On Lagrangian and Hamiltonian systems with homogeneous trajectories

Gábor Zsolt Tóth

KFKI Research Institute for Particle and Nuclear Physics,
Hungarian Academy of Sciences, P.O.B. 49, 1525 Budapest, Hungary

email: tgzs@rmki.kfki.hu, tgzs@cs.elte.hu

Abstract

Motivated by various results on homogeneous geodesics of Riemannian spaces, we study homogeneous trajectories, i.e. trajectories which are orbits of a one-parameter symmetry group, of Lagrangian and Hamiltonian systems. We present criteria under which an orbit of a one-parameter subgroup of a symmetry group G is a solution of the Euler-Lagrange or Hamiltonian equations. In particular, we generalize the 'geodesic lemma' known in Riemannian geometry to Lagrangian and Hamiltonian systems. We present results on the existence of homogeneous trajectories of Lagrangian systems. We study Hamiltonian and Lagrangian g.o. spaces, i.e. homogeneous spaces G/H with G-invariant Lagrangian or Hamiltonian functions on which every solution of the equations of motion is homogeneous. We show that the Hamiltonian g.o. spaces are related to the functions that are invariant under the coadjoint action of G. Riemannian g.o. spaces thus correspond to special $Ad^*(G)$-invariant functions. An $Ad^*(G)$-invariant function that is related to a g.o. space also serves as a potential for the mapping called 'geodesic graph'. As illustration we discuss the Riemannian g.o. metrics on $SU(3)/SU(2)$.

Keywords: g.o. space, homogeneous space, homogeneous geodesic, momentum map, Lagrangian and Hamiltonian systems with symmetry

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1 Introduction

Let M be a Riemannian manifold. A geodesic in M is called *homogeneous* if it is the orbit of a one-parameter group of isometries of M. A homogeneous Riemannian manifold $M = G/K$, where G is a connected Lie-group and K is a closed subgroup, is a *geodesic orbit (g.o.) space* with respect to G, if every geodesic in it is the orbit of a one-parameter subgroup of G.

The homogeneous space $M = G/K$ is called a *reductive space*, if there exists a direct sum decomposition (called *reductive decomposition*) $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ of the Lie algebra of G, where $\mathfrak{m}$ is an $ad(K)$-invariant linear subspace of $\mathfrak{g}$ and $\mathfrak{k}$ is the Lie algebra of K. It is known that all Riemannian homogeneous spaces are reductive. If $M = G/K$ is Riemannian and there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ such that each geodesic in M starting at the origin $o \in M$ is an orbit of a one-parameter subgroup of G generated by some element of $\mathfrak{m}$, then M is called a *naturally reductive space* with respect to G. The origin o is the image of K by the canonical projection $G \to G/K$.

Obviously, every naturally reductive space is a g.o. space as well. It was believed some decades ago, that the converse is also true, i.e. every g.o. space is isometric to some naturally reductive space. A counter example, however, was found by A. Kaplan [1], initiating the extensive study of g.o. spaces [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Pseudo-Riemannian g.o. spaces were also investigated [14, 15, 16].

In general, it is possible that a homogeneous Riemannian space $M = G/K$ is not naturally reductive with respect to G, but one can take larger groups $G' \supset G$ and $K' \supset K$ so that $M = G'/K'$ and M is naturally reductive with respect to G'. The same situation can occur for g.o. spaces as well. It is also possible in some cases that a g.o. space can be made naturally reductive by taking a larger symmetry group G', but there also exist g.o. spaces for which this is not possible, i.e. which are *in no way* naturally reductive. Kaplan's example is of the latter type.

Since Riemannian (and pseudo-Riemannian) manifolds can be viewed as a special class of the manifolds with a Lagrangian or Hamiltonian function, it is interesting to consider the generalization of the g.o. property to Lagrangian and Hamiltonian homogeneous spaces and to ask whether the known results for the Riemannian spaces can be generalized, and whether the techniques of Lagrangian or Hamiltonian dynamics can be used for the study of Riemannian g.o. spaces. In this paper we present the results that we obtained in relation to these questions.

A subject closely related to the study of g.o. spaces is the characterization of the homogeneous geodesics in Riemannian manifolds. Homogeneous geodesics are of interest also in Finsler geometry, pseudo-Riemannian geometry and in dynamics, and they appear in the literature under different names, e.g. "relative equilibria" and "stationary geodesics", as well. We refer the reader to [17, 18, 19, 20, 21, 22, 23, 24, 25] and further references therein. The present paper is also concerned with the characterization of homogeneous geodesics in the setting of Lagrangian and Hamiltonian dynamics, partly because this is necessary for the study of g.o. spaces. Since we were motivated mainly by Riemannian geometry, we use the name geodesic for the solutions of the Lagrangian or Hamiltonian equations. As we have already done above, we also refer to a manifold with a Lagrangian or Hamiltonian function as a Lagrangian or Hamiltonian space.

The paper is organized as follows. In section 2 we discuss the case of Lagrangian spaces. We describe criteria for an orbit of a one-parameter subgroup to be a geodesic, including the Lagrangian version of the 'geodesic lemma'. We also present results concerning the existence of homogeneous geodesics.

In section 3 we discuss the case of Hamiltonian spaces. We describe criteria for an orbit of a one-parameter subgroup to be a geodesic, including the Hamiltonian version of the geodesic lemma. Then we turn to the characterization of Hamiltonian g.o. spaces. In particular, we show that the Hamiltonian g.o. spaces are closely related to the functions which are invariant under the coadjoint action of G. Riemannian g.o. spaces correspond, of course, to special $Ad^*(G)$-invariant functions. An $Ad^*(G)$-invariant function that is related to a g.o. space also serves as a potential for the mapping called *geodesic graph*, which has proven to be useful for the description of Riemannian g.o. spaces. We describe a criterion based on the relation between g.o. spaces and $Ad^*(G)$-invariant functions that can be used to find g.o. Hamiltonians or metrics. We also describe a generalization of the notion of Hamiltonian g.o. space.

In section 4 we discuss the two-parameter family of Riemannian g.o. metrics on $SU(3)/SU(2)$ for the illustration of the results of section 3. We calculate the geodesic graph in a new way, utilizing the relation between g.o. spaces and $Ad^*(G)$-invariant functions.

2 Lagrangian spaces with homogeneous geodesics

Let M be a connected manifold with a Lagrangian function $L : TM \to \mathbb{R}$ on it. The solutions $\gamma : I \to M$, where I is an interval, of the corresponding Euler-Lagrange equations will be called *geodesics*. A Lagrangian function is *regular* if the solution of the Euler-Lagrange equations is unique for given initial data $(x, v) \in TM$.

In the following we assume that L is invariant under the action of a connected Lie group G on M. We use the Einstein summation convention. We denote the Lie derivative with respect to a vector field Z as $\mathcal{L}_E$. We use the notation $\circ$ for the composition of two functions, i.e. if f and g are two functions, then $f \circ g$ is the function for which $(f \circ g)(x) = f(g(x))$.

