Solutions of the Yang-Baxter equation: descendants of the six-vertex model from the Drinfeld doubles of dihedral group algebras

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Abstract

The representation theory of the Drinfeld doubles of dihedral groups is used to solve the Yang-Baxter equation. Use of the 2-dimensional representations recovers the six-vertex model solution. Solutions in arbitrary dimensions, which are viewed as descendants of the six-vertex model case, are then obtained using tensor product graph methods which were originally formulated for quantum algebras. Connections with the Fateev-Zamolodchikov model are discussed.

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1 Introduction

The Yang-Baxter equation in its spectral parameter-dependent form has well-known applications in a variety of areas of mathematical physics, including exactly solvable classical models of two-dimensional statistical mechanics and integrable quantum systems. The origins of the field trace back to the influential works by McGuire concerning many-body systems with delta function interactions [1], Yang's solution for interacting fermions [2], and Baxter's solution of the eight-vertex model [3]. Since then numerous solutions have been obtained including those leading to the Perk-Schultz model [4], the Andrews-Baxter-Forrester model [5], the chiral Potts model [6, 7] and the Hubbard model [8, 9] to name a few.

One systematic method for solving the Yang-Baxter equation was found with the advent of affine quantum algebras, which are deformations of the universal enveloping algebras of affine classical Lie algebras [10, 11]. Through the Drinfeld double construction [10], the affine quantum algebras can be seen to belong to a larger class of algebras which are referred to as *quasi-triangular Hopf* algebras. The double construction provides a universal prescription for constructing a quasi-triangular Hopf algebra, denoted $D(H)$, from any Hopf algebra H and its dual algebra H^*. Via this construction there exists a canonical element in the tensor product algebra of the Drinfeld double, $R \in D(H) \otimes D(H)$, known as the universal R-matrix. This element provides an algebraic solution of the Yang-Baxter equation. Thus for each matrix representation of $D(H)$, a matrix solution of the Yang-Baxter equation is automatically obtained. Solutions of the Yang-Baxter equation with spectral parameter arise whenever matrix representations of $D(H)$ can be parameterised by one or more continuous variables. In the context of affine quantum algebras, the spectral parameter arises naturally by consideration of an evaluation homomorphism from the affine quantum algebra to its non-affine counterpart. In this manner the representations of the affine quantum algebras which give solutions with spectral parameter are the loop representations. In principle the evaluation homomorphism provides a route to algebraic solutions of the Yang-Baxter equation with spectral parameter. However it is a technically challenging task to undertake and has only been implemented explicitly for some low rank cases [12, 13].

Often, a more tractable approach is to start with a constant matrix solution of the Yang-Baxter equation and then ask the question of whether a spectral parameter-dependent generalisation exists, a procedure which is commonly referred to as *Baxterisation*. For example, in cases where the solution of the Yang-Baxter equation without spectral parameter affords a representation of the Hecke algebra, there exists a procedure for constructing the associated spectral parameter-dependent solution [14]. This also applies in cases where the constant solution gives rise to a representation of the Birman-Wenzl-Murakami algebra [15]. For solutions associated with representations of quantum algebras, a general prescription for Baxterising constant solutions of the Yang-Baxter equation (subject to satisfying certain conditions) was developed in [16, 17, 18] using the notion of the *tensor product graph* method.

An alternative route to building solutions of the Yang-Baxter equation is in the framework of descendants. The most notable instances were found by Bazhanov and Stroganov [19], who showed that the chiral Potts model is a descendant of the six-vertex model, and Hasegawa and Yamada [20], who showed that the Kashiwara-Miwa model is a descendant of the zero-field eight-vertex model. In both cases the connection was found by constructing an appropriate algebraic structure through an L-operator. In [19] the algebraic structure is closely related to that of the quantum algebra $U_q(sl(2))$ with q a root of unity, and in [20] the L-operator is expressed in terms of the Sklyanin algebra [21]. In this approach use of the L-operator allows for the construction of descendants through higher dimensional representations of the algebra. It is known [22] that the six-vertex and the zero-field eight-vertex models intersect at the zero-field six-vertex model whose descendant is the Fateev-Zamolodchikov model [23], which is precisely the intersection between the chiral Potts and Kashiwara-Miwa models.

Another principal class of quasi-triangular Hopf algebras is the class of Drinfeld doubles of finite group algebras. These algebras have recently received attention in relation to the description of anyonic symmetries in quantum systems, specifically in terms of the braiding properties of anyonic quasiparticle excitations which are described by constant solutions of the Yang-Baxter equation [24, 25, 26, 27]. However these algebras these have not been investigated in the context of solving the Yang-Baxter equation with spectral parameter, apart from some preliminary investigations reported in [28, 29]. Arguably the simplest family of Drinfeld double algebras for finite groups is that associated with the dihedral groups, denoted D_n. The Drinfeld doubles of these algebras, denoted $D(D_n)$, are finite-dimensional quasi-triangular Hopf algebras with a finite number of irreducible representations. For $D(D_n)$ all the irreducible representations are known [28]. In the case where n is odd, the irreducible representations can only have dimensions 1, 2 or n, whereas for n even the dimensions are 1, 2 or $n/2$. The constant R-matrices associated with the 2-dimensional representations can be Baxterised to reproduce the six-vertex model at roots of unity [28].

Our goal below is to obtain spectral parameter-dependent solutions of the Yang-Baxter equation associated with the n-dimensional (n odd) and $n/2$-dimensional (n even) representations. The general strategy we follow is to view these solutions as descendants of the six-vertex model, by firstly determining the appropriate L-operators. Once this is achieved, the technical aspects for determining the explicit form of the descendants is accommodated by adapting the tensor product graph method developed for quantum algebras [16, 17, 18] to the present case. It turns out that solutions we obtain for n odd are limiting cases of the Fateev-Zamolodchikov models [30]. However we stress that the anticipated $U_q(sl(2))$ symmetry with q a root of unity degenerates in this limit, with the $D(D_n)$ symmetry emerging.

2 Preliminaries

We first define some notation which is used extensively throughout this article. We often work modulo $n \in \mathbb{N}$, and hence define the following map: given $a \in \mathbb{Z}$ then $0 \leq \overline{a} \leq n-1$ is the integer which satisfies $a \equiv \overline{a} \pmod{n}$. We also use the following two delta functions:

$$\delta_i^j = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad \bar{\delta}_i^j = \begin{cases} 1, & i \equiv j \pmod{n}, \\ 0, & i \not\equiv j \pmod{n}. \end{cases}$$

We use $e_{i,j}$ to denote an elementary matrix in $M_{n\times n}(\mathbb{C})$ whose indices are considered modulo n. We adopt the convention that $e_{0,0}$ corresponds to the matrix with an entry in the nth row and nth column. These matrices obey the relation

$$e_{i,j}e_{k,l} = \bar{\delta}_j^k e_{i,l}.$$

We use the following convention for the product symbol:

$$\prod_{i=j}^{k} a_i = \left\{ \begin{array}{cc} 1, & k < j, \\ a_j a_{j+1} \ldots a_k, & k \geq j. \end{array} \right.$$

2.1 The Yang-Baxter equation with spectral parameter

We will use three variants of the Yang-Baxter equation (YBE), which is a non-linear matrix equation in End $(V \otimes V \otimes V)$ for some vector space V. The first of these forms we call the constant YBE, given by

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where the subscripts of R refer to which vector spaces the operator is acting upon. That is, given

$$R = \sum_i a_i \otimes b_i \in \text{End } (V \otimes V),$$

we have

$$R_{12} = \sum_i a_i \otimes b_i \otimes I, \qquad R_{13} = \sum_i a_i \otimes I \otimes b_i, \qquad \text{etc.}$$

Here I denotes the identity matrix. The second form is known as the spectral parameter-dependent YBE and is given by

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x), \tag{1}$$

where $x, y \in \mathbb{C}$. Given a parameter-dependent solution, $R(z)$, constant solutions are recovered when we take the limits $z \to 0, 1$ and ∞. We refer to invertible solutions of either form as *R-matrices*. As this article deals primarily with the parameter-dependent YBE we shall henceforth refer to it simply as the Yang-Baxter equation.

Now suppose $r(z) \in$ End $(V \otimes V)$ and $R(z) \in$ End $(W \otimes W)$ are R-matrices with $\dim V < \dim W$. We describe $R(z)$ as a *descendant* of $r(z)$ provided that there exists a non-trivial invertible operator $L(z) \in$ End $(V \otimes W)$ (referred to as an *L-operator*) which satisfies

$$r_{12}(xy^{-1})L_{13}(x)L_{23}(y) = L_{23}(y)L_{13}(x)r_{12}(xy^{-1}), \quad \forall x, y \in \mathbb{C}, \tag{2}$$

and

$$L_{12}(x)L_{13}(y)R_{23}(x^{-1}y) = R_{23}(x^{-1}y)L_{13}(y)L_{12}(x), \quad \forall x, y \in \mathbb{C}. \tag{3}$$

In this article we start with an R-matrix associated with the zero-field six-vertex model and use the Drinfeld doubles of dihedral groups to generate L-operators and descendants.

The third variant of the YBE used in this paper is

$$\check{R}_{12}(x)\check{R}_{23}(xy)\check{R}_{12}(y) = \check{R}_{23}(y)\check{R}_{12}(xy)\check{R}_{23}(x).$$

Given a matrix solution $R(z)$ to Equation (1), a solution to this form of the YBE is given by

$$\check{R}(z) = P\,R(z),$$

where P is the permutation operator, i.e.

$$P = \sum_{i,j} e_{i,j} \otimes e_{j,i}.$$

2.2 Drinfeld doubles of finite groups

Next we recall relevant results from the Drinfeld double construction applied to finite group algebras, following [31, 32]. Given a group G, with identity e, we consider the algebra $\mathbb{C}G$, whose basis vectors are the elements of the group. The multiplication and unit of $\mathbb{C}G$ are inherited from the group in the natural way. We equip $\mathbb{C}G$ with a coproduct, counit and antipode defined respectively by

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1 \qquad \text{and} \qquad \gamma(g) = g^{-1}, \qquad \forall g \in G.$$

With these maps $\mathbb{C}G$ becomes a Hopf algebra. We next consider the dual space of $\mathbb{C}G$,

$$(\mathbb{C}G)_o = \mathbb{C}\{g^* | g \in G\}.$$

The multiplication and unit are given, respectively, by

$$g^* h^* = \delta_g^h \qquad \text{and} \qquad u(1) = \sum_{g \in G} g^*.$$

The costructure and antipode are defined by

$$\Delta(g^*) = \sum_{h \in G} (hg)^* \otimes (h^{-1})^*, \qquad \epsilon(g^*) = \delta_g^e \qquad \text{and} \qquad \gamma(g^*) = (g^{-1})^*, \qquad \forall g \in G.$$

Under these maps $(\mathbb{C}G)_o$ is also a Hopf algebra. Using the dual and the original algebra we can construct the Drinfeld double,

$$D(G) = \mathbb{C}\{gh^* | g, h \in G\}.$$

We impose the relation

$$h^* g = g(g^{-1}hg)^*$$

and adopt the required remaining structure from $\mathbb{C}G$ and $(\mathbb{C}G)_o$. Furthermore it is known that $D(G)$ is a quasi-triangular Hopf algebra, containing the canonical element

$$\mathcal{R} = \sum_{g \in G} g \otimes g^*.$$

This element satisfies the following relations:

$$\begin{aligned} \mathcal{R}\Delta(a) &= \Delta^T(a)\mathcal{R}, \quad \forall a \in D(G), \\ (\Delta \otimes \mathrm{id})\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}, \\ (\mathrm{id} \otimes \Delta)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12}, \end{aligned}$$

where Δ^T denotes the opposite coproduct. Consequently matrix representations of $\mathcal{R}$ provide solutions of the constant Yang-Baxter equation.

In this paper we use the dihedral group D_n, which is the symmetry group of a regular polygon with n vertices. That is,

$$D_n = \{\sigma, \tau | \sigma^n = \tau^2 = e, \sigma\tau\sigma = \tau\},$$

and has order $2n$. For ease of calculation we divide the description of the representation theory into different cases.

