INTEGRABLE SYSTEMS AND LOCAL FIELDS

A. N. Parshin

To the memory of
Alexey Ivanovich Kostrikin

In 70's there was discovered a construction how to attach to some algebraic-geometric data an infinite-dimensional subspace in the space $k((z))$ of the Laurent power series. The construction was successfully used in the theory of integrable systems, particularly, for the KP and KdV equations [12, 26]. There were also found some applications to the moduli of algebraic curves [2, 4]. Now it is known as the Krichever correspondence or the Krichever map [2, 17, 1, 24, 5]. The original work by I. M. Krichever has also included commutative rings of differential operators as a third part of the correspondence.

The map we want to study here was first described in an explicit way by G. Segal and G. Wilson [26]. They have used an analytical version of the infinite dimensional Grassmanian introduced by M. Sato [25, 23]. In the sequel we consider a purely algebraic approach as developed in [17].

Let us just note that the core of the construction is an embedding of the affine coordinate ring on an algebraic curve into the field $k((z))$ corresponding to the power decompositions in a point at infinity (the details see below in section 2). In number theory this corresponds to an embedding of the ring of algebraic integers to the fields $\mathbb{C}$ or $\mathbb{R}$. The latter one is well known starting from the XIX-th century. The idea introduced by Krichever was to insert the local parameter z. This trick looking so simple enormously extends the area of the correspondence. It allows to consider all algebraic curves simultaneously.

But there still remained a hard restriction by the case of curves, so by dimension 1. Recently, it was pointed out by the author [21, 22] that there are some connections between the theory of the KP-equations and the theory of n-dimensional local fields [19], [7]. From this point of view it becomes clear that the Krichever construction should have a generalization to the case of higher dimensions. This generalization is suggested in the paper for the case of algebraic surfaces (see theorem 4 in section 4). A further generalization to the case of arbitrary dimension was recenly proposed by D. V. Osipov [18].

We start with description of a connection between the KP hierarchy in the Lax form and the vector fields on infinite Grassmanian manifolds (section 1). These results are known but we prove them here in more transparent and simple way (we have used basically [17] and [15]). In appendix 1 we remind how to get the standard KP and KdV equations from the Lax operator form. In appendix 2 we ouline a construction of the semi-infinite monomes for the field $k((z))$ which is an important part of the theory of Sato Grassmanian.

An important feature of all these considerations is their purely algebraic character. Everything can be done over an arbitrary field k of characteristic zero[1].

Let us also note that the construction of the restricted adelic complex in section 3 is of an independent interest, also in arithmetics. It has already appeared in a description of vector bundles on algebraic surfaces [20].

This work was partly done during my visit to the Institut für Algebra und Zahlentheorie der Technische Universität Braunschweig in 1999. I am very much grateful to Professor Hans Opolka for the hospitality[2].

1 Sato correspondence for dimension 1

This correspondence connects two seemingly distant objects: infinite Grassmanian manifolds and rings of pseudo-differential operators.

Let $K = k((z))$ be the field of Laurent power series with filtration $K(n) = z^n k[[z]]$. If $V \cong k((z))$ is a vector space of dimension 1 over K then we can choose a filtration $V(n)$ such that $K(n)V(m) \subset V(n+m)$. Let $V_1 := V(0)$.

[1]Further developement of different aspects of the Krichever correspondence for dimension two see in [27, 13, 14]

[2]The text was published in *Communications in Algebra*, 29(2001), No.9, 4157-4181. This version includes a corrected proof of the proposition 2. I'm grateful to Alexander Zheglov for the correction of a mistake in the original proof. Also, we include some additional remarks on the deduction of concrete equations from the Lax hierarchy and appendix 2.

Denote by $\mathrm{Gr}(V)$ the set of the subspaces W in V such that the complex

$$W \oplus V_1 \to V \tag{1}$$

is a Fredholm one. It is a (infinite-dimensional) projective variety with connected components marked by the Euler characteristic of the complex (1) [1, 11](see also, appendix 2).