In the derivation of the results of this section the Euler-Lagrange equation, an equation expressing the invariance of L and equations characterizing the velocity and acceleration of orbits have important role.

Euler-Lagrange equation

The *Euler-Lagrange equation* for a curve $\gamma : I \to M$, where I is an interval, is

$$\frac{\partial L}{\partial x_i}(\gamma(t), \dot{\gamma}(t)) = \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}(\gamma, \dot{\gamma})\right)(t) \qquad \forall t \in I, \tag{1}$$

or, expanding the right hand side,

$$\frac{\partial L}{\partial x_i}(\gamma(t), \dot{\gamma}(t)) = \frac{\partial^2 L}{\partial x_i \partial v_j}(\gamma(t), \dot{\gamma}(t))\dot{\gamma}_j(t) + \frac{\partial^2 L}{\partial v_i \partial v_j}(\gamma(t), \dot{\gamma}(t))\ddot{\gamma}_j(t). \tag{2}$$

Symmetry condition for L
Let $Z_a : M \to TM$ and $\hat{Z}_a : TM \to TTM$, where $a \in \mathfrak{g}$, be the infinitesimal generator vector fields for the action of G on M and TM, respectively. Their coordinate form is

$$Z_a(x) = \frac{\partial \phi_i^a}{\partial \tau}(0,x)\frac{\partial}{\partial x_i}, \qquad x \in M \tag{3}$$

and

$$\hat{Z}_a(x,v) = \frac{\partial \phi_i^a}{\partial \tau}(0,x)\frac{\partial}{\partial x_i} + \frac{\partial^2 \phi_i^a}{\partial \tau \partial x_j}(0,x)v_j\frac{\partial}{\partial v_i}, \qquad (x,v) \in TM, \tag{4}$$

where $\phi^a : \mathbb{R} \times M \to M$ is the action of the one-parameter subgroup generated by $a \in \mathfrak{g}$ and τ denotes the first variable of ϕ^a.

The invariance of L under the action of G implies the following *symmetry condition*:

$$\mathcal{L}_{\hat{Z}_a} L(x,v) = \frac{\partial L}{\partial x_i}(x,v)\frac{\partial \phi_i^a}{\partial \tau}(0,x) + \frac{\partial L}{\partial v_i}(x,v)\frac{\partial^2 \phi_i^a}{\partial \tau \partial x_j}(0,x)v_j = 0, \tag{5}$$

where $a \in \mathfrak{g}$. This equation holds for all $(x,v) \in TM$.

Velocity and acceleration of an orbit
The orbit of the one-parameter subgroup generated by $a \in \mathfrak{g}$ in M with initial point x is the curve $\gamma : I \to M,\ t \mapsto \phi^a(t,x)$. For the velocity

$$\dot{\gamma}(t) = \frac{\partial \phi^a}{\partial \tau}(t,x) \tag{6}$$

of this orbit the equation

$$\dot{\gamma}_i(t) = \frac{\partial \phi_i^a}{\partial x_j}(t,x)\dot{\gamma}_j(0) = \frac{\partial \phi_i^a}{\partial x_j}(t,x)\frac{\partial \phi_j^a}{\partial \tau}(0,x) \tag{7}$$

holds because of the group property. For the acceleration we have

$$\ddot{\gamma}_i(t) = \frac{\partial^2 \phi_i^a}{\partial \tau \partial x_j}(t,x)\frac{\partial \phi_j^a}{\partial \tau}(0,x) = \frac{\partial^2 \phi_i^a}{\partial \tau^2}(t,x). \tag{8}$$

Theorem 2.1 *The orbit of a one-parameter subgroup of G starting at $x \in M$ is a geodesic of the (not necessarily regular) Lagrangian L if and only if x is a critical point of the function $L \circ Z_a$, i.e.*

$$d(L \circ Z_a)(x) = 0, \tag{9}$$

where Z_a is the generator vector field of the subgroup.

Proof. Because of the invariance of the Lagrangian an orbit of a one-parameter symmetry group is a geodesic if and only if it satisfies the Euler-Lagrange equations at the initial point. First, let us assume that the orbit is a geodesic. Differentiating the symmetry condition (5) with respect to v_j yields

$$\begin{aligned} 0 &= \frac{\partial}{\partial v_j}\mathcal{L}_{\hat{Z}_a} L(x,v) = \frac{\partial^2 L}{\partial x_i \partial v_j}(x,v)\frac{\partial \phi_i^a}{\partial \tau}(0,x) \\ &\quad + \frac{\partial^2 L}{\partial v_i \partial v_j}(x,v)\frac{\partial^2 \phi_i^a}{\partial \tau \partial x_k}(0,x)v_k + \frac{\partial L}{\partial v_i}(x,v)\frac{\partial^2 \phi_i^a}{\partial \tau \partial x_j}(0,x). \end{aligned} \tag{10}$$

Substituting the right hand side of (8) for $\ddot{\gamma}$ in the Euler-Lagrange equation (2) at $t = 0$ gives

$$\frac{\partial L}{\partial x_j}(x,v) = \frac{\partial^2 L}{\partial x_i \partial v_j}(x,v)\frac{\partial \phi_i^a}{\partial \tau}(0,x) + \frac{\partial^2 L}{\partial v_i \partial v_j}(x,v)\frac{\partial^2 \phi_i^a}{\partial \tau \partial x_j}(0,x)\frac{\partial \phi_j^a}{\partial \tau}(0,x), \tag{11}$$

where $v = \dot{\gamma}(0)$. Setting $v = \dot{\gamma}(0)$ also in (10) and subtracting from (11) gives

$$\frac{\partial L}{\partial x_j}(x,v) + \frac{\partial L}{\partial v_i}(x,v)\frac{\partial^2 \phi_i^a}{\partial \tau \partial x_j}(0,x) = 0, \tag{12}$$

where $v = \dot{\gamma}(0)$, which is just the coordinate form of (9). Considering the reverse direction of the statement, it is clear now that if (12) and (10) hold, then (11) follows. $\square$

A similar theorem is stated in [22] (see also [21]). However, our proof is different from those given in [22] and [21].

Definition 2.2 An element a of $\mathfrak{g}$ is called a *geodesic vector* at $x \in M$ if the orbit of the one-parameter subgroup of G generated by a and starting at x is a geodesic.

In Riemannian geometry the interesting geodesic vectors are, of course, those which generate orbits that are not single points in M.

The set of geodesic vectors at x is invariant under G_x, the stabilizer of x. If $gx = y$ for some $x, y \in M$ and $g \in G$, then the set of geodesic vectors at y can be obtained from that at x by the action of g.

As regards the existence of geodesic vectors, the following corollary of theorem 2.1 can be stated.