2.3 Representations of $D(D_n)$

2.3.1 The case when n is odd

We first consider $D(D_n)$ for the case where n is odd. As stated in [28] the representations of the double of a group are naturally partitioned by the conjugacy classes of the group. For these representations we consider $w \in \mathbb{C}$ to be a primitive nth root of unity. Then the irreducible representations (irreps) are given in Table 1 below:

Irrep π	Constraints	$\pi(\sigma)$	$\pi(\tau)$	$\pi(g^*),\ g \in D_n$
$\pi_1^{\pm}$		1	± 1	δ_g^e
$\pi_2^{(0,k)}$	$1 \leq k \leq \frac{n-1}{2}$	$\begin{pmatrix} w^k & 0 \\ 0 & w^{-k} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \delta_g^e & 0 \\ 0 & \delta_g^e \end{pmatrix}$
$\pi_2^{(l,k)}$	$1 \leq l \leq \frac{n-1}{2}$, $0 \leq k \leq n-1$	$\begin{pmatrix} w^k & 0 \\ 0 & w^{-k} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \delta_g^{\sigma^l} & 0 \\ 0 & \delta_g^{\sigma^{-l}} \end{pmatrix}$
$\pi_n^{\pm}$		$\sum_{i=1}^n e_{i+1,i}$	$\pm \sum_{i=1}^n e_{i,2-i}$	$\delta_g^{\sigma^{2j}\tau} e_{j+1,j+1}$

Table 1: The irreps of $D(D_n)$ when n is odd.

Here $\pi_1^{\pm}, \pi_2^{(a,b)}$ and $\pi_n^{\pm}$ have dimensions 1, 2 and n respectively, and their associated modules are denoted $V_1^{\pm}, V_2^{(a,b)}$ and $V_n^{\pm}$.

Also associated with these representations are the solutions of the Yang-Baxter equation arising from the canonical element. If we apply a two-dimensional irrep we find the R-matrix

$$(\pi_2^{(l,k)} \otimes \pi_2^{(l,k)})\mathcal{R} = \begin{pmatrix} w^{kl} & 0 & 0 & 0 \\ 0 & w^{-kl} & 0 & 0 \\ 0 & 0 & w^{-kl} & 0 \\ 0 & 0 & 0 & w^{kl} \end{pmatrix}.$$

It was shown in [28] that this constant solution leads to the parameter-dependent solution

$$r(z) = \begin{pmatrix} w^{2kl}z^{-1} - w^{-2kl}z & 0 & 0 & 0 \\ 0 & z^{-1} - z & w^{2kl} - w^{-2kl} & 0 \\ 0 & w^{2kl} - w^{-2kl} & z^{-1} - z & 0 \\ 0 & 0 & 0 & w^{2kl}z^{-1} - w^{-2kl}z \end{pmatrix}. \quad (4)$$

This R-matrix corresponds to the six-vertex model with zero-field. Also of interest is the representation of the canonical element using the n-dimensional irreps,

$$(\pi_n^{\pm} \otimes \pi_n^{\pm})\mathcal{R} = \pm \sum_{i,j=0}^{n-1} e_{i+j,i-j} \otimes e_{i,i}.$$

In our construction of the descendant of $r(z)$ we will use a linear combination of projection operators. It is known [32] that there exist operators which project $D(D_n)$

onto its ideals. These projection operators are defined by

$$E_\alpha = \frac{d[\alpha]}{|G|} \sum_{g,h\in G} \chi_\alpha(h^* g^{-1}) g h^*,$$

where α is an irrep, $d[\alpha]$ its dimension and χ_α is the group character defined by

$$\chi_\alpha(a) = \mathrm{tr}\ \pi_\alpha(a), \quad \forall a \in D(D_n).$$

We also consider the projection operators

$$p_n^\alpha = (\pi_n^\pm \otimes \pi_n^\pm)\Delta(E_\alpha). \tag{5}$$

These operators project from $\pi_n^\pm \otimes \pi_n^\pm$ onto copies of the irrep associated with α in the decomposition of $\pi_n^\pm \otimes \pi_n^\pm$. Note that we consider $\pi_n^+ \otimes \pi_n^+$ and $\pi_n^- \otimes \pi_n^-$ together since

$$(\pi_n^+ \otimes \pi_n^+)\Delta(a) = (\pi_n^- \otimes \pi_n^-)\Delta(a), \qquad \forall a \in D(D_n).$$

We calculate p_n^α explicitly, using the expression given in Equation (5), obtaining

$$p_n^\alpha = \frac{d[\alpha]}{2n} \sum_{g\in D_n} \sum_{i,j=0}^{n-1} \chi_\alpha((\sigma^{2j})^* g^{-1}) \pi_n^\pm(g) e_{i-j,i-j} \otimes \pi_n^\pm(g) e_{i,i}.$$

From this we can see that the only irreps with non-zero projection operators will be associated with the conjugacy classes $\{e\}$ and $\{\sigma^i, \sigma^{-i}\}$ for $1 \le i \le \frac{n-1}{2}$. This implies that $\pi_n^\pm \otimes \pi_n^\pm$ decomposes only into one and two-dimensional irreps.

For convenience, we slightly modify our notation for the irreps. Instead of using π_1^+ and $\pi_2^{(l,k)}$, we use ordered pairs corresponding only to irreps that appear in the direct sum decomposition of $\pi_n^\pm \otimes \pi_n^\pm$. The correspondence is summarised in Table 2:

α	Irrep	Constraint on a	Constraint on b
$(0,0)$	π_1^+		
$(0,b)$	$\pi_2^{(0,2b)}$		$1 \le b \le \lfloor \frac{n-1}{4} \rfloor$
$(0,b)$	$\pi_2^{(0,n-2b)}$		$\lfloor \frac{n+3}{4} \rfloor \le b \le \frac{n-1}{2}$
(a,b)	$\pi_2^{(2a,\overline{2b})}$	$1 \le a \le \lfloor \frac{n-1}{4} \rfloor$	$0 \le b \le n-1$
(a,b)	$\pi_2^{(n-2a,\overline{n-2b})}$	$\lfloor \frac{n+3}{4} \rfloor \le a \le \frac{n-1}{2}$	$0 \le b \le n-1$

Table 2: The ordered pairs labelling the irreps for $D(D_n)$, n odd.

Here $\lfloor a \rfloor$ denotes the floor of a. Then the projection operator for irrep $\alpha = (a,b)$ is given by

$$p_n^\alpha = \frac{c^\alpha}{n} \sum_{i,j=0}^{n-1} [w^{2bj} e_{i+a+j,i+a} \otimes e_{i+j,i} + w^{-2bj} e_{i-a+j,i-a} \otimes e_{i+j,i}]$$

where

$$c^\alpha = \begin{cases} \frac{1}{2}, & \alpha = (0,0), \\ 1, & \alpha \neq (0,0). \end{cases}$$

We have calculated non-zero projection operators for $(n^2-1)/2$ two-dimensional irreps and one 1-dimensional irrep. Using a counting argument, it is clear that these provide a complete decomposition of $\pi_n^\pm \otimes \pi_n^\pm$ into irreps for n odd.

2.3.2 The case when n is even

We similarly catalogue the irreps of $D(D_n)$ for the case when n is even. We set $n = 2m$ and let $w \in \mathbb{C}$ be a primitive $2m$th root of unity.

Irrep π	Constraints	$\pi(\sigma)$	$\pi(\tau)$	$\pi(g^*),\ g \in D_{2m}$
$\pi_{1,e}^{(a,b)}$	$a,b \in \{0,1\}$	$(-1)^b$	$(-1)^a$	δ_g^e
$\pi_{1,\sigma^m}^{(a,b)}$	$a,b \in \{0,1\}$	$(-1)^b$	$(-1)^a$	$\delta_g^{\sigma^m}$
$\pi_2^{(0,k)}$	$1 \leq k < m$	$\begin{pmatrix} w^k & 0 \\ 0 & w^{-k} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \delta_g^e & 0 \\ 0 & \delta_g^e \end{pmatrix}$
$\pi_2^{(m,k)}$	$1 \leq k < m$	$\begin{pmatrix} w^k & 0 \\ 0 & w^{-k} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \delta_g^{\sigma^m} & 0 \\ 0 & \delta_g^{\sigma^m} \end{pmatrix}$
$\pi_2^{(l,k)}$	$1 \leq l \leq m-1$, $0 \leq k \leq 2m-1$	$\begin{pmatrix} w^k & 0 \\ 0 & w^{-k} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \delta_g^{\sigma^l} & 0 \\ 0 & \delta_g^{\sigma^{-l}} \end{pmatrix}$
$\pi_{m,\tau}^{(a,b)}$	$a,b \in \{0,1\}$	$\sum_{i=1}^m (-1)^{a\delta_i^1} e_{i,i-1}$	$\sum_{i=1}^m (-1)^{a\delta_i^1+b} e_{i,2-i}$	$\delta_g^{\sigma^{2k}\tau} e_{k+1,k+1}$
$\pi_{m,\sigma\tau}^{(a,b)}$	$a,b \in \{0,1\}$	$\sum_{i=1}^m (-1)^{a\delta_i^1} e_{i,i-1}$	$(-1)^b \sum_{i=1}^m e_{i,1-i}$	$\delta_g^{\sigma^{2k+1}\tau} e_{k+1,k+1}$

Table 3: The irreps of $D(D_{2m})$.

We again concern ourselves with the two-dimensional representations applied to the canonical element. We find the constant solution

$$(\pi_2^{(l,k)} \otimes \pi_2^{(l,k)})\mathcal{R} = \begin{pmatrix} w^{kl} & 0 & 0 & 0 \\ 0 & w^{-kl} & 0 & 0 \\ 0 & 0 & w^{-kl} & 0 \\ 0 & 0 & 0 & w^{kl}, \end{pmatrix},$$

which leads to the parameter-dependent solution

$$r(z) = \begin{pmatrix} w^{2kl}z^{-1} - w^{-2kl}z & 0 & 0 & 0 \\ 0 & z^{-1} - z & w^{2kl} - w^{-2kl} & 0 \\ 0 & w^{2kl} - w^{-2kl} & z^{-1} - z & 0 \\ 0 & 0 & 0 & w^{2kl}z^{-1} - w^{-2kl}z \end{pmatrix}.$$

This result only differs to that of $D(D_n)$ where n is odd by the possible choices of the root of unity. Note also the similarity in the representation of the canonical element, given by

$$(\pi_m \otimes \pi_m)\mathcal{R} = (-1)^b \sum_{i,j=0}^{m-1} e_{i+j,i-j} \otimes e_{i,i},$$

when π_m is either $\pi_{m,\tau}^{(0,b)}$ or $\pi_{m,\sigma\tau}^{(0,b)}$ for $b \in \{0,1\}$.

We again use Equation (5) to derive our projection operators. We first look at the case when $m = n/2$ is odd. We find that

$$p_m^\alpha = \frac{d[\alpha]}{4m} \sum_{g\in G} \sum_{i,j=0}^{m-1} \chi_\alpha((\sigma^{2j})^* g^{-1})\pi_m(g)e_{i-j,i-j} \otimes \pi_m(g)e_{i,i},$$

where π_m is any one of the m-dimensional irreps. As in the case of $D(D_n)$ for odd n, we introduce an ordered pair notation for the irreps that appear in the direct sum decomposition of $\pi_{m,\tau}^{(0,b)} \otimes \pi_{m,\tau}^{(0,b)}$ and $\pi_{m,\sigma\tau}^{(0,b)} \otimes \pi_{m,\sigma\tau}^{(0,b)}$ for $b \in \{0,1\}$. The correspondence is summarised in Table 4:

α	Irrep	Constraint on a	Constraint on b
$(0,0)$	$\pi_{1,e}^{+}$		
$(0,b)$	$\pi_2^{(0,2b)}$		$1 \le b \le \frac{m-1}{2}$
(a,b)	$\pi_2^{(2a,2b)}$	$1 \le a \le \frac{m-1}{2}$	$0 \le b \le m-1$

Table 4: The ordered pairs labelling the irreps for $D(D_{2m})$, m odd.