Let us now introduce the ring $E = k[[x]]((\partial^{-1}))$ of formal pseudo-differential operators with coefficients from the ring $k[[x]]$ of regular formal power series as the left $k[[x]]$-module of all expressions

$$L = \sum_{i>-\infty}^{n} a_i \partial^i, \ a_i \in k[[x]].$$

Then a multiplication can be defined according to the Leibnitz rule:

$$(\sum_i a_i \partial^i)(\sum_j b_j \partial^j) \sum_{i,j;k\geq 0} \binom{i}{k} a_i d^k(b_j) \partial^{i+j-k}.$$

Here we put

$$\binom{i}{k} = \frac{i(i-1)\dots(i-k+1)}{k(k-1)\dots 1}, \text{ if } k>0, \ \binom{i}{0} = 1$$

and d is the derivation by x.

Particularly, for $f \in k[[x]]$:

$$[\partial, f] = \partial f - f\partial = d(f),$$

(Heisenberg commutative relation),

$$[\partial^{-1}, f] = \partial^{-1} f - f\partial^{-1} - d(f)\partial^{-2} + d^2(f)\partial^{-3} - \dots .$$

It can be checked that $k[[x]]((\partial^{-1}))$ will be an associative ring (the details see in [22]).

There is a decomposition

$$E = E_+ + E_-,$$

where $E_- = \{L \in E : L = \sum_{n<0} a_n \partial^n\}$ and E_+ consists of the operators containing only ≥ 0 powers of ∂. The elements from $E_+ =: D$ are the differential operators and the elements from E_- are the Volterra operators.

From the commutation relations we see that $E = k((\partial^{-1})) \oplus Ex$ and thus the map $E \to E/Ex = V$ (we identify the image of ∂^{-1} with z) defines a linear action of the ring E on V and also on $\mathrm{Gr}(V)$. The subspace V_1 is transformed by the action of operators from E into a subspace V' commesurable with V_1 (it means that the quotient $V' + V_1/V' \cap V_1$ is of finite dimension). Thus the Fredholm condition from the definition of Grassmanian manifold will be preserved.

We will call the map $E \to V$ by the Sato map.

Proposition 1 *(*LEMMA ON A STABILIZER*). Let $P \in E$ and $W_0 = k[z^{-1}] \in Gr(V)$ if $V = k((z))$.*

Then $PW_0 \subset W_0$ if and only if $P \in D$.

PROOF. Since W_0 is equal to the image of E_+ under the Sato map (by lemma 1 below) we can replace the first condition by the following one

$$P \cdot E_+ \subset E_+ + Ex \tag{2}$$

and work in the ring E. Since E_+ is a ring we have $P \cdot E_+ \subset E_+$ for $P \in E_+ = D$.

Now assume (1) and decompose P as $P = P_+ + P_-$. We will also use the notation $L \sim M$ for $L, M \in E$ such that $L - M \in Ex$.

Lemma 1 . *In the ring E, $x^n \partial^n \sim c_n$ where $c_n = (-1)^n n!$.*

PROOF. By commutation relations $x^n\partial^n = \partial^n x^n +$(some coefficient)$\partial^{n-1}x^{n-1} + \ldots + c_n$ and applying this operator to x^{-1} we get the value of c_n.

Returning to our proposition we see that (2) implies

$$P_-E_+ \in E_+ + E_-x. \tag{3}$$

To prove that $P_- = 0$ it is enough to show $P_- \in E_-x^n$ for all $n \geq 1$.

First, $1 \in E_+$ and consequently $P_- \in E_+ + E_-xE_+ \oplus E_-x$. We see that $P_- \in E_-x$.

Proceeding by induction we assume $P_- = Q_-x^n$ with $Q_- \in E_-$. We have

$$Q_-x^n\partial^n \in E_+ + E_-x, \text{ by (3)}$$

$$Q_-x^n\partial^n \in Q_-Ex + Q_- \subset Ex + Q_-, \text{ by lemma.}$$

Taking all together we get $Q_- \in E_+ + Ex \subset E_+ \oplus E_-x$. It means $Q_- \in E_-x$ and $P_- \in E_-x^{n+1}$.

