \bmod \mathbb{Z}^2/\mathbb{Z}^2$.

In fact, most of the theta-constants in (60) are proportional to $\theta[\Delta + \eta_i](0)$, $i = 1, \dots, 6$ and therefore vanish. Namely, in the first sum in the right hand side of (60) all the theta-constants are non-zero if and only if η_i is different from η_{b_1}, η_{b_2} , and $\eta_6 = 0$. In the second sum, if η_i or η_j coincides with η_{b_1} or η_{b_2} , then either the first or the second theta-constant is zero. Otherwise, if $\{\eta_i, \eta_j\} \cap \{\eta_{b_1}, \eta_{b_2}\} = \emptyset$, then, in view of the relations (44), the third theta-constant is proportional to $\theta[\Delta + \eta_k](0)$, for a certain $k \in \{1, \dots, 6\}$ and, therefore, equals zero.

Since for the case of genus 2 there are no even theta-functions which vanish for zero value of the argument (see [3]), one concludes that the above sum contains only 3 non-zero theta-products:

$$\begin{aligned} & \sum_{\eta_i \neq \eta_{b_1}, \eta_{b_2}, 0} \theta[\Delta + \eta_i](z') \theta[\Delta + \eta_i] \left(\frac{\mathcal{V}_1}{2} \right) \theta[\Delta + \eta_i] \left(\frac{\mathcal{V}_2}{2} \right) \theta[\Delta + \eta_i] \left(\frac{\mathcal{V}_1 + \mathcal{V}_2}{2} \right) \\ &= \sum_{\alpha=1}^3 \theta[\Delta + \eta_{c_\alpha}](z') \theta[\Delta + \eta_{c_\alpha}] \left(\frac{\mathcal{V}_1}{2} \right) \theta[\Delta + \eta_{c_\alpha}] \left(\frac{\mathcal{V}_2}{2} \right) \theta[\Delta + \eta_{c_\alpha}] \left(\frac{\mathcal{V}_1 + \mathcal{V}_2}{2} \right). \end{aligned}$$

Now, assume (for the moment) the following ordering of the Weierstrass points:

$$E_1 = b_1, \quad E_2 = c_1, \quad E_3 = b_2, \quad E_4 = c_2, \quad E_5 = c_3. \quad (61)$$

Then, in view of (40) and the identities (44), the above sum, up to a constant common factor, can be written as

$$\begin{aligned} & \varepsilon_1 \theta[\Delta + \eta_{c_1}](z') \theta[\eta_{b_1} + \eta_{c_2}](0) \theta[\eta_{b_2} + \eta_{c_2}](0) \theta[\eta_{c_1} + \eta_{c_3}](0) \\ & + \varepsilon_2 \theta[\Delta + \eta_{c_2}](z') \theta[\eta_{b_1} + \eta_{c_1}](0) \theta[\eta_{b_2} + \eta_{c_1}](0) \theta[\eta_{c_2} + \eta_{c_3}](0) \\ & + \varepsilon_3 \theta[\Delta + \eta_{c_3}](z') \theta[\eta_{b_2}](0) \theta[\eta_{b_1}](0) \theta0, \quad (62) \end{aligned}$$

where now ε_α are certain quartic roots of 1.

Now we are going to show that the denominator in the theta-function solution (48) coincides with (62) up to multiplication by an exponent of z' .

Namely, in view of (52), the sum $\sum_{\alpha=1}^3 k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z)$ can be written as a product of

$$\text{const} \sqrt{\theta[\Delta](z - q/2) \theta[\Delta](z + q/2)}$$

and the expression

$$\mathcal{G} = \frac{\sqrt{-(\lambda_1 - c_1)(\lambda_2 - c_1)}}{\sqrt{(c_1 - c_2)(c_1 - c_2)}} \sqrt{\frac{c_2 - c_3}{c_1 - c_2}} + \frac{\sqrt{-(\lambda_1 - c_2)(\lambda_2 - c_2)}}{\sqrt{(c_2 - c_1)(c_2 - c_3)}} \sqrt{\frac{c_3 - c_1}{c_1 - c_2}} + \frac{\sqrt{-(\lambda_1 - c_3)(\lambda_2 - c_3)}}{\sqrt{(c_3 - c_1)(c_3 - c_2)}}$$

(there is no second radical in the third summand!). Now, make the projective transformation $\lambda \rightarrow \nu = \lambda/(\lambda - E_6) = \lambda/(\lambda - b_3)$, which sends the Weierstrass points c_α, b_1, b_2, b_3 on Γ to $\bar{c}_\alpha, \bar{b}_1, \bar{b}_2$, and ∞ . The two infinite points over $\lambda = \infty$ are mapped to 2 points over $\nu = 1$. This change leaves the sum \mathcal{G} almost invariant: it becomes the product of $\text{const}/\sqrt{(\nu_1 - 1)(\nu_2 - 1)}$ and the sum

$$\bar{\mathcal{G}} = \frac{\sqrt{-(\nu_1 - \bar{c}_1)(\nu_2 - \bar{c}_1)}}{\sqrt{(\bar{c}_1 - \bar{c}_2)(\bar{c}_1 - \bar{c}_2)}} \sqrt{\frac{\bar{c}_2 - \bar{c}_3}{\bar{c}_1 - \bar{c}_2}} + \frac{\sqrt{-(\nu_1 - \bar{c}_2)(\nu_2 - \bar{c}_2)}}{\sqrt{(\bar{c}_2 - \bar{c}_1)(\bar{c}_2 - \bar{c}_3)}} \sqrt{\frac{\bar{c}_3 - \bar{c}_1}{\bar{c}_1 - \bar{c}_2}} + \frac{\sqrt{-(\nu_1 - \bar{c}_3)(\nu_2 - \bar{c}_3)}}{\sqrt{(\bar{c}_3 - \bar{c}_1)(\bar{c}_3 - \bar{c}_2)}}$$