Then the projection operator associated with irrep $\alpha = (a,b)$ is given by

$$p_m^\alpha = \frac{c^\alpha}{m} \sum_{i,j=0}^{m-1} [w^{2bj} e_{i+a+j,i+a} \otimes e_{i+j,i} + w^{-2bj} e_{i-a+j,i-a} \otimes e_{i+j,i}]$$

where

$$c^\alpha = \begin{cases} \frac{1}{2}, & \alpha = (0,0), \\ 1, & \alpha \neq (0,0). \end{cases}$$

We note that these projection operators here match those arising in the case of $D(D_n)$ where is n odd.

Similarly, when $m = n/2$ is even we have the projection operators

$$p_m^\alpha = \frac{d[\alpha]}{4m} \sum_{g\in G} \sum_{i,j=0}^{m-1} \chi_\alpha((\sigma^{2j})^* g^{-1})\pi_m(g)e_{i-j,i-j} \otimes \pi_m(g)e_{i,i},$$

for any m-dimensional irrep π_m. The ordered pair notation for the irreps occuring in the decomposition of $\pi_{m,\tau}^{(0,b)} \otimes \pi_{m,\tau}^{(0,b)}$ for $b \in \{0,1\}$ is given in Table 5, whereas those occuring in the decomposition of $\pi_{m,\sigma\tau}^{(0,b)} \otimes \pi_{m,\sigma\tau}^{(0,b)}$ for $b \in \{0,1\}$ are given in Table 6.

α	Irrep	Constraint on a	Constraint on b
$(0, b\frac{m}{2})$	$\pi_{1,e}^{(0,b)}$		$b \in \{0, 1\}$
$(\frac{m}{2}, b\frac{m}{2})$	$\pi_{1,\sigma^m}^{(0,b)}$		$b \in \{0, 1\}$
$(0, b)$	$\pi_2^{(0,2b)}$		$1 \leq b \leq \frac{m}{2} - 1$
$(\frac{m}{2}, b)$	$\pi_2^{(m,2b)}$		$1 \leq b \leq \frac{m}{2} - 1$
(a, b)	$\pi_2^{(2a,2b)}$	$1 \leq a \leq \frac{m}{2} - 1$	$0 \leq b \leq m - 1$

Table 5: The ordered pairs labelling the irreps for $D(D_{2m})$, m even, case (i).

α	Irrep	Constraint on a	Constraint on b
$(0, b\frac{m}{2})$	$\pi_{1,e}^{(b,b)}$		$b \in \{0, 1\}$
$(\frac{m}{2}, b\frac{m}{2})$	$\pi_{1,\sigma^m}^{(b,b)}$		$b \in \{0, 1\}$
$(0, b)$	$\pi_2^{(0,2b)}$		$1 \leq b \leq \frac{m}{2} - 1$
$(\frac{m}{2}, b)$	$\pi_2^{(m,2b)}$		$1 \leq b \leq \frac{m}{2} - 1$
(a, b)	$\pi_2^{(2a,2b)}$	$1 \leq a \leq \frac{m}{2} - 1$	$0 \leq b \leq m - 1$

Table 6: The ordered pairs labelling the irreps for $D(D_{2m})$, m even, case (ii).

We obtain the same projection operators irrespective of which of the two cases we consider. In particular, the projection operator corresponding to the irrep $\alpha = (a, b)$ is

$$p_m^\alpha = \frac{c^\alpha}{m} \sum_{i,j=0}^{m-1} [w^{2bj} e_{i+a+j,i+a} \otimes e_{i+j,i} + w^{-2bj} e_{i-a+j,i-a} \otimes e_{i+j,i}]$$

where

$$c^\alpha = \begin{cases} \frac{1}{2}, & a \in \{0, \frac{m}{2}\} \text{ and } b \in \{0, \frac{m}{2}\}, \\ 1, & \text{otherwise.} \end{cases}$$

Again, a counting argument verifies that these are the only non-zero projection operators. As the projection operators are the same for the different m-dimensional irreps, we can without loss of generality consider only $\pi_{m,\tau}^{(0,0)}$.

3 Descendants associated with $D(D_n)$ - n odd

In this section we construct a family of solutions of the Yang-Baxter equation using $D(D_n)$, where n is odd and $n > 2$.

3.1 Construction of the L-operator

We begin by constructing an operator $L(z) \in \text{End}(V_2^{(l,k)} \otimes V_n^{\pm})$. We observe that $V_2^{(l,k)} \otimes V_n^{\pm} \cong V_n^+ \oplus V_n^-$, suggesting that the L-operator will be a linear combination of two intertwining operators with one parameter-dependent coefficient (after scaling). Hence we adopt the following ansatz for the L-operator:

$$L(z) = (\pi_2^{(l,k)} \otimes \pi_n^{\pm})[\mathcal{R} + h(z)(\mathcal{R}^T)^{-1}]$$

where $(\mathcal{R}^T)^{-1} = \sum_{g\in G} g^* \otimes g^{-1}$. Applying the representations leads to the following expression for $L(z)$:

$$L(z) = \sum_{i=0}^{n-1} \left\{ (w^{2(i-1)k} e_{1,2} + w^{-2(i-1)k} e_{2,1}) \otimes e_{i,i} + h(z) \left[e_{1,1} \otimes e_{i-l,i} + e_{2,2} \otimes e_{i+l,i} \right] \right\}.$$

Performing a basis transformation which leaves $r(z)$ invariant, we obtain

$$L(z) = \sum_{i=0}^{n-1} \left\{ (w^{2ik} e_{1,2} + w^{-2ik} e_{2,1}) \otimes e_{i,i} + h(z) \left[e_{1,1} \otimes e_{i-l,i} + e_{2,2} \otimes e_{i+l,i} \right] \right\}.$$

Substituting $r(z)$ and $L(z)$ into Equation (2), we find only one constraint on $h(z)$, namely

$$h(x)y - h(y)x = 0,$$

which has the solution

$$h(z) = Cz,$$

for any $C \in \mathbb{C}$. We are free to rescale the parameter z without affecting the descendants given in the next section, so without loss of generality we can choose $C = 1$. We have therefore shown the following:

Proposition 3.1. *The L-operator given explicitly by*

$$L(z) = \sum_{i=0}^{n-1} \left\{ (w^{2ik} e_{1,2} + w^{-2ik} e_{2,1}) \otimes e_{i,i} + z \left[e_{1,1} \otimes e_{i-l,i} + e_{2,2} \otimes e_{i+l,i} \right] \right\}, \qquad (6)$$

and the $r(z)$ given in Equation (4) together satisfy Equation (2).

It is important to comment that, up to a basis transformation, the L-operator (6) is a particular limit of the general L-operator discussed in [19] in relation to the chiral Potts model. However, the associated $U_q(sl(2))$ structure with $q^3 = 1$ described in [19] is lost in this limit. Specifically, Equation (2.9) in [19] does not hold. Amongst the defining relations of the generalised $U_q(sl(2))$ algebra this means, in particular, that

$$[e, f] = 0$$

where e and f are the raising and lowering generators respectively. The above relation is indicative of the fact that the $U_q(sl(2))$ structure degenerates in this limit. Instead we find that the symmetry algebra of $D(D_n)$ emerges when the L-operator is given by Equation (6).

3.2 Construction of the descendants

Here we put forth a predicted form of a descendant, $R(z)$, and determine the constraints on this form. We then impose additional constraints enforcing that the descendant inherits certain properties from representations of the canonical element from $D(D_n)$. For convenience we shall use an alternate form of Equation (3):

$$\check{R}_{23}(xy^{-1})L_{13}(x)L_{12}(y) = L_{13}(y)L_{12}(x)\check{R}_{23}(xy^{-1}), \tag{7}$$

where

$$R(z) = P\check{R}(z)$$

and $L(z)$ is given by Equation (6). As the descendants $\check{R}(z)$ commute with the action of the coproduct, we assume they are of the form

$$\check{R}(z) = \sum_{\alpha\in S} f_\alpha(z)p^\alpha, \tag{8}$$

where p^α are the projection operators previously calculated, $f_\alpha(z)$ are continuous functions and S is the set of ordered pairs which correspond to non-zero projection operators. For convenience we rescale our projection operators, henceforth using

$$\tilde{p}^\alpha = \sum_{i,j=0}^{n-1}[w^{2jb}e_{i+a+j,i+a}\otimes e_{i+j,i} + w^{-2bj}e_{i-a+j,i-a}\otimes e_{i+j,i}], \quad \alpha=(a,b)\in S.$$

Applying $\tilde{p}^\alpha_{23}$ and $\tilde{p}^\beta_{23}$ to the left- and right-hand sides respectively of Equation (7), we find

$$f_\alpha(xy^{-1})\tilde{p}^\alpha_{23}L_{13}(x)L_{12}(y)\tilde{p}^\beta_{23} = f_\beta(xy^{-1})\tilde{p}^\alpha_{23}L_{13}(y)L_{12}(x)\tilde{p}^\beta_{23}. \tag{9}$$

We calculate that

$$L_{13}(x)L_{12}(y) = e_{11}\otimes A(x,y) + e_{12}\otimes B(x,y) + e_{21}\otimes C(x,y) + e_{22}\otimes D(x,y),$$

where

$$\begin{aligned}
A(x,y) &= \sum_{i,j=0}^{n-1} w^{2(j-i)k}e_{i,i}\otimes e_{j,j} + xy\ e_{i-l,i}\otimes e_{j-l,j},\\
B(x,y) &= \sum_{i,j=0}^{n-1} xw^{2ik}e_{i,i}\otimes e_{j-l,j} + yw^{2jk}e_{i+l,i}\otimes e_{j,j},\\
C(x,y) &= \sum_{i,j=0}^{n-1} yw^{-2jk}e_{i-l,i}\otimes e_{j,j} + xw^{-2ik}e_{i,i}\otimes e_{j+l,j},\\
D(x,y) &= \sum_{i,j=0}^{n-1} w^{2(i-j)k}e_{i,i}\otimes e_{j,j} + xy\ e_{i+l,i}\otimes e_{j+l,j}.
\end{aligned} \tag{10}$$

Substituting these functions into Equation (9), we find the following constraints on $f_\alpha(xy^{-1})$:

$$\begin{aligned} f_\alpha(xy^{-1})\tilde{p}^\alpha A(x,y)\tilde{p}^\beta &= f_\beta(xy^{-1})\tilde{p}^\alpha A(x,y)\tilde{p}^\beta, & (11)\\ f_\alpha(xy^{-1})\tilde{p}^\alpha B(x,y)\tilde{p}^\beta &= f_\beta(xy^{-1})\tilde{p}^\alpha B(y,x)\tilde{p}^\beta, & (12)\\ f_\alpha(xy^{-1})\tilde{p}^\alpha C(x,y)\tilde{p}^\beta &= f_\beta(xy^{-1})\tilde{p}^\alpha C(y,x)\tilde{p}^\beta, & (13)\\ f_\alpha(xy^{-1})\tilde{p}^\alpha D(x,y)\tilde{p}^\beta &= f_\beta(xy^{-1})\tilde{p}^\alpha D(x,y)\tilde{p}^\beta. & (14) \end{aligned}$$

Note that Equations (11,14) are automatically satisfied.

The above relations provide a tensor product graph [16, 17, 18]. Assigning each irrep to a vertex of a graph, we connect the vertices labelled α and β by an edge if

$$\tilde{p}^\alpha \chi(x,y)\tilde{p}^\beta \neq 0$$

for either $\chi = B$ or $\chi = C$. The tensor product graph for the case $l = k = 1$ is depicted in Figure 1. An edge connecting vertices α and β signifies that the functions $f_\alpha(x)$ and $f_\beta(x)$ are constrained by Equations (12,13).