Proposition 2 *(*TRANSITIVITY THEOREM*). Let $W \in Gr(V)$ and $W \oplus V_1 = V$. Then there exists an unique operator $S \in 1 + E_-$ such that $W = S^{-1}W_0$.*

PROOF. If W satisfies the conditions of the theorem then W is a union of subspaces $W \cap V_{-n}$ and one can choose basis w_n in W such that $w_n = z^{-n} + v_n,\ v_n \in V_1$. We want to construct an operator $S = 1 + P \in 1 + E_-$ such that $Sz^{-n} = w_n + c_0 w_0 + \ldots + c_{n-1}w_{n-1}$ for some $c_0, \ldots c_{n-1} \in k$, or $Pz^{-n} = v_n + c_0 w_0 + \ldots + c_{n-1}w_{n-1}$ for all $n \geq 0$.

The Sato map $E \to V$ transforms ∂^n into z^{-n} and we can find $Q_n \in E_-$ such that Q_n goes to v_n by the map. Thus in order to construct our S from the conclusion of the theorem we have to find an operator $P \in E_-$ such that

$$P\partial^n = Q_n + c_0(1 + Q_0) + \ldots + c_{n-1}(\partial^{n-1} + Q_{n-1}) + Ex, \tag{4}$$

where Q_n is a given sequence of operators from E_-. From now on we work in the ring E. Put $P_0 = Q_0$.

CLAIM. For $n \geq 1$ there exist operators $P_n \in E_-$ such that

i) $P_n\partial^n + P_0\partial^n + \ldots + P_{n-1}\partial^n \in Q_n + c_0(1 + Q_0) + \ldots + c_{n-1}(\partial^{n-1} + Q_{n-1}) + Ex,$ for some $c_0, \ldots, c_{n-1} \in k$

ii) $P_n, P_n\partial, \ldots, P_n\partial^{n-1} \in Ex$

iii) $P_n \in Ex^n$

First, we show that the claim implies the existence of the P with the property (4). We put $P = P_0 + P_1 + \ldots$. Then

$$\begin{array}{rcl} P_0\partial^n + P_1\partial^n + \ldots + P_n\partial^n + P_{n+1}\partial^n + \ldots & \sim & \text{(by ii)} \\ P_0\partial^n + P_1\partial^n + \ldots + P_n\partial^n & \sim & \text{(by i)} \\ Q_n + c_0(1 + Q_0) + \ldots + c_{n-1}(\partial^{n-1} + Q_{n-1}). & & \end{array}$$

Here we use the notation $\sim$ from the lemma 1. The property iii) implies the convergence of the series for P by the following

Lemma 2 . *In the ring E, the series $\sum_{n\geq 0} P_n, P_n \in E_-$ will converge if $P_n \in E_- x^n$.*

Now we prove the claim and then the lemma.

We can define P_n by induction. Obviously, by induction we have

$$P_0\partial^n + \ldots + P_{n-1}\partial^n \in c_0(1 + Q_0) + \ldots + c_{n-1}(\partial^{n-1} + Q_{n-1}) + Ex + E_-$$

for some $c_0, \ldots, c_{n-1} \in k$. So, the image of the operator $P_0\partial^n + \ldots + P_{n-1}\partial^n - c_0(1 + Q_0) + \ldots + c_{n-1}(\partial^{n-1} + Q_{n-1})$ under the Sato map lies in V_1 and we can find an operator $Q'_n \in E_-$ such that Q'_n goes to this image by the map.

Then we can take $P_n = (Q_n - Q'_n)x^n \in E_-$. It gives iii) and

$$P_n\partial^n = (Q_n - Q'_n)x^n\partial^n \sim Q_n - Q'_n \sim$$

$$Q_n + c_0(1 + Q_0) + \ldots + c_{n-1}(\partial^{n-1} + Q_{n-1}) - P_0\partial^n - \ldots - P_{n-1}\partial^n$$

by lemma 1 and

$$P_n\partial^k = Q'_n x^n \partial^k = Q'_n x^{n-k} x^k \partial^k \sim Q'_n x^{n-k} \sim 0,\ k = 0, 1, \ldots, n-1$$

again by lemma 1 and we are done.

The uniqueness of the operator $S \in 1 + E_-$ such that $W = S^{-1}W_0$ follows from proposition 1. Indeed, let $SW_0 = W_0$. Then S must belong to E_+ and thus $S = 1$.