Theorem 2.3 *Let M, G, L be as in the theorem 2.1 and let M be compact. For any $a \in \mathfrak{g}$ there exists at least one geodesic of L in M, which is the orbit of the one-parameter subgroup generated by a. If there exists an $a \in \mathfrak{g}$ such that $Z_a(x) \neq 0 \ \ \forall x \in M$, then there exists at least one geodesic of L in M, which is the orbit of the one-parameter subgroup generated by a and is not a single point in M. If, in addition, M is also homogeneous with respect to the action of G, then there exists at least one nonzero geodesic vector at every point in M, which generates an orbit that is not a single point.*

In the rest of this section we consider the case when M is a homogeneous space. For a homogeneous space $M = G/K$ there is a linear map $f_x : \ \mathfrak{g} \to T_xM, \ a \mapsto Z_a(x)$ for each point $x \in M$. We use the notation f for f_o (i.e. we omit the subscript o denoting the origin in G/K).

The dual of a vector space V will be denoted by V^*. The contraction (or natural pairing) between V and V^* will be denoted in the following way: $(w|v)$, where $w \in V^*$ and $v \in V$. The transpose of a linear map $A : V \to W$ will be denoted by A^* (it is defined as $A^* : W^* \to V^*, \ w \mapsto w \circ A$).

The following lemma, which concerns homogeneous manifolds with invariant Lagrangians and is the generalization of the known 'geodesic lemma' for the Riemannian case [2] (see also for example [18, 3, 4]), gives a condition for an element of $\mathfrak{g}$ to be a geodesic vector at o. This is a local condition in the sense that it is given in terms of L restricted to T_oM, the elements of $\mathfrak{g}$, and the values of the infinitesimal generator vector fields at o. In Riemannian geometry the geodesic lemma has proven to be very useful in the study of homogeneous geodesics.

Lemma 2.4 (Geodesic lemma) *Let $M = G/K$ be a homogeneous space with a G-invariant Lagrangian $L : TM \to \mathbb{R}$. An element $a \in \mathfrak{g}$ is a geodesic vector at o if and only if*

$$(\ dL_o(f(a)) \ | \ f([a,b]) \ ) = 0 \qquad \forall b \in \mathfrak{g}, \tag{13}$$

where L_o is L restricted to T_oM. In particular, if L corresponds to a Riemannian metric, then (13) takes the form

$$\langle f([a,b]) \ , \ f(a) \rangle = 0 \qquad \forall b \in \mathfrak{g}, \tag{14}$$

or, equivalently,

$$\langle [a,b]_{\mathfrak{m}} \ , \ a_{\mathfrak{m}} \rangle = 0 \qquad \forall b \in \mathfrak{g}, \tag{15}$$

where the index $\mathfrak{m}$ denotes the $\mathfrak{m}$-component related to a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, and $\mathfrak{m}$ is assumed to be identified with T_oM by f.

Proof. Let us assume first, that a is a geodesic vector. (9) in theorem 2.1 is equivalent to $\mathcal{L}_{Z_b}(L \circ Z_a)(o) = 0 \ \ \forall b \in \mathfrak{g}$. In coordinate form

$$\begin{aligned} \mathcal{L}_{Z_b}(L \circ Z_a)(o) &= \frac{\partial \phi_i^b}{\partial \tau}(0,o) \frac{\partial L}{\partial x_i}(o, \frac{\partial \phi^a}{\partial \tau}(0,o)) \\ &+ \frac{\partial \phi_i^b}{\partial \tau}(0,o) \frac{\partial L}{\partial v_j}(o, \frac{\partial \phi^a}{\partial \tau}(0,o)) \frac{\partial^2 \phi_j^a}{\partial \tau \partial x_i}(0,o) = 0. \end{aligned} \tag{16}$$

Taking the symmetry condition (5) at the point $(o, \frac{\partial \phi^a}{\partial \tau}(0,o))$ we get

$$\begin{aligned} \mathcal{L}_{\hat{Z}_b} L(o, \frac{\partial \phi^a}{\partial \tau}(0,o)) &= \frac{\partial \phi_i^b}{\partial \tau}(0,o) \frac{\partial L}{\partial x_i}(o, \frac{\partial \phi^a}{\partial \tau}(0,o)) \\ &+ \frac{\partial L}{\partial v_i}(o, \frac{\partial \phi^a}{\partial \tau}(0,o)) \frac{\partial^2 \phi_i^b}{\partial \tau \partial x_j}(0,o) \frac{\partial \phi_j^a}{\partial \tau}(0,o) = 0. \end{aligned} \tag{17}$$

Subtracting these two equations gives

$$\frac{\partial L}{\partial v_j}(o, \frac{\partial \phi^a}{\partial \tau}(0,o)) \left[\frac{\partial \phi_i^b}{\partial \tau}(0,o) \frac{\partial^2 \phi_j^a}{\partial \tau \partial x_i}(0,o) - \frac{\partial \phi_i^a}{\partial \tau}(0,o) \frac{\partial^2 \phi_j^b}{\partial \tau \partial x_i}(0,o) \right] = 0, \qquad (18)$$

which is the coordinate expression for (13). Conversely, assuming that (18) holds and using (17) one obtains (16). The second part of the lemma concerning the Riemannian case follows obviously from the first part. □

The formula (15) for Riemannian spaces is well known and is also a generalization of Arnold's result about homogeneous geodesics of left-invariant metrics on Lie-groups [24].

Let $r : \mathbb{R} \to \mathfrak{g}$ be the adjoint orbit starting at a and generated by b. $f([a,b])$ is the tangent vector of the curve $f \circ r$ at the point $f(a)$. Equation (13) means that the derivative of L_o at $f(a)$ along this tangent vector is 0.

The following theorems 2.5 and 2.6 are about the existence of homogeneous geodesics.

Theorem 2.5 *Let $M = G/K$ be a homogeneous space with a G-invariant Lagrangian $L : TM \to \mathbb{R}$. If G is compact, then each adjoint orbit of G contains at least one geodesic vector at o, and each adjoint orbit of G that is not contained entirely by $\mathfrak{k}$ contains at least one geodesic vector at o which generates an orbit that is not a single point.*

Proof. Any adjoint orbit O of G is compact. $f(O)$ is also compact and L_o is continuous on it, thus there exists at least one $\tilde{v} \in f(O)$ so that $L_o|_{f(O)}$ is minimal or maximal at $\tilde{v}$. Because of this extremality the derivative of L_o is zero at $\tilde{v}$ along any curve that lies in $f(O)$ and passes through $\tilde{v}$. It is clear from the remark after the proof of the geodesic lemma that any element of $f^{-1}(\tilde{v}) \cap O$ is a geodesic vector at o.