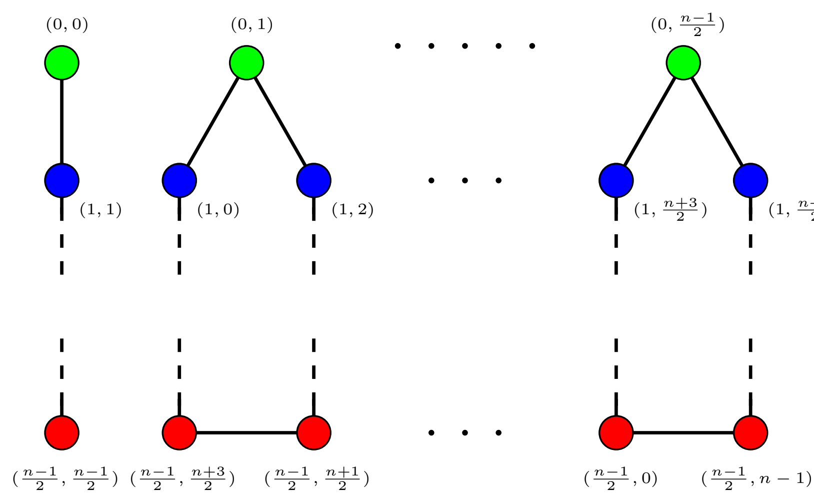

Figure 1: Tensor product graph for odd n when $l = k = 1$.

We now give a series of propositions and lemmas which result in a solution to Equation (7).

Proposition 3.2. *For $\check{R}(z)$ defined by Equation (8) to be a solution of Equation (7), the following four constraints must be satisfied:*

$$\begin{aligned} 0 &= \bar{\delta}^d_{k+b}\bar{\delta}^c_{a+l}\left[f_{(a,b)}(z)(zw^{2((-l-a)k-bl)}+1) - f_{(c,d)}(z)(w^{2((-l-a)k-bl)}+z)\right],\\ 0 &= \bar{\delta}^d_{k-b}\bar{\delta}^{l-a}_c\left[f_{(a,b)}(z)(zw^{2((-l+a)k+bl)}+1) - f_{(c,d)}(z)(w^{2((-l+a)k+bl)}+z)\right],\\ 0 &= \bar{\delta}^{-d}_{k+b}\bar{\delta}^{l+a}_{-c}\left[f_{(a,b)}(z)(zw^{2((-l-a)k-bl)}+1) - f_{(c,d)}(z)(w^{2((-l-a)k-bl)}+z)\right],\\ 0 &= \bar{\delta}^{-d}_{k-b}\bar{\delta}^{a-l}_c\left[f_{(a,b)}(z)(zw^{2((-l+a)k+bl)}+1) - f_{(c,d)}(z)(w^{2((-l+a)k+bl)}+z)\right], \end{aligned}$$

where the pairs (a,b) and (c,d) belong to S.

Proof. We consider constraint Equation (12). Given the operators

$$B(x,y)=\sum_{i,j=0}^{n-1} xw^{2ik}e_{i,i}\otimes e_{j-l,j}+yw^{2jk}e_{i+l,i}\otimes e_{j,j},$$

and

$$\tilde{p}^{\alpha}=\sum_{i,j=0}^{n-1}[w^{2jb}e_{i+a+j,i+a}\otimes e_{i+j,i}+w^{-2bj}e_{i-a+j,i-a}\otimes e_{i+j,i}],$$

we first calculate

$$\begin{aligned}
&\tilde{p}^{(a,b)}B(x,y)\\
&\quad=\sum_{i,j,s,t=0}^{n-1}[w^{-2bj}e_{i-a+j,i-a}\otimes e_{i+j,i}+w^{2bj}e_{i+a+j,i+a}\otimes e_{i+j,i}][xw^{2sk}e_{s,s}\otimes e_{t-l,t}+yw^{2tk}e_{s+l,s}\otimes e_{t,t}]\\
&\quad=\sum_{j,t=0}^{n-1}[xw^{2((t-l-a)k-bj)}e_{t-l-a+j,t-l-a}\otimes e_{t-l+j,t}+xw^{2((t-l+a)k+bj)}e_{t-l+a+j,t-l+a}\otimes e_{t-l+j,t}\\
&\qquad\quad+yw^{2(tk-bj)}e_{t-a+j,t-a-l}\otimes e_{t+j,t}+yw^{2(tk+bj)}e_{t+a+j,t+a-l}\otimes e_{t+j,t}].
\end{aligned}$$

This leads to

$$\begin{aligned}
&\tilde{p}^{(a,b)}B(x,y)\tilde{p}^{(c,d)}\\
=\;&\sum_{j,t=0}^{n-1}[xw^{2((t-l-a)k-bj)}e_{t-l-a+j,t-l-a}\otimes e_{t-l+j,t}+xw^{2((t-l+a)k+bj)}e_{t-l+a+j,t-l+a}\otimes e_{t-l+j,t}\\
&\quad+yw^{2(tk-bj)}e_{t-a+j,t-a-l}\otimes e_{t+j,t}+yw^{2(tk+bj)}e_{t+a+j,t+a-l}\otimes e_{t+j,t}]\tilde{p}^{(c,d)}\\
=\;&\sum_{j,t,v=0}^{n-1}[xw^{2((t-l-a)k-bj-dv)}\bar{\delta}^{c}_{a+l}e_{t-l-a+j,t-c-v}\otimes e_{t-l+j,t-v}\\
&\quad+xw^{2((t-l+a)k+bj-dv)}\bar{\delta}^{l-a}_{c}e_{t-l+a+j,t-c-v}\otimes e_{t-l+j,t-v}\\
&\quad+yw^{2(tk-bj-dv)}\bar{\delta}^{a+l}_{c}e_{t-a+j,t-c-v}\otimes e_{t+j,t-v}+yw^{2(tk+bj-dv)}\bar{\delta}^{l-a}_{c}e_{t+a+j,t-c-v}\otimes e_{t+j,t-v}\\
&\quad+xw^{2((t-l-a)k-bj+dv)}\bar{\delta}^{l+a}_{-c}e_{t-l-a+j,t+c-v}\otimes e_{t-l+j,t-v}\\
&\quad+xw^{2((t-l+a)k+bj+dv)}\bar{\delta}^{a-l}_{c}e_{t-l+a+j,t+c-v}\otimes e_{t-l+j,t-v}\\
&\quad+yw^{2(tk-bj+dv)}\bar{\delta}^{a+l}_{-c}e_{t-a+j,t+c-v}\otimes e_{t+j,t-v}+yw^{2(tk+bj+dv)}\bar{\delta}^{a-l}_{c}e_{t+a+j,t+c-v}\otimes e_{t+j,t-v}]\\
=\;&n\sum_{j,v=0}^{n-1}\Big\{w^{2(dv-bj)}\left[xw^{2((-l-a)k-bl)}+y\right]\bar{\delta}^{d}_{k+b}\bar{\delta}^{c}_{a+l}e_{j-a,v-c}\otimes e_{j,v}\\
&\quad+w^{2(dv+bj)}\left[xw^{2((-l+a)k+bl)}+y\right]\bar{\delta}^{d}_{k-b}\bar{\delta}^{l-a}_{c}e_{j+a,v-c}\otimes e_{j,v}\\
&\quad+w^{-(2dv+bj)}\left[xw^{2((-l-a)k-bl)}+y\right]\bar{\delta}^{-d}_{k+b}\bar{\delta}^{l+a}_{-c}e_{j-a,v+c}\otimes e_{j,v}\\
&\quad+w^{-(2dv-bj)}\left[xw^{2((-l+a)k+bl)}+y\right]\bar{\delta}^{-d}_{k-b}\bar{\delta}^{a-l}_{c}e_{j+a,v+c}\otimes e_{j,v}\Big\}.
\end{aligned}$$

We now substitute the above result into Equation (12),

$$p^{\alpha}[f_{\alpha}(xy^{-1})B(x,y) - f_{\beta}(xy^{-1})B(y,x)]p^{\beta} = 0,$$

and finally arrive at the desired four constraints.

Very similar calculations for Equation (13) leads to the same four constraint equations.

□

Lemma 3.3. Let w^2 be a primitive nth root of unity and l, k be integers such that $gcd(l,n) = gcd(k,n) = 1$. Then

$$\prod_{j=1}^{a}\left(\frac{z + w^{2l((2j-1)k+b)}}{1 + zw^{2l((2j-1)k+b)}}\right) = \prod_{j=1}^{\overline{a}}\left(\frac{z + w^{2l((2j-1)k+b)}}{1 + zw^{2l((2j-1)k+b)}}\right),$$

for all $a \in \mathbb{N}$, $b \in \mathbb{Z}$.

The proof is omitted. Henceforth we choose l, k satisfying $gcd(l,n) = gcd(k,n) = 1$ so that we may use the above lemma.

Proposition 3.4. *Let $S' = \{(a,b) | a \geq 0, b \in \mathbb{Z}\}$, and consider a set of functions $\{f_{(a,b)} | (a,b) \in S'\}$. If the functions satisfy the relations*

$$\begin{aligned}
f_{(a+l,b+k)}(z) &= \left(\frac{z + w^{2((a+l)k+bl)}}{1 + zw^{2((a+l)k+bl)}}\right) f_{(a,b)}(z), \\
f_{(0,b)}(z) &= f_{(0,-b)}(z), \\
f_{(a,b)}(z) &= f_{(a,b+n)}(z), \\
f_{(a,b)}(z) &= f_{(a+n,b)}(z), \\
f_{(\overline{a},b)}(z) &= f_{(n-\overline{a},-b)}(z), \quad (a,b) \in S',
\end{aligned}$$

then the functions also satisfy the constraints given in Proposition 3.2 for all $(a,b) \in S$. Moreover, every set of functions satisfying the conditions of Proposition 3.2 can be extended in a unique way to a set of functions defined on S' satisfying the above conditions. Hence the two sets of constraints are equivalent.

The proof is straightforward and omitted.

Lemma 3.5. The set of functions

$$f_{(a,b)}(z) = \prod_{j=1}^{\overline{al^{-1}}}\left(\frac{z + w^{2l((2j-1)k+b-akl^{-1})}}{1 + zw^{2l((2j-1)k+b-akl^{-1})}}\right) f_{(0,b-akl^{-1})}(z)$$

satisfies the conditions in Proposition 3.4 given

$$f_{(0,b)}(z) = f_{(0,-b)}(z) = f_{(0,b+n)},$$

for $b \in \mathbb{Z}$.

The proof follows from Lemma 3.3.

Substituting the functions given in Lemma 3.5 and their associated projection operators into our form for $\check{R}(z)$ gives the operator

$$\check{R}(z) = \frac{1}{n} \sum_{i,j,a,b=0}^{n-1} w^{2bj} \prod_{p=1}^{\overline{al^{-1}}} \left(\frac{z + w^{2l((2p-1)k+b-akl^{-1})}}{1 + zw^{2l((2p-1)k+b-akl^{-1})}} \right) f_{(0,b-akl^{-1})}(z) e_{i+a+j,i+a} \otimes e_{i+j,i},$$

which satisfies Equation (7). Note that in these calculations the only property that we have used is that w^2 is a primitive nth root of unity, irrespective of whether n is odd or even. This allows us to use these calculations later on without alteration.