Proof of Lemma 2. Let $P_n = P'_n x^n$ where

$$P'_n = \sum_{m \geq 0} a_m^{(n)} \partial^{-m} \text{ and } P_n = \sum_{m \geq 0} b_m^{(n)} \partial^{-m}.$$

We see $b_m^{(n)} = a_m^{(n)} x^n \pm a_{m-1}^{(n)} d(x^n) \pm a_{m-2}^{(n)} d^2(x^n) \pm \ldots$ and for every m the series $\sum_{n \geq 0} b_m^{(n)}$ will converge in $k[[x]]$.

If $W \in \mathrm{Gr}(V)$ then we have $T_W = \mathrm{Hom}(W, V/W)$ for the tangent space in the point W and there is a natural map $\mathrm{Hom}(V, V) \to T_W$. For $n \in \mathbb{Z}$ we can define a vector field T_n on $\mathrm{Gr}(V)$. It is equal to the image of the multiplication operator by z^{-n} in the space V.

The KP hierarchy is a dynamical system defined on an affine space $E' := \partial + E_-$. The tangent space to any point $L \in E'$ is canonically equal to E_-.

Definition 1. The n-th vector field of the KP hierarchy on E' is defined as $KP_n = [(L^n)_+, L]$.

Since $(L^n)_+ = L^n - (L^n)_-$ the field KP_n belongs to E_-.

The set $G = 1 + E_-$ is a group carrying the vector fields $-(S\partial^n S^{-1})_- S$. The tangent space to any point from G is again E_-.

At last we denote by Gr_+ the big cell from the Grassmanian manifold,

$$Gr_+(V) = \{W \in Gr(V) : W \oplus V(1) = V\}. \tag{5}$$

All these spaces, E', G, Gr_+ have the distinguished points: $\partial, 1, W_0 = k[z^{-1}]$(if $V = k((z))$).

Theorem 1 *(*Sato correspondence*). The maps*

$$E' \overset{\varphi}{\leftarrow} G \overset{\psi}{\rightarrow} Gr_+(V),$$

where $\varphi(S) = S\partial S^{-1}, \psi(S) = S^{-1}(W_0)$ *have the following properties*

i) for any $S \in G$, $L = \varphi(S) \in E'$ *and* $W = \psi(S) \in Gr_+$ *the diagram*

$$\begin{array}{ccccc}
\text{‘}T_L & \overset{d\varphi_S}{\leftarrow} & T_S & \overset{d\psi_S}{\rightarrow} & T_W \\
 & & & & \| \\
\| & & \| & & Hom(W, V/W) \\
 & & & & \uparrow \\
E_- & \overset{\varphi'}{\leftarrow} & E_- & \overset{\psi'}{\rightarrow} & Hom(V,V)
\end{array}$$

commutes. Here $d\varphi_S$ *and* $d\psi_S$ *are jacobian maps of the maps* φ, ψ *on the tangent space* T_S*, and*

$$\varphi'(A) = [AS^{-1}, L],$$

$$\psi'(A) = -S^{-1}A \text{ acting on } V \text{ by the Sato map}$$

with $A \in E_-$*.*

ii) for any $S \in G$, $L = \varphi(S) \in E'$ *and* $W = \psi(S) \in Gr_+$ *and any* $n \geq 1$

$$d\varphi_S(-(S\partial^n S^{-1})_- S) = KP_n,$$

$$d\psi_S(-(S\partial^n S^{-1})_- S) = T_n$$

Proof. First we consider the left hand side of the diagram. If $A \in E_- = T_S$ and $R = S + A$ is an infinitely small deformation of S then up to the higher powers of A

$$R = S(1 + S^{-1}A) = (1 + AS^{-1})S,$$

$$R^{-1} = (1 - S^{-1}A)S^{-1} = S^{-1}(1 - AS^{-1}),$$

$$R\partial R^{-1} = (1 + AS^{-1})S\partial S^{-1}(1 - AS^{-1})$$

$$(S\partial S^{-1} + A\partial S^{-1})(1 - AS^{-1})$$

$$S\partial S^{-1} + A\partial S^{-1} - S\partial S^{-1}AS^{-1}$$

$$L + [AS^{-1}, L] = L + d\varphi_S(A).$$

Using this result we can check up the statement on the vector fields:

$$d\varphi_S(-(S\partial^n S^{-1})_- S) = [-(S\partial^n S^{-1})_- S S^{-1}, L] = [(L^n)_+, L].$$