If an adjoint orbit O is not contained entirely by $\mathfrak{k}$, then $f(O) \neq \{0\}$, thus there exists at least one $\tilde{v} \in f(O)$ so that $\tilde{v} \neq 0$ and $L_o|_{f(O)}$ is minimal or maximal at $\tilde{v}$. Any element of $f^{-1}(\tilde{v}) \cap O$ is a geodesic vector at o that generates an orbit that is not a single point. □

Theorem 2.6 *Let $M = G/K$ be a homogeneous space with a G-invariant Lagrangian $L : TM \to \mathbb{R}$. If G is solvable and the image space of $dL_o|_{T_oM\setminus\{0\}}$ contains vectors of arbitrary direction, than there exists at least one geodesic vector at o, which generates an orbit that is not a single point.*

Proof. Consider the derived series of $\mathfrak{g}$, i.e. the sequence

$$\mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \cdots \supset g^{(i)} \supset \ldots,$$

where $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ for $i = 1, 2, \ldots$. Because of the solvability of G, the derived series strictly decreases and ends in the null space. Consequently, there exists an index $r \geq 0$ such that $f(\mathfrak{g}^{(r)}) = T_oM$, but $f(\mathfrak{g}^{(r+1)})$ is a proper subspace of T_oM. The connected subgroup $G^{(r)}$ corresponding to $\mathfrak{g}^{(r)}$ still acts transitively on M, therefore it is necessary and sufficient for a vector to be geodesic that (13) hold for all $b \in \mathfrak{g}^{(r)}$. The condition imposed on dL_o in the proposition ensures that there exists an $\tilde{v} \in T_oM \setminus \{0\}$ such that $(\ dL_0(\tilde{v}) \mid f([\mathfrak{g}^{(r)}, \mathfrak{g}^{(r)}])\ ) = 0$, implying that any element of $f^{-1}(\tilde{v}) \cap \mathfrak{g}^{(r)}$ is a geodesic vector. $\square$

This theorem is similar to some parts of proposition 3 of [18]. It is clear from the proof that the solvability of G is not necessary, it can be replaced by the weaker condition that there exists an element $\mathfrak{g}^{(r+1)}$ of the derived series of $\mathfrak{g}$ such that $f(\mathfrak{g}^{(r+1)})$ is a proper subspace of T_oM.

The condition of regularity has not been imposed on the Lagrangians so far. It is assumed, however, in the following two propositions 2.8 and 2.10, which characterize Lagrangian g.o. spaces.

Definition 2.7 Let $M = G/K$ a homogeneous space and $L : M \to \mathbb{R}$ a G-invariant Lagrangian function. M is a called a *Lagrangian geodesic orbit (g.o.) space with respect to* G, if every geodesic in it is an orbit of a one-parameter subgroup of G.

Proposition 2.8 *Let $M = G/K$ and L be as in definition 2.7, and assume that L is regular. Then (M, L) is a g.o. space with respect to G if and only if for all $v \in T_oM$ there exists an $a \in \mathfrak{g}$ such that $f(a) = v$ and a is a geodesic vector.*

Definition 2.9 Let $M = G/K$ be a Lagrangian g.o. space with respect to G. A mapping $\xi : T_oM \to \mathfrak{g}$ with the properties that $f(\xi(v)) = v$ and $\xi(v)$ is a geodesic vector at o for all $v \in T_oM$ is called a *geodesic graph.* Obviously, there exists at least one geodesic graph for every g.o. space. $f(\xi(v)) = v$ means that the velocity of the orbit generated by $\xi(v)$ is v at o.

In Riemannian geometry the geodesic graph is very useful for studying g.o. spaces. Important results about its properties were obtained in [4, 19]. We note that there is a minor difference between our definition and the usual definition; in the usual definition one has a direct sum decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, and one takes the $\mathfrak{k}$-component of $\xi(v)$ as the value of the geodesic graph at v.

The following consequence of proposition 2.8 and of the geodesic lemma, in particular of (13), applying to the special case $M = G$, is well known [25].

Theorem 2.10 *If $M = G$, i.e. L is a regular left-invariant Lagrangian on G, then M is a g.o. space with respect to G if and only if $L_e = L|_{T_eG}$ (where e is the unit element of G) is invariant under the adjoint action of G. Any function on T_eG can be extended uniquely to a left-invariant function on G, therefore the Lagrangians on G that have the g.o. property with respect to G are in one-to-one correspondence with the regular Ad-invariant functions on $\mathfrak{g}$.*

We note that in the case $M = G$ the equation (13) expresses the $Ad(G)$-invariance of L_e.

In the next section we turn to the Hamiltonian formalism, which is better suited to the characterization of g.o. spaces than the Lagrangian formalism.

3 Hamiltonian spaces with homogeneous geodesics

Let M be a manifold with a Hamiltonian function $H : T^*M \to \mathbb{R}$. The solutions $\gamma \equiv (x,p) : I \to T^*M$, where I is an interval, of the Hamiltonian equations will be called *geodesics*. In the following we assume that H is invariant under the action of a connected Lie-group G on M.

Hamiltonian equations
The *Hamiltonian equations* for a curve $\gamma : I \to T^*M$ are the following:

$$X_H(\gamma(t)) = \dot{\gamma}(t) \qquad \forall t \in I, \tag{19}$$

or equivalently

$$\frac{\partial H}{\partial p_i}(x,p) \;=\; \dot{x}_i \tag{20}$$

$$-\frac{\partial H}{\partial x_i}(x,p) \;=\; \dot{p}_i, \tag{21}$$

where the Hamiltonian flow (vector field) generated by the function H on the symplectic manifold T^*M is denoted by X_H.

Let $\hat{Z}_a^* : T^*M \to TT^*M$, $a \in \mathfrak{g}$, be the infinitesimal generator vector fields for the action of G on T^*M. Their coordinate form is

$$\hat{Z}_a^*(x,p) = \frac{\partial \phi_i^a}{\partial \tau}(0,x)\frac{\partial}{\partial x_i} - \frac{\partial^2 \phi_j^a}{\partial \tau \partial x_i}(0,x)p_j\frac{\partial}{\partial p_i}, \tag{22}$$

where ϕ^a is the same object as in section 2.

Symmetry condition for H
The invariance of H implies the following *symmetry condition*:

$$\mathcal{L}_{\hat{Z}_b^*}H(x,p) = \frac{\partial H}{\partial x_i}(x,p)\frac{\partial \phi_i^b}{\partial \tau}(0,x) - \frac{\partial H}{\partial p_i}(x,p)\frac{\partial^2 \phi_j^b}{\partial \tau \partial x_i}(0,x)p_j = 0, \tag{23}$$

where $b \in \mathfrak{g}$. This equation holds for all $(x,p) \in T^*M$.

We recall that the *momentum map* for the action of G on T^*M is $P : T^*M \to \mathfrak{g}^*$, $(x,p) \mapsto f_x^*(p)$, where f_x is the linear mapping introduced in section 2 after theorem 2.3. Clearly P is linear on each cotangent space T_x^*M, $x \in M$, and it is also equivariant. P restricted to the cotangent space T_x^*M at $x \in M$ is the transpose of f_x. P has the property that

$$X_{(P|a)} = \hat{Z}_a^* \qquad \forall a \in \mathfrak{g}, \tag{24}$$

where X_F denotes the Hamiltonian flow generated by a function $F : T^*M \to \mathbb{R}$ and $(P|a)$ denotes the function $(x, p) \mapsto (f_x^*(p)|a)$. This property implies $[X_{(P|a)}, X_{(P|b)}] = X_{(P|[a,b])}$, where $[,]$ on the left hand side denotes the Lie bracket of vector fields. The functions $(P|a)$, $a \in \mathfrak{g}$, are conserved quantities, i.e. the function P (and thus $(P|a)$, for all $a \in \mathfrak{g}$) is constant along the geodesics of H.