For $\check{R}(z)$ to be a descendent, it must also satisfy the Yang–Baxter equation. For the moment we will set $l = k = 1$, and consider the more general case in Section 5.1. Then the functions become

$$f_{(a,a+b)}(z) = \prod_{j=1}^{a} \left(\frac{z + w^{2(2j-1+b)}}{1 + zw^{2(2j-1+b)}} \right) f_{(0,b)}(z). \tag{15}$$

This gives the operator

$$\begin{aligned} \check{R}(z) \;&=\; f_{(0,0)}(z) \left\{ \sum_{a=0}^{\frac{n-1}{2}} \prod_{j=1}^{a} \left(\frac{z + w^{2(2j-1)}}{1 + zw^{2(2j-1)}} \right) p^{(a,a)} \right\} \\ &+ \sum_{b=1}^{\frac{n-1}{2}} f_{(0,b)}(z) \left\{ p^{(0,b)} + \sum_{a=1}^{\frac{n}{2}-1} \left[\prod_{j=1}^{a} \left(\frac{z + w^{2(2j-1+b)}}{1 + zw^{2(2j-1+b)}} \right) p^{(a,\overline{a+b})} \right.\right. \\ &\left.\left. + \prod_{j=1}^{a} \left(\frac{z + w^{2(2j-1-b)}}{1 + zw^{2(2j-1-b)}} \right) p^{(a,\overline{a-b})} \right] \right\}, \end{aligned}$$

or equivalently

$$\check{R}(z) = \sum_{i,j,a=0}^{n-1} \left[\frac{1}{n} \sum_{b=0}^{n-1} w^{2bj} \prod_{p=1}^{a} \left(\frac{z + w^{2(2p-1+b-a)}}{1 + zw^{2(2p-1+b-a)}} \right) f_{(0,b-a)}(z) \right] e_{i+a+j,i+a} \otimes e_{i+j,i}. \tag{16}$$

In $\check{R}(z)$ there remain $\frac{n+1}{2}$ arbitrary functions, one for each disconnected component in the tensor product graph of Figure 1. This is in some contrast to the examples studied in [16, 17, 18] for which the tensor product graphs are connected. To ensure $R(z)$ inherits properties from the representation of the canonical element, we enforce additional conditions. Firstly, we impose that $R(z)$ is self-adjoint, which is equivalent to

$$\check{R}_{12}(z) = \check{R}^{\dagger}_{21}(z), \qquad z \in \mathbb{R},$$

where † is the adjoint operator.

Proposition 3.6. *The following three statements about the matrix $\check{R}(z)$ of Equation (16) and coefficient functions of Equation (15) are equivalent:*

(i) $\check{R}_{12}(z) = \check{R}^{\dagger}_{21}(z)$.

(ii) $(f_{(a,b)}(z))^* = f_{(a,-b)}(z), \quad \forall (a,b) \in S'$.

(iii) $(f_{(0,b)}(z))^* = f_{(0,b)}(z) = f_{(0,b+2c)}(z), \quad \forall z \in \mathbb{R}$ *and* $b, c \in \mathbb{Z}$.

Here * *denotes complex conjugation.*

Proof. Assume (i) holds. Looking at the general form of the projection operators we find that

$$(\tilde{p}^{(a,b)}_{21})^{\dagger} = (\tilde{p}^{(a,n-b)}_{12}).$$

Recalling the form of $\check{R}(z)$ given in Equation (8) and using the linear independence of the projection operators, this implies that

$$(f_{(a,b)}(z))^* = f_{(a,-b)}(z), \quad \forall (a,b) \in S'.$$

Conversely, (i) follows directly from (ii), and hence statements (i) and (ii) are equivalent.

Now assume (ii) holds. Equation (15) gives us

$$f_{(1,1+b)}(z) = \left(\frac{z + w^{2(b+1)}}{1 + zw^{2(b+1)}} \right) f_{(0,b)}(z),$$

and

$$f_{(1,-1-b)}(z) = \left(\frac{z + w^{-2(b+1)}}{1 + zw^{-2(b+1)}} \right) f_{(0,-b-2)}(z).$$

We see that

$$(f_{(1,-1-b)}(z))^* = \left(\frac{z + w^{2(b+1)}}{1 + zw^{2(b+1)}} \right) (f_{(0,-b-2)}(z))^* = \left(\frac{z + w^{2(b+1)}}{1 + zw^{2(b+1)}} \right) f_{(0,b+2)}(z),$$

which implies that

$$f_{(0,b)}(z) = f_{(0,b+2)}(z).$$

Combining this with the constraint of $f_{(0,b)}(z)$ we find

$$(f_{(0,b)}(z))^* = f_{(0,b)}(z) = f_{(0,b+2c)}(z),$$

for $b, c \in \mathbb{Z}$.

Conversely, suppose (iii) holds. It follows that

$$f_{(a,b)}(z) = \prod_{j=1}^{a} \left(\frac{z + w^{2(2j-1+b-a)}}{1 + zw^{2(2j-1+b-a)}} \right) f_{(0,b-a)}(z).$$

Hence

$$\begin{aligned}
(f_{(a,b)}(z))^* &= \left(\prod_{j=1}^{a}\left(\frac{z+w^{2(2j-1+b-a)}}{1+zw^{2(2j-1+b-a)}}\right)f_{(0,b-a)}(z)\right)^* \\
&= \prod_{j=1}^{a}\left(\frac{z+w^{-2(2j-1+b-a)}}{1+zw^{-2(2j-1+b-a)}}\right)\left(f_{(0,b-a)}(z)\right)^* \\
&= \prod_{j=1}^{a}\left(\frac{z+w^{2(2(-j)+1-b+a)}}{1+zw^{2(2(-j)+1-b+a)}}\right)f_{(0,0-b-a)}(z) \\
&= \prod_{j=1}^{a}\left(\frac{z+w^{2(2j-1-b-a)}}{1+zw^{2(2j-1-b-a)}}\right)f_{(0,-b-a)}(z) \\
&= f_{(a,-b)}(z).
\end{aligned}$$

This completes the proof.

□

As the second index of the function can be considered modulo n and n is odd, imposing self-adjointness implies that there is only one arbitrary function left. Without loss of generality we consider it to be $f_{(0,0)}(z)$. This can be seen as an overall scalar of our operator thus we are able to set it to a constant. Now we impose the limiting condition

$$\lim_{z\to 0} R(z) = \pm(\pi_n^{\pm}\otimes\pi_n^{\pm})\mathcal{R}. \tag{17}$$

This sets the scalar to be $f_{(0,0)}(z) = 1$, so the functions become

$$f_{(a,b)}(z) = \prod_{j=1}^{a}\left(\frac{z+w^{2(2j-1+b-a)}}{1+zw^{2(2j-1+b-a)}}\right).$$

Hence

$$\check{R}(z) = \sum_{b=0}^{\frac{n-1}{2}} p^{(0,b)} + \sum_{a=1}^{\frac{n-1}{2}}\sum_{b=0}^{n-1}\prod_{j=1}^{a}\left(\frac{z+w^{2(2j-1+b-a)}}{1+zw^{2(2j-1+b-a)}}\right)p^{(a,b)},$$

or equivalently

$$\check{R}(z) = \sum_{i,j,a=0}^{n-1}\left[\frac{1}{n}\sum_{b=0}^{n-1} w^{2bj}\prod_{p=1}^{a}\left(\frac{z+w^{2(2p-1+b-a)}}{1+zw^{2(2p-1+b-a)}}\right)\right] e_{i+a+j,i+a}\otimes e_{i+j,i}.$$

To simplify this we define the functions

$$g_{(a,j)}(z) = \frac{1}{n}\sum_{b=0}^{n-1} w^{2bj} f_{(a,b)}(z) = \frac{1}{n}\sum_{b=0}^{n-1} w^{2bj}\prod_{p=1}^{a}\left(\frac{z+w^{2(2p-1+b-a)}}{1+zw^{2(2p-1+b-a)}}\right),$$

where $a \in \mathbb{N}$ and $j \in \mathbb{Z}$. In terms of these new functions $g_{(a,j)}(z)$, our operator becomes

$$\check{R}(z) = \sum_{i,j,a=0}^{n-1} g_{(a,j)}(z) e_{i+a+j,i+a} \otimes e_{i+j,i},$$

or

$$R(z) = \sum_{i,j,a=0}^{n-1} g_{(a,j)}(z) e_{i+j,i+a} \otimes e_{i+a+j,i}. \tag{18}$$

Remark. The operator $R(z)$ of Equation (18) satisfies the following properties:

(i) $R(z)^* = R(z),\ \forall z \in \mathbb{R}$,

(ii) $R^t(z) = R(z),\ \forall z \in \mathbb{C}$,

(iii) $R^{-1}(z) = R(z),\ \forall z \in \mathbb{C}$,

(iv) $R_{12}(z)R_{21}(z^{-1}) = I \otimes I,\ \forall z \in \mathbb{C}$,

(v) $\lim_{z\to 0} R(z) = \pm(\pi_n^{\pm} \otimes \pi_n^{\pm})\mathcal{R}$,

(vi) $\lim_{z\to 1} R(z) = P$.

Proposition 3.7. *The operator $R(z)$ of Equation (18) is a descendant if and only if*

$$\sum_{k=0}^{n-1} g_{(a,k-d)}(x) g_{(k,a-b)}(xy) g_{(b,c-k)}(y) = \sum_{k=0}^{n-1} g_{(c,k-b)}(x) g_{(k,c-d)}(xy) g_{(d,a-k)}(y),$$

for $0 \leq a, b, c, d \leq n-1$.

This follows directly from the Yang–Baxter equation.

The matrix $R(z)$ of Equation (18) agrees with the R-matrix obtained in [28] when $n = 3$. It has also been verified computationally for all odd $n \leq 17$ that the functions $g(z)$ satisfy the equation given in Proposition 3.7 above, and thus that $R(z)$ is a solution to the Yang–Baxter equation. The computations were performed with z treated as an arbitrary complex number and w as an arbitrary primitive nth root of unity.

Conjecture 3.8. The matrix $R(z)$ given in Equation (18) is a descendant of the zero-field six-vertex model with $D(D_n)$ symmetry for all odd n.

4 Descendants associated with $D(D_{2m})$

In this section we construct a family of solutions of the Yang-Baxter equation using $D(D_{2m})$, treating the cases where m is even and odd separately. It will be shown that although in both cases the representations have the same general form, the case where m is odd has more in common with the solutions from the previous section associated with $D(D_n)$ where n is odd.

Throughout this entire section, we consider w to be a primitive $2m$th root of unity and we use the $r(z)$ stated previously and associated with $\pi_2 = \pi_2^{(k,l)}$. The L-operator arising from the cases of m odd or even can be considered at once, since the approach is similar to that of Section 3.1.

4.1 Construction of the L-operator

In this subsection we construct an operator $L(z) \in \mathrm{End}(V_2 \otimes V_m)$ using a similar approach as before in Section 3.1. Again, we assume that the L-operator is of the form

$$L(z) = (\pi_2 \otimes \pi_m)[R + h(z)(R^T)^{-1}],$$

where π_m is either $\pi_{m,\tau}^{(0,b)}$ or $\pi_{m,\sigma\tau}^{(0,b)}$ with $b \in \{0,1\}$. Applying this representation and an appropriate basis transformation on the two-dimensional space we find

$$L(z) = \sum_{i=0}^{m-1} \left\{ (w^{2ik} e_{1,2} + w^{-2ik} e_{2,1}) \otimes e_{i,i} + h(z) \left[e_{1,1} \otimes e_{i-l,i} + e_{2,2} \otimes e_{i+l,i} \right] \right\}.$$

We note that the basis transformation is of the same general form used in Section 3.1, as are the operators $r(z)$ and $L(z)$. Thus it follows that our L-operator is of the form

$$L(z) = \sum_{i=0}^{m-1} \left\{ (w^{2ik} e_{1,2} + w^{-2ik} e_{2,1}) \otimes e_{i,i} + z \left[e_{1,1} \otimes e_{i-l,i} + e_{2,2} \otimes e_{i+l,i} \right] \right\}.$$

4.2 Construction of the descendants when m is even

Here we consider the case of $D(D_{2m})$ where m is even. We use $r(z)$ associated with $\pi_2^{(l,k)}$ with w a primitive $2m$th root of unity. As before, we look for descendants of the form

$$PR(z) = \check{R}(z) = \sum_{\alpha \in S} f_\alpha(z) p^\alpha,$$

where p^α are the projection operators previously calculated, $f_\alpha(z)$ are continuous functions and S is the set of ordered pairs which correspond to non-zero projection operators. We require that $\check{R}(z)$ satisfies Equation (7). As before, we use rescaled projection operators:

$$\tilde{p}^\alpha = \sum_{i,j=1}^{m} [w^{2jb} e_{i+a+j,i+a} \otimes e_{i+j,i} + w^{-2bj} e_{i-a+j,i-a} \otimes e_{i+j,i}], \quad \alpha = (a,b) \in S.$$

As these operators and the L-operator are equivalent to those found earlier, we can use the previous calculations to arrive at the tensor product graph shown in Figure 2.