It remains to consider the right hand side of the diagram. Let $W = S^{-1}W_0$ as in the diagram and let $R = S + A$ be as above. Then $W' : R^{-1}W_0$ and we get

$$W' = R^{-1}SS^{-1}W_0 = R^{-1}SW = (1 - S^{-1}A)S^{-1}SW = (1 - S^{-1}A)W.$$

Since W, W' are two spaces from the big cell

$$W \oplus V(1) = V,$$

$$W' \oplus V(1) = V$$

and the space W' defines a linear map

$$W \to V \to V/W' \overset{\sim}{\leftarrow} V(1) \overset{\sim}{\to} V/W$$

which is an element of the tangent space T_W corresponding to the deformation W' of W (see, for example, [11]). It is easy to see that for our space W' the linear map will coincide with the action of operator $-S^{-1}A$ through the Sato map.

The last step of the proof is to check that $d\psi_S$ takes $-(S\partial^n S^{-1})_- S$ into T_n. But we have

$$-S^{-1}(-(S\partial^n S^{-1})_- S) = S^{-1}L^n S - S^{-1}(L^n)_+ S\partial^n - S^{-1}(L^n)_+ S$$

and we have to show that the second term is trivial in T_W.

This can be seen from the commutative diagram

$$\begin{array}{ccccccccc} & W_0 & \to & V & \to & V & \to & V/W_0 \\ S^{-1} & \downarrow & S^{-1} & \downarrow & S^{-1} & \downarrow & S^{-1} & \downarrow \\ & W & \to & V & \to & V & \to & V/W \end{array}$$

The bottom map from V to V is equal to the Sato image of $S^{-1}(L^n)_+S$ and the corresponding top horisontal map is the Sato action of the operator $(L^n)_+$. By the proposition 1 the last one is trivial in $T_{W_0} = Hom(W_0, V/W_0)$.

The theorem is proved.

REMARK 1. The maps φ and ψ can be deduced from the corresponding actions of the group G on the manifolds E' and $Gr(V)$. One has to consider the actions on the orbits going through ∂ and W_0, respectively. The first orbit is one of the co-adjoint orbits for the (infinite dimensional) Lie group G.

Corollary 1 . *The Sato correspondence induces the diagram of bijections*

$$E' \stackrel{\varphi}{\leftarrow} G/G_0 \stackrel{\psi}{\rightarrow} Gr_+(V)/k[[z]]^*,$$

where $G_0 = G \cap k((\partial^{-1}))$ *and the action of* $k[[z]]^*$ *on* $Gr(V)$ *is defined by the module structure on* V *over* K.

Proof for the map ψ easily follows from the definitions, theorem 1 and proposition 2. To check up the bijectivity of the map φ one has to apply theorem 1 from [22].

2 Krichever correspondence for dimension 1

We first discuss the adelic complexes for the case of dimension 1. Concerning a definition of the adelic notions we refer to [7],[10]. We also note that the sign $\prod$ denotes the adelic product.

Let C be an projective algebraic curve over a field k, P be a smooth point and η a general point on C. For every point (in Grothendieck's sense) $\alpha \in C$ we have a field K_α. K_α is a quotient ring of the completed local ring $\hat{\mathcal{O}}_\alpha$ of the point α. It is a 1-dimensional local field.

Let $\mathcal{F}$ be a torsion free coherent sheaf on C. We denote by $\hat{\mathcal{F}}_\alpha = \mathcal{F} \otimes \hat{\mathcal{O}}_\alpha$ the completed fiber at the point $\alpha \in C$. The fiber $\mathcal{F}_\eta$ at a general point is also the space of all rational sections of the sheaf $\mathcal{F}$.

Proposition 3 . *The following complexes are quasi-isomorphic:*

i) adelic complex

$$\mathcal{F}_\eta \oplus \prod_{x \in C} \hat{\mathcal{F}}_x \longrightarrow \prod_{x \in C} (\hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_x} K_x)$$

ii) the complex

$$W \oplus \hat{\mathcal{F}}_P \longrightarrow \hat{\mathcal{F}}_P \otimes_{\hat{\mathcal{O}}_P} K_P$$

where $W = \Gamma(C - P, \mathcal{F}) \subset \hat{\mathcal{F}}_\eta$.