Definition 3.1 An element a of $\mathfrak{g}$ is called a *geodesic vector* at $(x, p) \in T^*M$ if the orbit of the corresponding one-parameter subgroup starting at (x, p) is a geodesic.

Since the momentum map is constant along geodesics, if $a \in \mathfrak{g}$ is a geodesic vector at $(x, p) \in T^*M$, then a is an element of the stabilizer subgroup of $P(x, p)$ with respect to the coadjoint action of G.

Lemma 3.2 *Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian function that is invariant under the action of a connected Lie group G. $a \in \mathfrak{g}$ is a geodesic vector at $(x, p) \in T^*M$ if and only if*

$$X_H(x, p) = \hat{Z}_a^*(x, p), \tag{25}$$

or, equivalently,

$$d(H - (P|a))(x, p) = 0, \tag{26}$$

where P is the momentum mapping for the action of G on T^*M.

The proof of this lemma can be found in [25] (proposition 4.3.7.), for instance.

The following version of the geodesic lemma can be stated for Hamiltonian homogeneous spaces.

Lemma 3.3 (Geodesic lemma) *Let $M = G/K$ be a homogeneous space and $H : T^*M \to \mathbb{R}$ a G-invariant Hamiltonian function. An element $a \in \mathfrak{g}$ is a geodesic vector at (o, p), where o denotes the origin, if and only if*

$$dH_o(p) = f(a) \tag{27}$$

and

$$(\ f^*(p) \mid [a, b]\ ) = 0 \qquad \forall b \in \mathfrak{g} \tag{28}$$

hold, where H_o is H restricted to T_o^*M. (28) is equivalent to the condition that the one-parameter subgroup generated by a is contained by the stabilizer subgroup of $f^*(p) \in \mathfrak{g}^*$ with respect to the coadjoint action of G.

Proof. Assume first that a is a geodesic vector. (27) is just the first of the two Hamiltonian equations at the initial point and in coordinate form it reads as follows:

$$\frac{\partial \phi_i}{\partial \tau}(x, 0) = \frac{\partial H}{\partial p_i}(x, p). \tag{29}$$

The second Hamiltonian equation at the initial point is

$$\frac{\partial^2 \phi_j}{\partial\tau\partial x_i}(0,x)p_j = -\frac{\partial H}{\partial x_i}(x,p). \tag{30}$$

Substituting the left hand sides of (29) and (30) for the right hand sides of (29) and (30) in (23) gives

$$p_j \left[\frac{\partial \phi_i^a}{\partial\tau}(0,x)\frac{\partial^2 \phi_j^b}{\partial\tau\partial x_i}(0,x) - \frac{\partial \phi_i^b}{\partial\tau}(0,x)\frac{\partial^2 \phi_j^a}{\partial\tau\partial x_i}(0,x) \right] = 0 \qquad \forall b \in \mathfrak{g}, \tag{31}$$

which is just the coordinate form of the equation

$$(\ p \mid [Z_a, Z_b](o)\ ) = 0 \qquad \forall b \in \mathfrak{g}. \tag{32}$$

This is equivalent to (28), because $[Z_a, Z_b](o) = f([a,b])$ and $(p \mid f([a,b])) = (f^*(p) \mid [a,b])$. Considering the reverse direction, it is clear that (30) can be obtained from (31), (29) and (23). $\square$

Those points of T^*M where there exists a geodesic vector are called *relative equilibria* of the Hamiltonian function. If (x,p) is a relative equilibrium and $\mu = P(x,p)$ is a regular value of P, then (x,p) is a critical point of the *reduced Hamiltonian* H_μ (see [25], section 4.3).

Definition 3.4 Let $M = G/K$ a homogeneous space and $H : T^*M \to \mathbb{R}$ a G-invariant Hamiltonian function. M is a called a *geodesic orbit (g.o.) space with respect to* G, if every geodesic in it is an orbit of a one-parameter subgroup of G.

If M is a g.o. space, then every point of T^*M is a relative equilibrium, and thus the reduced Hamiltonian H_μ is constant on the connected components of the *reduced phase spaces* (the definition of the reduced phase space can be found e.g. in [25], section 4.3).

In the following propositions 3.5 and 3.6 elementary conditions are given under which a Hamiltonian homogeneous space has the g.o. property. They are direct consequences of lemma 3.2 and lemma 3.3.

Proposition 3.5 *Let $M = G/K$ be a homogeneous space and $H : T^*M \to \mathbb{R}$ a G-invariant Hamiltonian function. This space has the g.o. property with respect to G if and only if*

$$dH(o,p) \in \{d(P|b)(o,p) : b \in \mathfrak{g}\} \qquad \forall (o,p) \in T_o^*M \tag{33}$$

or, equivalently,

$$X_H(o,p) \in \{\hat{Z}_b^*(o,p) : b \in \mathfrak{g}\} \qquad \forall (o,p) \in T_o^*M. \tag{34}$$

Proposition 3.6 *Let M and H be the same as in the previous proposition. M is a g.o. space with respect to G if and only if for all $p \in T_o^*M$ there exists an $a \in \mathfrak{g}$ such that*

$$dH_o(p) = f(a) \tag{35}$$

and

$$(\ f^*(p) \mid [a, b]\ ) = 0 \qquad \forall b \in \mathfrak{g} \tag{36}$$

hold.

Definition 3.7 Let $M = G/K$ be a Hamiltonian g.o. space with respect to G. A mapping $\xi : T_o^*M \to \mathfrak{g}$ with the property that $\xi(p)$ is a geodesic vector at (o, p) for all $p \in T_o^*M$ is called a *geodesic graph.* Obviously, there exists at least one geodesic graph for every Hamiltonian g.o. space.

The following last part of the section we describe the relation between g.o. spaces and $Ad^*(G)$-invariant functions, and we describe how an $Ad^*(G)$-invariant function that corresponds to a g.o. space can be used to obtain a geodesic graph. We also describe a criterion that can be used to find g.o. Hamiltonians or metrics. Finally, we discuss briefly a generalization of the notion of Hamiltonian g.o. space.

Lemma 3.8 *Let $M = G/K$ be a homogeneous space and $H :\ T^*M \to \mathbb{R}$ a G-invariant Hamiltonian function that has the g.o. property with respect to G. If P is constant along a smooth curve $\gamma : I \to T^*M$, then H is also constant along this curve.*

Proof. The derivative $\frac{d(H\circ\gamma)}{dt}$ of H along γ at $t \in I$ equals $(dH(\gamma(t))|\dot{\gamma}(t))$. It is sufficient to show that this number is zero for any $t \in I$. Let t be a fixed element of I. It follows from proposition 3.5. that $(dH(\gamma(t))|\dot{\gamma}(t)) = (d(P|b)(\gamma(t))|\dot{\gamma}(t))$ for some $b \in \mathfrak{g}$. Since P is constant along γ, the derivative of P along γ is zero, therefore the derivative of $(P|b)$ is also zero, thus $(d(P|b)(\gamma(t))|\dot{\gamma}(t)) = 0$. □

The following theorem is a direct consequence of lemma 3.8.