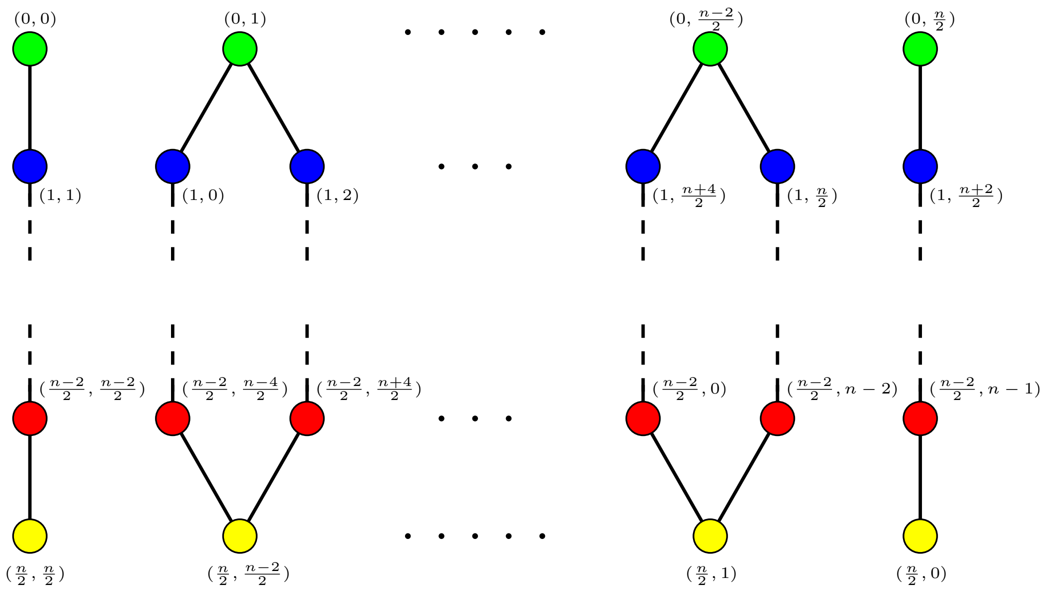

Figure 2: Tensor product graph for even n when $l = k = 1$.

We again define the set

$$S' = \{(a, b) | a \in \mathbb{N}, b \in \mathbb{Z}\},$$

and extend the functions $f_\alpha(z)$ to that set in a way analogous to that of Proposition 3.4. We restrict ourselves to k, l satisfying $\gcd(l, m) = \gcd(k, m) = 1$. Following the calculations of the previous section, we determine that if $R(z)$ is a descendant then the functions $f_{(a,b)}(z)$ must satisfy

$$\begin{aligned} f_{(a,akl^{-1}+b)}(z) &= \prod_{j=1}^{\overline{al^{-1}}} \left(\frac{z + w^{2l((2j-1)k+b)}}{1 + zw^{2l((2j-1)k+b)}} \right) f_{(0,b)}(z), \\ f_{(0,b)}(z) &= f_{(0,b+m)}(z), \\ f_{(0,b)}(z) &= f_{(0,-b)}(z), \qquad \forall (a, b) \in S'. \end{aligned}$$

We now choose to set $l = k = 1$; this gives the functions

$$f_{(a,a+b)}(z) = \prod_{j=1}^{a} \left(\frac{z + w^{2(2j-1+b)}}{1 + zw^{2(2j-1+b)}} \right) f_{(0,b)}(z).$$

This produces the operator

$$
\begin{aligned}
\check{R}(z) \;=\;& f_{(0,0)}(z)\left\{\sum_{a=0}^{\frac{m}{2}}\prod_{j=1}^{a}\left(\frac{z+w^{2(2j-1)}}{1+zw^{2(2j-1)}}\right)p^{(a,a)}\right\}\\
&+f_{(0,\frac{m}{2})}(z)\left\{\sum_{a=0}^{\frac{m}{2}}\prod_{j=1}^{a}\left(\frac{z+w^{2(2j-1+\frac{m}{2})}}{1+zw^{2(2j-1+\frac{m}{2})}}\right)p^{(a,\overline{\frac{m}{2}+a})}\right\}\\
&+\sum_{b=1}^{\frac{m}{2}-1} f_{(0,b)}(z)\left\{p^{(0,b)}+\sum_{a=1}^{\frac{m}{2}-1}\left[\prod_{j=1}^{a}\left(\frac{z+w^{2(2j-1+b)}}{1+zw^{2(2j-1+b)}}\right)p^{(a,\overline{a+b})}\right.\right.\\
&\left.\left.+\prod_{j=1}^{a}\left(\frac{z+w^{2(2j-1-b)}}{1+zw^{2(2j-1-b)}}\right)p^{(a,\overline{a-b})}\right]+\prod_{j=1}^{a}\left(\frac{z+w^{2(2j-1+\frac{m}{2})}}{1+zw^{2(2j-1+\frac{m}{2})}}\right)p^{(a,\overline{a+\frac{m}{2}})}\right\}
\end{aligned}
$$

or equivalently

$$
R(z)=\sum_{i,j,a=0}^{m-1}\left[\frac{1}{n}\sum_{b=0}^{m-1}w^{2bj}\prod_{p=1}^{a}\left(\frac{z+w^{2(2p-1+b-a)}}{1+zw^{2(2p-1+b-a)}}\right)f_{(0,b-a)}(z)\right]e_{i+j,i+a}\otimes e_{i+a+j,i}.
$$

Here we again enforce that our operator is self-adjoint. We are able to use the previous calculations and recall that

$$
(f_{(0,b)}(z))^{*}=f_{(0,b)}(z)=f_{(0,b+2c)}(z),
$$

$\forall z\in\mathbb{R}$ and $b,c\in\mathbb{N}$. The second index of the function can be considered modulo m and as m is even the functions are partitioned into two sets. We see that we have only two functions in which we have any freedom left; without loss of generality we consider them to be $f_{(0,0)}(z)$ and $f_{(0,1)}(z)$. As there are two functions we find that we cannot consider them an overall scalar, so we need to impose additional conditions. As the m-dimensional representations of the canonical element are self-adjoint, we enforce that $R(z)$ is unitary . From this it follows that

$$
f_{(0,0)}(z)=\pm1 \text{ and } f_{(0,1)}(z)=\pm1.
$$

Imposing the limiting condition given in Equation (17) sets

$$
f_{(0,0)}(z)=f_{(0,1)}(z)=1.
$$

This yields the operator

$$
\check{R}(z)=\sum_{i,j,a=0}^{m-1}\left[\frac{1}{m}\sum_{b=0}^{m-1}w^{2bj}\prod_{p=1}^{a}\left(\frac{z+w^{2(2p-1+b-a)}}{1+zw^{2(2p-1+b-a)}}\right)\right]e_{i+a+j,i+a}\otimes e_{i+j,i}. \tag{19}
$$

To simplify this we define the functions

$$g_{(a,j)}(z) = \frac{1}{m}\sum_{b=0}^{m-1} w^{2bj} f_{(a,b)}(z) = \frac{1}{m}\sum_{b=0}^{m-1} w^{2bj} \prod_{k=1}^{a} \left(\frac{1 + zw^{2(a-b+1-2k)}}{z + w^{2(a-b+1-2k)}} \right),$$

where $a \in \mathbb{N}$ and $j \in \mathbb{Z}$. With the use of these new functions our operator becomes

$$R(z) = \sum_{i,j,a=0}^{m-1} g_{(a,j)}(z) e_{i+j,i+a} \otimes e_{i+a+j,i}. \tag{20}$$

Remark. The operator $R(z)$ of Equation (20) satisfies the following properties:

(i) $R(z)^* = R(z), \ \forall z \in \mathbb{R}$,

(ii) $R^t(z) = R(z), \ \forall z \in \mathbb{C}$,

(iii) $R^{-1}(z) = R(z), \ \forall z \in \mathbb{C}$,

(iv) $R_{12}(z)R_{21}(z^{-1}) = I \otimes I, \ \forall z \in \mathbb{C}$,

(iv) $\lim_{z \to 0} R(z) = \pm(\pi_n^{\pm} \otimes \pi_n^{\pm})\mathcal{R}$,

(v) $\lim_{z \to 1} R(z) \neq P$.

Note that the last property shows that $R(z)$ does not satisfy regularity, unlike the $R(z)$ constructed from $D(D_n)$ when n is odd. We do, however, have the following analogue of Proposition 3.7:

Proposition 4.1. *The operator of Equation (20) is a descendant if and only if the following constraint is satisfied:*

$$\sum_{k=0}^{m-1} g_{(a,k-d)}(x) g_{(k,a-b)}(xy) g_{(b,c-k)}(y) = \sum_{k=0}^{m-1} g_{(c,k-b)}(x) g_{(k,c-d)}(xy) g_{(d,a-k)}(y),$$

for $0 \leq a, b, c, d \leq m-1$.

It has been computationally verified that the functions $g(z)$ satisfy the conditions in Proposition 4.1 above for all even $m \leq 16$, and hence that $R(z)$ is a solution to the Yang–Baxter equation.

Conjecture 4.2. The matrix $R(z)$ given by Equation (20) is a descendent of the zero-field six-vertex model with $D(D_{2m})$ symmetry for all even m.

4.3 Construction of the descendants when m is odd

For completeness, we include the descendants associated with $D(D_{2m})$ for odd m. In our construction of the descendant of $r(z)$ we use a linear combination of projection operators found in Subsection 2.3.2. Using these projections, for a self-adjoint descendant of $r(z)$ which limits to a representation of the canonical element, we obtain the operator

$$R(z) = \sum_{i,j,a=0}^{m-1} \left[\frac{1}{m} \sum_{b=0}^{m-1} w^{bj} \prod_{p=1}^{a} \left(\frac{z + w^{2(2p-1+b-a)}}{1 + zw^{2(2p-1+b-a)}} \right) \right] e_{i+j,i+a} \otimes e_{i+a+j,i}.$$

This is equivalent to the operator $R(z)$ found using $D(D_m)$ where m is odd. Hence it obeys the properties stated in the remark in Section 3.2. Moreover, $R(z)$ is a descendant of the zero-field six-vertex model with $D(D_{2m})$ symmetry for odd $m \leq 17$ and conjectured to be a descendant for all odd $m \geq 3$.

5 Generalised descendants

In this section we investigate some of the choices we made in order to obtain descendants. Specifically, we investigate the effects of choosing a different initial 2-dimensional irrep and of imposing fewer conditions on the descendant.