Under the Abel map (38), the radicals in $\bar{\mathcal{G}}$ can be expressed completely in terms of the theta-functions and theta-constants of Γ : Applying the theta-formulae of Frobenius and Thomae for the case when one of the Weierstrass points of the curve lies at infinity (see, e.g., [21, 18]) and keeping the ordering (61), we have

$$\begin{aligned} \frac{\sqrt{(\nu_1 - \bar{c}_\alpha)(\nu_2 - \bar{c}_\alpha)}}{\sqrt{(\bar{c}_\alpha - \bar{c}_\beta)(\bar{c}_\alpha - \bar{c}_\gamma)}} &= \pm \varrho_1 \frac{\theta[\Delta + \eta_{c_\beta} + \eta_{\bar{c}_\gamma}](0)}{\theta[\Delta + \eta_{c_1} + \eta_{c_2} + \eta_{c_3}](0)} \frac{\theta[\Delta + \eta_{c_\alpha}](z)}{\theta[\Delta](z)}, \quad (\alpha, \beta, \gamma) = (1, 2, 3), \\ \sqrt{\frac{\bar{c}_2 - \bar{c}_3}{\bar{c}_1 - \bar{c}_2}} &= \varrho_2 \frac{\theta[\eta_{c_2} + \eta_{b_2}](0) \theta[\eta_{c_2} + \eta_{b_1}](0)}{\theta[\eta_{c_1} + \eta_{c_2} + \eta_{c_3} + \eta_{b_2}](0) \theta[\eta_{c_1} + \eta_{c_2} + \eta_{c_3} + \eta_{b_1}](0)}, \\ \sqrt{\frac{\bar{c}_3 - \bar{c}_1}{\bar{c}_1 - \bar{c}_2}} &= \varrho_3 \frac{\theta[\eta_{c_1} + \eta_{b_1}](0) \theta[\eta_{c_1} + \eta_{b_2}](0)}{\theta[\eta_{c_1} + \eta_{c_2} + \eta_{c_3} + \eta_{b_2}](0) \theta[\eta_{c_1} + \eta_{c_2} + \eta_{c_3} + \eta_{b_1}](0)}, \end{aligned} \quad (63)$$

where $\eta_{c_\alpha}, \eta_{b_1}, \eta_{b_2}$ are the same as above and ϱ_i are the appropriate quartic roots of 1. Lastly, we have

$$\sqrt{(\nu_1 - 1)(\nu_2 - 1)} = \text{const} \frac{\sqrt{\theta[\Delta](z - q/2) \theta[\Delta](z + q/2)}}{\theta[\Delta](z)}, \quad (64)$$

where q is the same as in (45), or, in terms of the new coordinate ν on Γ , $q/2 = \int_\infty^{(1,0)} \bar{\omega}$.

Combining the above expressions, we see that in the quotient $\mathcal{G} = \bar{\mathcal{G}}/\sqrt{(\nu_1 - 1)(\nu_2 - 1)}$ the term $\theta[\Delta](z)$ is canceled and in the product $\mathcal{G} \sqrt{\theta[\Delta](z - q/2) \theta[\Delta](z + q/2)}$ the square root (64) is canceled. Now, simplifying the theta-characteristics in (63) by using (44) and ignoring common constant factors, we eventually find

$$\begin{aligned} \text{const} \sum_{\alpha=1}^3 k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z) &= \bar{\varepsilon}_1 \theta[\Delta + \eta_{c_1}](z) \theta[\eta_{c_1} + \eta_{c_3}](0) \theta[\eta_{c_2} + \eta_{b_1}](0) \theta[\eta_{c_2} + \eta_{b_2}](0) \\ &\quad + \bar{\varepsilon}_2 \theta[\Delta + \eta_{c_2}](z) \theta[\eta_{c_2} + \eta_{c_3}](0) \theta[\eta_{c_1} + \eta_{b_1}](0) \theta[\eta_{c_1} + \eta_{b_2}](0) \\ &\quad + \bar{\varepsilon}_3 \theta[\Delta + \eta_{c_3}](z) \theta[\eta_{b_2}](0) \theta[\eta_{b_1}](0) \theta(0), \end{aligned} \quad (65)$$

$\bar{\varepsilon}_i$ also being certain quartic roots of 1. The latter expression have the same structure as the sum (62). Lastly, note that under the shift of z by an appropriate complete period in $\text{Jac}(\Gamma)$ the roots $\bar{\varepsilon}_i$ can be made proportional to any combination of roots ε_α in (62). (This corresponds to choosing an appropriate origin in $\widetilde{\text{Jac}}(\Gamma)$.) Hence, we proved the theorem for the chosen ordering (61).

To complete the proof for the other possible orderings of b_i, c_α it remains to modify the theta-characteristics in (62), (65). \square

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