Theorem 3.9 *Let $M = G/K$ be a homogeneous space and $H :\ T^*M \to \mathbb{R}$ a G-invariant Hamiltonian function that has the g.o. property with respect to G. If the connected components of the level sets of the momentum mapping P have the property that any two point in them can be connected by a piecewise smooth curve, then H is constant on the connected components of the level sets of P. If, in addition, H takes the same value on all connected components of any level set of P, then H takes the form*

$$H = h \circ P, \tag{37}$$

where $h : \mathfrak{g}^* \to \mathbb{R}$ is an $Ad^*(G)$-invariant function.

P is an analytic function, therefore its rank is maximal on an open dense subset N of T^*M, which is G-invariant. It follows that in N the level sets of P are submanifolds, therefore the condition of theorem 3.9 is satisfied and thus H is constant on the connected components of the level sets of $P|_N$. This can also be inferred from the fact, noted after definition 3.4, that for a g.o. space the reduced Hamiltonian is constant on the connected components of the reduced phase spaces. It is also obvious that the function h that appears in (37) is almost the same as the reduced Hamiltonian, i.e. the constant value of the reduced Hamiltonian H_μ at a fixed value of μ is equal to $h(\mu)$.

The formula $H = h \circ P$ always holds locally in N; if (o, p) is in N, then there exists an open neighborhood O of (o, p) in N so that in this neighbourhood H takes the form $H = h \circ P$, where h is a (locally) $Ad^*(G)$-invariant smooth function on $P(O)$. Furthermore, it follows from the proof of theorem 3.11, that $\xi : p' \mapsto dh(P(o, p'))$ is a smooth (locally) K-equivariant geodesic graph in an open neighbourhood of p in $T^*_o M$.

The following theorem is a converse of theorem 3.9.

Theorem 3.11 *Let $h : \mathfrak{g}^* \to \mathbb{R}$ be an $Ad^*(G)$-invariant function with the properties that $h \circ f^*$ is smooth and h is differentiable at the points of the image space of f^* (which is $f^*(T^*_o M)$). The Hamiltonian function defined as*

$$H = h \circ P \tag{38}$$

is G-invariant and has the g.o. property. The vector

$$dh(P(o,p)) = \frac{\partial h}{\partial g_n}(P(o,p))dg_n, \tag{39}$$

where the g_n are some linear coordinates on $\mathfrak{g}^*$, is a geodesic vector at $(o,p) \in T^*M$, thus the mapping

$$\xi = dh \circ f^* \ : T^*_o M \to \mathfrak{g}, \ p \mapsto dh(P(o,p)) \equiv (dh \circ f^*)(p) \tag{40}$$

is a K-equivariant geodesic graph.

Proof. We note that $P(o, p) = f^*(p)$, by definition. H is obviously G-invariant. The property that $h \circ f^*$ is smooth implies the smoothness of H. We have

$$dH = \frac{\partial h}{\partial g_n}\frac{\partial P_n}{\partial x_j}dx_j + \frac{\partial h}{\partial g_n}\frac{\partial P_n}{\partial p_j}dp_j, \tag{41}$$

where P_n are the components of P with respect to the coordinates g_n. This shows that at $(o, p) \in T^*M$ the vector $b \in \mathfrak{g}$ that has the components $\frac{\partial h}{\partial g_n}(P(o,p))$ has the property that $dH(o, p) = d(P|b)(o, p)$, thus the condition of proposition 3.5 is fulfilled. Clearly $\frac{\partial h}{\partial g_n}(P(o,p))$ are just the components of $dh(P(o, p))$ with respect to the coordinates g_n. □

It is also clear from the proof of theorem 3.11 that

Proposition 3.12 *If $h : \mathfrak{g}^* \to \mathbb{R}$ is an $Ad^*(G)$-invariant function, $H = h \circ P$ is a smooth Hamiltonian function and h is differentiable at $P(o, p)$ for some $p \in T^*_o M$, then $dh(P(o, p))$*

is a geodesic vector at (o, p)*.*

The condition imposed on h in theorem 3.11 could probably be weakened, in particular we do not expect that the differentiability of h in every point of $f^*(T_o^*M)$ is necessary for $h \circ P$ to be a g.o. Hamiltonian.

The following propositions 3.13 and 3.14 are about Riemannian g.o. spaces.

Proposition 3.13 *If* h *is a smooth* $Ad^*(G)$*-invariant function on* $\mathfrak{g}^*$ *and* $h \circ f^*$ *is a homogeneous positive definite quadratic polynomial, then* h *defines a naturally reductive space.*

Proof. h gives rise to a Riemannian metric, since $h \circ f^*$ is a homogeneous positive definite quadratic polynomial. If h is a smooth function on $\mathfrak{g}^*$, then the geodesic graph (40) is K-equivariant and is also smooth, however, according to the results of [19] (reformulated in the introduction of [4]), any K-equivariant geodesic graph of a not naturally reductive Riemannian g.o. space is not differentiable at $p = 0$. □

Proposition 3.14 *If* h *is a quadratic* $Ad^*(G)$*-invariant polynomial on* $\mathfrak{g}^*$ *and the polynomial* $h \circ f^*$ *is homogeneous, quadratic and nondegenerate, then* h *gives rise to a Riemannian or pseudo-Riemannian metric on* $M = G/K$*. On* T_o^*M *the quadratic polynomial that corresponds to the metric is* $h \circ f^*$*. The geodesic graph* $\xi : p \mapsto dh(P(o, p))$ *is linear in this case. If* $h \circ f^*$ *is positive definite, then the metric is Riemannian, and the linearity of the geodesic graph obviously implies that it is naturally reductive.*

It seems plausible to conjecture that the naturally reductive metrics are those which can be obtained from h functions that are quadratic polynomials.

In section 4 we discuss an example where h is a complicated function, nevertheless $h \circ f^*$ is a homogeneous quadratic polynomial and it is also positive definite, thus h still gives rise to a Riemannian metric on G/K. This metric has the g.o. property, but the geodesic graph is not linear and is not differentiable at $p = 0$, and the metric is not naturally reductive with respect to G, in accordance with the results of [19] (see also the introduction of [4]) mentioned in the proof of proposition 3.13.