5.1 Dependency on choice of irreducible representations

In earlier sections we constructed descendants by starting from the R-matrix associated with the two-dimensional irrep $\pi_2^{(1,1)}$. Here we explore the consequences of starting with a different two-dimensional irrep, namely any $\pi_2^{(l,k)}$ where $\gcd(l,n) = \gcd(k,n) = 1$. We consider n-dimensional descendants for any $n \geq 3$, with w a primitive nth root of unity. We recall that

$$r(z) = \begin{pmatrix} w^{kl}z^{-1} - w^{-kl}z & 0 & 0 & 0 \\ 0 & z^{-1} - z & w^{kl} - w^{-kl} & 0 \\ 0 & w^{kl} - w^{-kl} & z^{-1} - z & 0 \\ 0 & 0 & 0 & w^{kl}z^{-1} - w^{-kl}z \end{pmatrix}$$

and

$$L(z) = \sum_{i=0}^{n-1} \left\{ (w^{ik}e_{1,2} + w^{-ik}e_{2,1}) \otimes e_{i,i} + z\left[e_{1,1} \otimes e_{i-l,i} + e_{2,2} \otimes e_{i+l,i}\right] \right\}$$

where $0 \leq l \leq \frac{n}{2}$ and $0 \leq k \leq n-1$, although here w differs slightly from that used in Section 2.3. We now repeat our earlier calculations for our more general l and k, obtaining

$$\check{R}(z) = \frac{1}{n} \sum_{i,j,a,b=0}^{n-1} w^{bjk} \prod_{p=1}^{\overline{al^{-1}}} \left(\frac{z + w^{lk(2p-1+b-al^{-1})}}{1 + zw^{lk(2p-1+b-al^{-1})}} \right) f_{(0,bk-akl^{-1})}(z) e_{i+a+j,i+a} \otimes e_{i+j,i}.$$

We consider the change of basis on the n-dimensional space which yields

$$e_{i,j} \to e_{il^{-1},jl^{-1}}, \qquad i \in \mathbb{Z}.$$

Under this change of basis we find that our L-operator becomes

$$L(z) = \sum_{i=0}^{n-1} \left\{ (w^{ilk} e_{1,2} + w^{-ilk} e_{2,1}) \otimes e_{i,i} + z \left[e_{1,1} \otimes e_{i-1,i} + e_{2,2} \otimes e_{i+1,i} \right] \right\},$$

while $\check{R}(z)$ becomes

$$\check{R}(z) = \frac{1}{n} \sum_{i,j,a,b=0}^{n-1} w^{bjlk} \prod_{p=1}^{a} \left(\frac{z + w^{lk(2p-1+b-a)}}{1 + zw^{lk(2p-1+b-a)}} \right) f_{(0,(b-a)k)}(z) e_{i+a+j,i+a} \otimes e_{i+j,i}.$$

From this we see that these different choices of initial two-dimensional representation provide equivalent descendants. The different choices yield different basis transformations, a permutation on the arbitrary functions and a change of the root of unity, which must remain primitive. Thus any 2-dimensional irrep satisfying $\gcd(l, n) = \gcd(k, n) = 1$ results in equivalent descendants. This is unsurprising as $\check{R}(z)$ is real whenever $z \in \mathbb{R}$, and choosing different l, k effectively just changes the root of unity being used.

5.2 Descendants with an extra parameter associated with $D(D_{2m})$ when m is even.

Here we return to $D(D_{2m})$ where m is even and we find more descendants by imposing fewer constraints. We use the general form

$$\check{R}(z) = \sum_{i,j,a=0}^{m-1} \left[\frac{1}{m} \sum_{b=0}^{m-1} w^{bj} \prod_{p=1}^{a} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right) f_{(0,b-a)}(z) \right] e_{i+a+j,i+a} \otimes e_{i+j,i}$$

where w is now a primitive mth root of unity. We recall that previously we imposed that $R(z)$ is self-adjoint, unitary and obeys a limiting condition. If we ignore the limiting condition but enforce the other two conditions we find that

$$f_{(0,2b)}(z) = f_{(0,0)}(z) = \pm 1 \qquad \text{and} \qquad f_{(0,2b+1)}(z) = f_{(0,1)}(z) = \pm 1,$$

for all $b \in \mathbb{Z}$. Without loss of generality we can set

$$f_{(0,0)}(z) = 1.$$

This gives us two possibilities; the first choice is $f_{(0,1)}(z) = 1$, which yields

$$\check{R}^{+}(z) = \sum_{i,j,a=0}^{m-1} \left[\frac{1}{m} \sum_{b=0}^{m-1} w^{bj} \prod_{p=1}^{a} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right) \right] e_{i+a+j,i+a} \otimes e_{i+j,i}. \qquad (21)$$

This operator corresponds to the conjectured descendant given in Equation 19. The second option is $f_{(0,1)}(z) = -1$, which gives the operator

$$\check{R}^{-}(z) = \sum_{i,j,a=0}^{m-1} \left[\frac{1}{m} \sum_{b=0}^{m-1} (-1)^{a+b} w^{bj} \prod_{p=1}^{a} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right) \right] e_{i+a+j,i+a} \otimes e_{i+j,i}. \quad (22)$$

The matrices $\check{R}^{+}(z)$ and $\check{R}^{-}(z)$ are different; nonetheless they share many properties, including that they square to the identity and obey the unitarity condition. Moreover, we have the following proposition:

Proposition 5.1. *The descendant $\check{R}^{-}(z)$ of Equation (22) satisfies the Yang–Baxter equation if and only if $\check{R}^{+}(z)$ given by Equation (21) does.*

Proof. We use the identity

$$\prod_{p=c+1}^{c+\frac{m}{2}} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right) = (-1)^{a+b+\frac{m}{2}}.$$

This implies that

$$\check{R}^{-}(z) = (-1)^{\frac{m}{2}} \frac{1}{m} \sum_{i,j,a,b=0}^{m-1} \left[w^{bj} \prod_{p=1}^{a+\frac{m}{2}} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right) \right] e_{i+a+j,i+a} \otimes e_{i+j,i}.$$

We now consider a basis transformation. Given any $\lambda \in \mathbb{C}$, there exists a basis transformation which yields

$$e_{i,j} \to \lambda^{\bar{i}-\bar{j}} e_{i,j}.$$

We choose λ such that our operator becomes (after scaling)

$$\check{R}^{-}(z) = \frac{1}{m} \sum_{i,j,a,b=0}^{m-1} \left[w^{(b+\frac{m}{2})j} \prod_{p=1}^{a+\frac{m}{2}} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right) \right] e_{i+a+j,i+a} \otimes e_{i+j,i}.$$

We define the functions

$$g_{(a,j)}(z) = \sum_{b=0}^{m-1} w^{bj} \prod_{p=1}^{a} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right),$$

which allow to write the operators

$$\check{R}^{+}(z) = \sum_{i,j,a=0}^{m-1} g_{(a,j)}(z) e_{i+a+j,i+a} \otimes e_{i+j,i}$$

and

$$\check{R}^{-}(z) = \sum_{i,j,a=0}^{m-1} g_{(a+\frac{m}{2},j)}(z) e_{i+a+j,i+a} \otimes e_{i+j,i}.$$

Hence $\check{R}^-(z)$ differs from $\check{R}^+(z)$ only by a basis transformation and a permutation of the entries. We calculated previously that $\check{R}^+(z)$ satisfies the Yang–Baxter equation if and only if

$$\sum_{k=0}^{m-1} \left[g_{(a,k-d)}(x) g_{(k,a-b)}(xy) g_{(b,c-k)}(y) - g_{(c,k-b)}(x) g_{(k,c-d)}(xy) g_{(d,a-k)}(y) \right] = 0,$$

$0 \leq a, b, c, d \leq m-1$. Similarly we have that $\check{R}^-(z)$ satisfies the Yang–Baxter equation if and only if

$$\sum_{k=0}^{m-1} \left[g_{(a+\frac{m}{2},k-d)}(x) g_{(k+\frac{m}{2},a-b)}(xy) g_{(b+\frac{m}{2},c-k)}(y) - g_{(c+\frac{m}{2},k-b)}(x) g_{(k+\frac{m}{2},c-d)}(xy) g_{(d+\frac{m}{2},a-k)}(y) \right] = 0,$$

$0 \leq a, b, c, d \leq m-1$. As we can consider each of the indices of the functions modulo m we find that the two above constraints are equivalent, hence the result.

$\square$

Thus we have found another family of conjectured descendants. We explain the existence of $\check{R}^-(z)$ by considering the values of functions associated with the irreps. To obtain $\check{R}^+(z)$ we simply set every function associated with irreps from the conjugacy class $\{e\}$ equal to one, i.e. $f_{(0,b)} = 1$ for $b \in \mathbb{Z}$. Conversely, $\check{R}^-(z)$ is obtained by setting every function associated with irreps from the conjugacy class $\{\sigma^m\}$ equal to one, i.e. $f_{(\frac{m}{2},b)}(1) = 1$ for $b \in \mathbb{Z}$. We recall that e and σ^m are the central elements in D_{2m} when m is even. It is in part due to this enlarged centre that we obtain more general solutions in this case.

It is also possible for us to ignore the unitary condition and only impose that $R(z)$ is self-adjoint. This leads to the constraints

$$f_{(0,2b)}(z) = f_{(0,0)}(z) = (f_{(0,0)}(z))^* \qquad \text{and} \qquad f_{(0,2b+1)}(z) = f_{(0,1)}(z) = (f_{(0,1)}(z))^*,$$

for all $b \in \mathbb{Z}$ and $z \in \mathbb{R}$. Using these constraints we set

$$f_{(0,0)}(z) = 1 + f(z) \qquad \text{and} \qquad f_{(0,1)}(z) = 1 - f(z),$$

where $f(z)$ is an arbitrary real function. This yields the operator

$$\check{R}(z) = \check{R}^+(z) + f(z)\check{R}^-(z). \tag{23}$$

This is invertible provided

$$f(z) \neq \pm 1.$$

The function $f(z)$ is equivalent to a second parameter. That is, the operator

$$\check{R}(z, \mu) = \check{R}^+(z) + \mu \check{R}^-(z), \tag{24}$$

is invertible for $\mu \neq \pm 1$. Furthermore, we have the following:

Proposition 5.2. *The operator $\check{R}(z,\mu)$ given by Equation (24) satisfies*

$$\check{R}_{12}(x,\lambda)\check{R}_{23}(xy,\mu)\check{R}_{12}(y,\nu)=\check{R}_{23}(y,\nu)\check{R}_{12}(xy,\mu)\check{R}_{23}(x,\lambda), \tag{25}$$

if and only if $\check{R}(z)$ given by Equation (23) is a solution to the Yang–Baxter equation with no constraints on $f(z)$.

Proof. Consider a solution of the Yang-Baxter equation $R(z)\in\mathrm{End}(V_n\otimes V_n)$ which contains an arbitrary function $f(z)$. Furthermore suppose every entry of $R(z)$ can be written as a polynomial in terms of z and $f(z)$. Let

$$\Omega=R_{12}(x)R_{13}(xy)R_{23}(y)-R_{23}(y)R_{13}(xy)R_{12}(x).$$

We observe that every entry of Ω is expressible as

$$\sum_{i,j,k=0}^{\infty}h^l_{ijk}(x,y)f^i(x)f^j(xy)f^k(y),$$

where $h^l_{ijk}(x,y)$ are polynomials in x and y and l indexes the entry of Ω. As $R(z)$ is a solution of the YBE, i.e. $\Omega=0$, and $f(z)$ is an arbitrary function, we deduce that.

$$h^l_{ijk}(x,y)=0,\qquad \forall x,y\in\mathbb{C}/\{0,1,\infty\}\text{ s.t. }x\neq y.$$

Let $y_0\in\mathbb{C}/\{0,1,\infty\}$. There are at most we have four values of x for which $h^l_{ijk}(x,y_0)$ can be non-zero, but $h^l_{ijk}(x,y_0)$ is continuous in x. This means that

$$h^l_{ijk}(x,y_0)=0\qquad \forall x\in\mathbb{C}.$$

By symmetry

$$h^l_{ijk}(x,y)=0,\qquad \forall(x,y)\in(\mathbb{C}\times\mathbb{C})/(\{0,1,\infty\}\times\{0,1,\infty\}).$$

Thus there are at most 9 points in which $h^l_{ijk}(x,y)$ can be non-zero; however $h^l_{ijk}(x,y)$ is continuous in x and y. Hence

$$h^l_{ijk}(x,y)=0,\qquad \forall x,y\in\mathbb{C}.$$

Let $R(z,\mu)$ be the operator derived from $R(z)$ in which we have replaced the free function $f(z)$ with μ. Every entry of $R(z)$ must be expressible as a polynomial in terms of z and μ. If we let

$$\Omega=R_{12}(x,\lambda)R_{13}(xy,\mu)R_{23}(y,\nu)-R_{23}(y,\nu)R_{13}(xy,\mu)R_{12}(x,\lambda),$$

$x,y,\lambda,\mu,\nu\in\mathbb{C}$, then we find every entry of Ω can be written

$$\sum_{i,j,k=0}^{\infty}h^l_{ijk}(x,y)\lambda^i\mu^j\nu^k.$$

As shown previously

$$h_{ijk}^{l}(x,y)=0, \qquad \forall x,y\in\mathbb{C}.$$

Thus $\Omega=0$ and $R(z,\mu)$ satisfies

$$R_{12}(x,\lambda)R_{13}(xy,\mu)R_{23}(y,\nu)=R_{23}(y,\nu)R_{13,\mu}(xy)R_{12}(x,\lambda).$$

Conversely, suppose we have $R(z,\mu)$ which satisfies

$$R_{12}(x,\lambda)R_{13}(xy,\mu)R_{23}(y,\nu)=R_{23}(y,\nu)R_{13}(xy,\mu)R_{12}(x,\lambda).$$

If we consider $R(z)=R(z,f(z))$ where $f(z)$ is an arbitrary function then $R(z)$ must satisfy the YBE. To recover the result, we use the fact that $\check{R}(z)=P\ R(z)$ and $\check{R}(z,\mu)=P\ R(z,\mu)$. This proof works if the entries of $R(z)$ can be written as a polynomial $f(z)$ whose coefficients are rational functions of z. We are able to scale $R(z)$ by the product of all the denominators of the coefficients of $f(z)$, hence turning it into a polynomial.