As the example shows, in the Riemannian case the function h is not necessarily simple even though $H|_{T_oM} \equiv H_o = h \circ f^*$, and thus also $h|_{\mathfrak{m}^*}$, where $\mathfrak{m}^*$ is defined as $\mathfrak{m}^* = f^*(T_o^*M)$, is a quadratic polynomial. However, $h|_{\mathfrak{m}^*}$ is sufficient for determining H_o (since $H_o = h|_{\mathfrak{m}^*} \circ f^*$), and thus H. Therefore in order to specify a Riemannian g.o. space it is sufficient to specify the polynomial $h|_{\mathfrak{m}^*}$, for which we introduce the notation $h_o = h|_{\mathfrak{m}^*}$. The g.o. property implies that there is an open dense subset N_o of $\mathfrak{m}^*$ such that at any point $b \in N_o$ the derivative of h_o has to be zero in any direction $ad_a^*(b)$, where $a \in \mathfrak{g}$ is such that $ad_a^*(b) \in \mathfrak{m}^*$. That is to say, at any point $b \in N_o$ the equation

$$(dh_o(b) \mid ad_a^*(b)) = 0 \tag{42}$$

has to hold for all $a \in \mathfrak{g}$ for which $ad_a^*(b) \in \mathfrak{m}^*$. This equation can be used in practice for finding suitable h_o functions, i.e. for finding g.o. metrics or g.o. Hamiltonians, or to test whether a given metric or Hamiltonian function has the g.o. property. In terms of H_o, h_o is given as $h_o = H_o \circ (f^*)^{-1}$, of course.

The $Ad^*(K)$-invariance of h_o is necessary and sufficient for the G-invariance of the Hamiltonian function defined by h_o. If $a \in \mathfrak{k}$ and $b \in N_o$, then $ad^*_a(b) \in \mathfrak{m}^*$, thus (42) has to be satisfied. However, if h_o is $Ad^*(K)$-invariant, then (42) obviously holds if $a \in \mathfrak{k}$. The condition (42) is therefore interesting mainly for those elements a of $\mathfrak{g}$ which are not in $\mathfrak{k}$.

The construction of g.o. Hamiltonian functions as $H = h \circ P$ can be generalized in the following way.

Theorem 3.15 *Let M be a manifold and P a mapping $T^*M \to \mathfrak{g}^*$, where $\mathfrak{g}$ is a Lie algebra of a Lie group G, with the property $[X_{(P|a)}, X_{(P|b)}] = X_{(P|[a,b])}$ for all $a, b \in \mathfrak{g}$. Let h be a smooth $Ad^*(G)$-invariant function. The Hamiltonian function $H = h \circ P$ is G-invariant with respect to G in the sense that H is constant along the integral curves of $X_{(P|a)}$ for all $a \in \mathfrak{g}$. Any integral curve of X_H coincides with an integral curve of $X_{(P|a)}$ for some $a \in \mathfrak{g}$. In particular, the integral curve of X_H starting at the point $(x, p) \in T^*M$ coincides with the integral curve of $X_{(P|a)}$, where $a = dh(P(x, p))$, starting at (x, p).*

In a more general form of the theorem the condition that h should be smooth could be relaxed. Certain notable dynamical systems, for example the system of two pointlike bodies which interact by the Newtonian gravitational force (the Kepler problem) and the harmonic oscillator, admit a formulation in this framework with noncommutative groups G. Completely integrable systems can also be formulated in the framework of theorem 3.15 with commutative symmetry groups.

4 Example

In this section we discuss the example when $G = SU(3)$ and $K = SU(2)$ in order to give an illustration to the second part of section 3. This case was studied in [2], where the authors described a two-parameter family of invariant Riemannian g.o. metrics on $SU(3)/SU(2)$, of which only a one-parameter subfamily is naturally reductive with respect to $SU(3)$. Further results, in particular concerning the geodesic graph, were obtained in [4]. We note that these metrics belong to the type of g.o. metrics which are naturally reductive with respect to a suitable larger symmetry group [4].

The Lie algebras of $SU(3)$ and $SU(2)$ are the following:

$$su(3) = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$
$$su(2) = \mathfrak{k} = \mathrm{span}(A, B, C)$$
$$\mathfrak{m} = \mathrm{span}(E_1, E_2, E_3, E_4, Z)$$

$$
\begin{array}{lllll}
[A,B]=2C & [A,Z]=0 & [A,E_1]=-E_2 & [B,E_1]=E_3 & [C,E_1]=E_4 \\
[B,C]=2A & [B,Z]=0 & [A,E_2]=E_1 & [B,E_2]=E_4 & [C,E_2]=-E_3 \\
[C,A]=2B & [C,Z]=0 & [A,E_3]=E_4 & [B,E_3]=-E_1 & [C,E_3]=E_2 \\
 & & [A,E_4]=-E_3 & [B,E_4]=-E_2 & [C,E_4]=-E_1
\end{array}
$$

$$
\begin{array}{lll}
[Z,E_1]=E_2 & [E_1,E_2]=Z-\frac{1}{3}A & [E_2,E_4]=\frac{1}{3}B \\
[Z,E_2]=-E_1 & [E_1,E_3]=\frac{1}{3}B & [E_3,E_4]=Z+\frac{1}{3}A \\
[Z,E_3]=E_4 & [E_1,E_4]=\frac{1}{3}C & \\
[Z,E_4]=-E_3 & [E_2,E_3]=-\frac{1}{3}C. &
\end{array}
$$

There exists one (up to multiplication by a constant) quadratic homogeneous invariant polynomial on $su(3)$:

$$Y_1 = a'^2 + b'^2 + c'^2 + e_1^2 + e_2^2 + e_3^2 + e_4^2 + z^2, \tag{43}$$

where $a', b', c', e_1, e_2, e_3, e_4, z$ denote the coordinates corresponding to the basis vectors $A' = \frac{A}{\sqrt{3}}, B' = \frac{B}{\sqrt{3}}, C' = \frac{C}{\sqrt{3}}, E_1, E_2, E_3, E_4, Z$ of $su(3)$. Y_1 defines a positive definite Ad-invariant quadratic form on $su(3)$, allowing the identification of $su(3)$ and $su(3)^*$ and implying the equivalence of the coadjoint and adjoint actions of $SU(3)$. The basis $A', B', C', E_1, E_2, E_3, E_4, Z$ is orthonormal with respect to the quadratic form defined by Y_1. We use the same notation for the corresponding orthonormal basis in $su(3)^*$. Y_1 can now be taken as an invariant polynomial on $su(3)^*$ as well. f can be used to identify T_oM with $\mathfrak{m}$, and then the momentum mapping restricted to T_o^*M, i.e. f^*, is the trivial embedding $\mathfrak{m} \to \mathfrak{m} \oplus \mathfrak{k}$. The polynomial Y_1 composed with f^* thus takes the form

$$y_1 = Y_1 \circ f^* = e_1^2 + e_2^2 + e_3^2 + e_4^2 + z^2, \tag{44}$$

where we have introduced the notation y_1 for $Y_1 \circ f^*$. The metric on $SU(3)/SU(2)$ corresponding to y_1 is naturally reductive. In [2] it was found that the complete family of Riemannian g.o. metrics on $SU(3)/SU(2)$ is given on $T_o^*M \equiv \mathfrak{m}$ by

$$\alpha(e_1^2 + e_2^2 + e_3^2 + e_4^2) + \beta z^2, \qquad \alpha > 0,\ \beta > 0, \tag{45}$$

where α and β are real numbers. The metric (45) is naturally reductive if and only if $\alpha = \beta$ [2], which corresponds to $h = \alpha Y$. The family of polynomials (45) coincide with the complete family of positive definite $Ad^*(K)$-invariant quadratic homogeneous polynomials on $\mathfrak{m}$. It is not difficult to verify that the metrics (45) also satisfy the condition (42).