□

We are able to verify using Maple that $\check{R}(z)$ given by Equation (23) satisfies the Yang–Baxter equation for even n up to 12, and hence that $\check{R}(z,\mu)$ satisfies Equation (25). We conjecture that this holds true for all even n, and have shown it to hold in the limit $x=y=0$.

6 Connection to the Fateev–Zamolodchikov model

Closely associated with the Yang-Baxter equation is the star-triangle relation (STR), given by

$$\sum_{d=0}^{N-1}\bar{W}(x|a-d)W(xy|d,c)\bar{W}(y|d-b)=W(x|b-c)\bar{W}(xy|a-b)W(y|a-c),$$

for $0\le a,b,c\le N-1$. One well-known solution of the STR is the N-state Fateev–Zamolodchikov model [23], which has weights

$$W(z|l)=\prod_{j=1}^{l}\frac{\lambda^{2j-1}z-1}{\lambda^{2j-1}-z} \qquad \text{and} \qquad \bar{W}(z|l)=\prod_{j=1}^{l}\frac{\lambda^{2j-1}-\lambda z}{\lambda^{2j}z-1},$$

for $0\le l\le N-1$, where λ is a primitive $2N$th root of unity. These weights are extended by the relations

$$W(z|l)=W(z|N+l) \qquad \text{and} \qquad \bar{W}(z|l)=\bar{W}(z|N+l),$$

while also satisfying

$$W(z|l)=W(z|N-l) \qquad \text{and} \qquad \bar{W}(z|l)=\bar{W}(z|N-l),$$

for $l \in \mathbb{Z}$. These weights satisfy the STR and lead to the R-matrix defined by

$$R(\tilde{x}, \tilde{y}) = \sum_{a_1,a_2,b_1,b_2=1}^{N} R_{a_1 a_2}^{b_1 b_2}(\tilde{x}, \tilde{y}) e_{b_1,a_1} \otimes e_{b_2,a_2},$$

where

$$\tilde{x} = \binom{x_1}{x_2}, \qquad \tilde{y} = \binom{y_1}{y_2}$$

and

$$R_{a_1 a_2}^{b_1 b_2}(\tilde{x}, \tilde{y}) = \bar{W}(x_1 y_1^{-1}|a_1 - b_2) W(x_2 y_1^{-1}|a_1 - a_2) \bar{W}(x_2 y_2^{-1}|a_2 - b_1) W(x_1 y_2^{-1}|b_2 - b_1).$$

Through a private communication with V. Bazhanov and J. Perk [30] we learnt of a connection between the $D(D_n)$ solution and the Fateev–Zamolodchikov model. Specifically in the private communication a limiting case of the 3-state Fateev–Zamolodchikov model was shown to reduce to the $D(D_3)$ solution. Using the ideas presented in [30] we are able to establish a connection between the N-state Fateev–Zamolodchikov model and $D(D_n)$ (or equivalently $D(D_{2n})$) solution in the case where $N = n$ and n is odd.

To investigate which limit of the Fateev–Zamolodchikov model leads to the $D(D_n)$ model, we determine when the Fateev–Zamolodchikov R-matrix squares to the identity. Using certain properties of the weights it is possible to show that

$$R(\tilde{x}, \tilde{y}) R(\tilde{x}^{-T}, \tilde{y}^{-T}) \propto I \otimes I,$$

where

$$\tilde{x} = \binom{x_1}{x_2} \quad \text{and} \quad \tilde{x}^{-T} = \binom{x_2^{-1}}{x_1^{-1}}.$$

Thus the inverse of the R-matrix is known up to a scalar multiple. Furthermore we are able to show that if the R-matrix squares to the identity then it is equivalent to one which satisfies the constraint

$$\tilde{x} = \tilde{x}^{-T} \quad \text{and} \quad \tilde{y} = \tilde{y}^{-T}.$$

This implies that the R-matrix can be reduced to

$$R(x, y) = \sum_{a_1,a_2,b_1,b_2=1}^{N} R_{a_1 a_2}^{b_1 b_2}(x, y) e_{b_1,a_1} \otimes e_{b_2,a_2},$$

where

$$R_{a_1 a_2}^{b_1 b_2}(x, y) = \bar{W}(xy^{-1}|a_1 - b_2) W(x^{-1}y^{-1}|a_1 - a_2) \bar{W}(x^{-1}y|a_2 - b_1) W(xy|b_2 - b_1).$$

To obtain the *difference property* we set

$$R(z) = \lim_{x,y \to \infty} R(x, y), \tag{26}$$

where $z = \frac{x}{y}$. For odd $n \leq 11$ we are able to computationally verify that this R-matrix is equivalent (up to a basis transformation) to the $D(D_n)$ R-matrix while setting $\lambda = -w^{-1}$. For odd $n > 11$ we can verify that $R(0)$ is indeed equivalent to $(\pi_n^+ \otimes \pi_n^+)\mathcal{R}$.

We now briefly comment on the descendants obtained from $D(D_{2n})$, n even. There are two distinct R-matrices, $R^+(z)$ and $R^-(z)$, which both square to the identity. The multiplicities of the eigenvalues of $R^+(z)$ and $R^-(z)$ differ; furthermore neither eigenvalue spectrum matches that of $R(z)$ as defined by Equation (26). Hence unlike the case when n is odd, the R-matrix (23) is not equivalent up to basis transformation of a limit of the Fateev–Zamolodchikov R-matrix.

7 Summary

In this paper we used the framework of descendants to construct R-matrices from the Drinfeld doubles of dihedral groups. For $3 \leq n \leq 17$ and w a primitive nth root of unity,

$$R(z) = \sum_{i,j,a=0}^{n-1} \left[\frac{1}{n} \sum_{b=0}^{n-1} w^{bj} \prod_{p=1}^{a} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right) \right] e_{i+j,i+a} \otimes e_{i+a+j,i}.$$

is a descendant of the six-vertex model

$$r(z) = \begin{pmatrix} wz^{-1} - w^{-1}z & 0 & 0 & 0 \\ 0 & z^{-1} - z & w - w^{-1} & 0 \\ 0 & w - w^{-1} & z^{-1} - z & 0 \\ 0 & 0 & 0 & wz^{-1} - w^{-1}z \end{pmatrix},$$

with corresponding L-operator

$$L(z) = \sum_{i=0}^{n-1} \left\{ (w^i e_{1,2} + w^{-i} e_{2,1}) \otimes e_{i,i} + z \left[e_{1,1} \otimes e_{i-1,i} + e_{2,2} \otimes e_{i+1,i} \right] \right\}.$$

We conjecture that this holds true for all $n \geq 2$.

We also showed that when n is even we obtain R-matrices with a second, non-spectral parameter. Specifically, let $n > 2$ be an even integer and w a primitive nth root of unity. Given

$$R^+(z) = \sum_{i,j,a=0}^{n-1} \left[\frac{1}{n} \sum_{b=0}^{n-1} w^{bj} \prod_{p=1}^{a} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right) \right] e_{i+j,i+a} \otimes e_{i+a+j,i}$$

and

$$R^-(z) = \sum_{i,j,a=0}^{n-1} \left[\frac{1}{n} \sum_{b=0}^{n-1} (-1)^{b-a} w^{bj} \prod_{p=1}^{a} \left(\frac{z + w^{(2p-1+b-a)}}{1 + zw^{(2p-1+b-a)}} \right) \right] e_{i+j,i+a} \otimes e_{i+a+j,i},$$

then

$$R(z, \mu) = R^{+}(z) + \mu R^{-}(z)$$

satisfies Equation (25) for $n \leq 12$ and is conjectured to satisfy it for larger n. Moreover, $R(z, \mu)$ and $L(z)$ together satisfy

$$L_{12}(x)L_{13}(y)R_{23}(x^{-1}y, \mu) = R_{23}(x^{-1}y, \mu)L_{13}(y)L_{12}(x).$$

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References

[1] McGuire J B 1964 *J. Math. Phys.* **5** 622

[2] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312

[3] Baxter R J 1971 *Phys. Rev. Lett.* **26** 832

[4] Perk J H H and Schultz C L 1981 *Phys. Lett. A* **84** 407

[5] Andrews G E, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193

[6] Au-Yang H, McCoy B M, Perk J H H, Tang S and Yan M L 1987 *Phys. Lett. A* **123** 219

[7] Baxter R J, Perk J H H and Au-Yang H 1988 *Phys. Lett. A* **128** 128

[8] Shastry B S 1986 Phys. Rev. Lett. **56,** 1529

[9] Umeno Y, Shiroishi M and Wadati M 1998 J. Phys. Soc. Jpn. **67** 2242

[10] Drinfeld V G 1986 Quantum groups, *Proceedings of the International Congress of Mathematicians*, A.M. Gleason (ed) (Providence RI: American Mathematical Society), p. 798

[11] Jimbo M 1986 *Lect. Notes Phys.* **246** 335

[12] Zhang Y Z and Gould M D 1994 *Lett. Math. Phys.* **31** 101

[13] Bracken A J, Gould M D and Zhang Y Z 1995 *Bull. Aust. Math. Soc.* **51** 177

[14] Jones V F R 1990 *Int. J. Mod. Phys. B* **4** 701

[15] Cheng Y, Ge M-L and Xue K 1991 *Commun. Math. Phys.* **136** 195

[16] Zhang R B, Gould M D and Bracken A J 1991 *Nucl. Phys. B* **354** 625

[17] Delius G W, Gould M D and Zhang Y-Z 1994 *Nucl. Phys. B* **432** 377

[18] Delius G W , Gould M D and Zhang Y-Z 1996 *Int. J. Mod. Phys. A* **11** 3415

[19] Bazhanov V V and Stroganov Yu G 1990 *J. Stat. Phys.* bf 59 799

[20] Hasegawa K and Yamada Y 1990 *Phys. Lett. A* **146** 275

[21] Sklyanin E K 1982 *Funct. Anal. Appl.* **16** 27

[22] Baxter R J 2004 J. Stat. Phys. **116** 43

[23] Fateev V A and Zamolodchikov A B 1982 *Phys. Lett. A* **92** 37

[24] de Wild Propitius M and Bais F A 1998, *Discrete gauge theories.* In G. Semenoff and L. Vinet, (eds.), Particles and Fields, CRM Series in Mathematical Physics (Springer–Verlag, New York) 353

[25] Kitaev A Yu 2003 *Ann. Phys. (N.Y.)* **303** 2

[26] Mochon C 2003 *Phys. Rev. A* **67** 022315

[27] Nayak C, Simon S H, Stern A, Freedman M and Das Sarma S 2008 *Rev. Mod. Phys.* **80** 1083

[28] Dancer K A, Isaac P S and Links J 2006 *J. Math. Phys.* **47** 103511

[29] Dancer K A and Links J 2009 *J. Phys. A: Math. Theor.* **42** 042002

[30] Bazhanov V V and Perk J H H 2009 *private communication*

[31] Dijkgraaf R, Pasquier V and Roche P 1990 *Nucl. Phys. B (Proc. Supp.)* **18** 60

[32] Gould M D 1993 *Bull. Aust. Math. Soc.* **48** 275