By solving the partial differential equations that express the $Ad^*(G)$-invariance of a function we find that the $Ad^*(G)$-invariant functions are of the form $G(Y_1, Y_2)$, where G is an arbitrary function of two variables and Y_2 is the homogeneous third order polynomial

$$Y_2 = \sqrt{3}\sigma_3 + z(\sigma_2 - 2\sigma_1) + \frac{2}{3}z^3, \tag{46}$$

where σ_1, σ_2 and σ_3 are the following $Ad^*(K)$-invariant polynomials:

$$\begin{aligned}
\sigma_1 &= a'^2 + b'^2 + c'^2 && (47)\\
\sigma_2 &= e_1^2 + e_2^2 + e_3^2 + e_4^2 && (48)\\
\sigma_3 &= a'(e_1^2 + e_2^2 - e_3^2 - e_4^2) + 2b'(e_1 e_4 - e_2 e_3) - 2c'(e_1 e_3 + e_2 e_4). && (49)
\end{aligned}$$

We have

$$y_2 = Y_2 \circ f^* = z(e_1^2 + e_2^2 + e_3^2 + e_4^2) + \frac{2}{3} z^3, \tag{50}$$

where the notation y_2 is introduced for $Y_2 \circ f^*$. In order to get the G function for which $G(Y_1, Y_2) \circ f^*$ equals (45) one has to solve the equations (44) and (50) for $e_1^2 + e_2^2 + e_3^2 + e_4^2$ and z. This involves the solution of a third order algebraic equation, therefore the result is a complicated formula that we do not write here. This example shows that the function h (which is $G(Y_1, Y_2)$ in the present case) can be complicated even though $h \circ f^*$ is a quadratic polynomial.

The geodesic graph can be calculated directly by solving the equations in lemma 3.3 or in lemma 2.4, as is done in [4] (it is the equation (15) that is actually used); it is not necessary for this to know h. The result, which can be found written explicitly below in equation (63) and in [4], has a relatively simple form. The geodesic graph can also be calculated from the formula $\xi = dh \circ f^*$, where the necessary derivatives of h can be determined from (42). As a third approach, one can utilize the knowledge of the invariant polynomials Y_1 and Y_2 to calculate $dh \circ f^*$. Here we calculate the geodesic graph in this way, using (44), (50) and (45). We have

$$d(G(Y_1, Y_2)) = \frac{\partial G}{\partial Y_1} dY_1 + \frac{\partial G}{\partial Y_2} dY_2, \tag{51}$$

thus we have to calculate the partial derivatives of G. (45), (44) and (50) can be written as

$$\begin{aligned}
G(y_1, y_2) &= \alpha r^2 + \beta z^2 && (52)\\
y_1 &= z^2 + r^2 && (53)\\
y_2 &= \frac{2}{3} z^3 + z r^2, && (54)
\end{aligned}$$

where

$$r^2 = e_1^2 + e_2^2 + e_3^2 + e_4^2. \tag{55}$$

We have

$$\begin{aligned}
\frac{\partial G}{\partial y_1} &= \frac{\partial G}{\partial r}\frac{\partial r}{\partial y_1} + \frac{\partial G}{\partial z}\frac{\partial z}{\partial y_1} && (56)\\
\frac{\partial G}{\partial y_2} &= \frac{\partial G}{\partial r}\frac{\partial r}{\partial y_2} + \frac{\partial G}{\partial z}\frac{\partial z}{\partial y_2}. && (57)
\end{aligned}$$

For $\frac{\partial G}{\partial r}$ and $\frac{\partial G}{\partial z}$ we obtain

$$\frac{\partial G}{\partial r} = 2\alpha r \qquad \frac{\partial G}{\partial z} = 2\beta z \tag{58}$$

from (52). The partial derivatives $\frac{\partial r}{\partial y_1}$, $\frac{\partial r}{\partial y_2}$, $\frac{\partial z}{\partial y_1}$ and $\frac{\partial z}{\partial y_2}$ can be calculated by taking partial derivatives of the equations (53) and (54) with respect to y_1 and y_2, and then solving the obtained four equations for $\frac{\partial r}{\partial y_1}$, $\frac{\partial r}{\partial y_2}$, $\frac{\partial z}{\partial y_1}$ and $\frac{\partial z}{\partial y_2}$. The result is

$$\frac{\partial r}{\partial y_1} = \frac{z^2}{r^3} + \frac{1}{2r} \qquad \frac{\partial r}{\partial y_2} = \frac{z}{r^3} \tag{59}$$

$$\frac{\partial z}{\partial y_1} = -\frac{z}{r^2} \qquad \frac{\partial z}{\partial y_2} = -\frac{1}{r^2}. \tag{60}$$

Taking into consideration (56) and (57) and using the results (58), (59) and (60) we obtain for $\frac{\partial G}{\partial y_1}$ and $\frac{\partial G}{\partial y_2}$ that

$$\frac{\partial G}{\partial y_1} = \alpha + (\alpha - \beta)\frac{2z^2}{r^2} \tag{61}$$

$$\frac{\partial G}{\partial y_2} = -(\alpha - \beta)\frac{2z}{r^2}. \tag{62}$$

dY_1 and dY_2 are straightforward to calculate, and the result for the geodesic graph is

$$\begin{aligned}[dG(Y_1, Y_2) \circ f^*](e_1E_1 + e_1E_2 + e_3E_3 + e_4E_4 + zZ) = \\ 2\alpha(e_1E_1 + e_2E_2 + e_3E_3 + e_4E_4) + 2\beta zZ \\ +(\beta - \alpha)\frac{2\sqrt{3}z}{r^2}[(e_1^2 + e_2^2 - e_3^2 - e_4^2)A' \\ +2(e_1e_4 - e_2e_3)B' - 2(e_1e_3 + e_2e_4)C'],\end{aligned} \tag{63}$$

which agrees with the result obtained in [4], if we take into consideration the differences between the definitions in this paper and in [4]. One difference that is worth noting is that in [4] the geodesic graph is defined in such a way that only the $\mathfrak{k}$-component is kept, i.e. the obvious $2\alpha(e_1E_1 + e_2E_2 + e_3E_3 + e_4E_4) + 2\beta zZ$ part is subtracted.

(63) is well defined on an open dense subset of T_o^*M, but it does not have well-defined values at $r = 0$ if $\alpha \neq \beta$. It can be verified using (27) and (28) that at zZ (i.e. when $r = 0$) all vectors $2\beta zZ + aA' + bB' + cC'$, $a, b, c \in \mathbb{R}$, are geodesic vectors.

Several other examples of Riemannian g.o. spaces can be found in the literature (see e.g. [2, 4, 3]), which would also be interesting to discuss in a similar way.

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