Proof will be done in two steps. First, the adelic complex contains a trivial exact subcomplex

$$\prod_{x \in U} \hat{\mathcal{F}}_x \longrightarrow \prod_{x \in U} \hat{\mathcal{F}}_x,$$

where $U = C - P$. The quotient-complex is equal to

$$\mathcal{F}_\eta \oplus \hat{\mathcal{F}}_P \longrightarrow \prod_{x \in U} (\hat{\mathcal{F}}_x \otimes K_x)/\hat{\mathcal{F}}_x \oplus \hat{\mathcal{F}}_P.$$

It has a surjective homomorphism to the exact complex

$$\mathcal{F}_\eta / W \longrightarrow \prod_{x \in U} (\hat{\mathcal{F}}_x \otimes K_x)/\hat{\mathcal{F}}_x.$$

The exactness of the complex is the strong approximation theorem for the curve C [6][ch.II, §3, corollary of prop. 9; ch. VII, §, prop. 2]. The kernel of this surjection will be the second complex from proposition.

Let us now explain the Krichever correspondence for dimension 1.

Definition 2.

$\mathcal{M}_1$	$:=$	$\{C, P, z, \mathcal{F}, e_P\}$
C		projective irreducible curve $/k$
$P \in C$		a smooth point
z		formal local parameter at P
$\mathcal{F}$		torsion free rank r sheaf on C
e_P		a trivialization of $\mathcal{F}$ at P

Independently, we have the field $K = k((z))$ of Laurent power series with filtration $K(n) = z^n k[[z]]$. Let $K_1 := K(0)$. If $V = k((z))^{\oplus r}$ then $V(n) = K(n)^{\oplus r}$ and $V_1 := V(0)$.

Theorem 2 *[17]. There exists a canonical map*

$$\Phi_1 : \mathcal{M}_1 \longrightarrow \{\textit{vector subspaces } A \subset K, W \subset V\}$$

such that

i) the cohomology of complexes

$$A \oplus K_1 \longrightarrow K, \; W \oplus V_1 \longrightarrow V$$

are isomorphic to $H^{\cdot}(C, \mathcal{O}_C)$ and $H^{\cdot}(C, \mathcal{F})$, respectively

ii) if $(A, W) \in Im\, \Phi_1$ then $A \cdot A \subset A, A \cdot W \subset W$,

iii) if $m, m' \in \mathcal{M}_1$ and $\Phi_1(m)\Phi_1(m')$ then m is isomorphic to m'

Proof. If $m = (C, P, z, \mathcal{F}, e_P) \in \mathcal{M}_1$ then we put

$$A := \Gamma(C - P, \mathcal{O}_C),$$
$$W := \Gamma(C - P, \mathcal{F}).$$

Also we have by the choice of z and e_P

$$\hat{\mathcal{O}}_P = k[[z]],\ K_P = k((z)),$$
$$\mathcal{F}_P = \mathcal{O}_P e_P = \mathcal{O}_P^{\oplus r},\ \hat{\mathcal{F}}_P = \hat{\mathcal{O}}_P^{\oplus r}.$$

This defines the point $\Phi_1(m) \in \mathcal{M}_1$. Indeed, for the subspace W we have the following canonical identifications

$$\Gamma(C - P, \mathcal{F}) \subset \mathcal{F}_\eta \otimes_{\mathcal{O}_P} K_P = \hat{\mathcal{F}}_P \otimes K_P = \hat{\mathcal{O}}_P^{\oplus r} \otimes K_P = k((z))^{\oplus r}.$$

The same works for the subspace A.

The property ii) is obvious, the property i) follows from the proposition 3. To get iii) let us start with a point $\Phi_1(m) = (A, W)$. The standard valuation on K gives us increasing filtrations $A(n) = A \cap K(n)$ and $W(n) = W \cap V(n)$ on the spaces A and W. Then we have

$$\begin{array}{rcl} C - P & = & \mathrm{Spec}(A), \\ C & = & \mathrm{Proj}(\oplus_n A(n)), \\ \mathcal{F} & = & \mathrm{Proj}(\oplus_n W(n)), \end{array}$$

by lemma 9. Thus we can reconstruct the quintiple m from the point $\Phi_1(m)$.

Remark 2. It is possible to replace the ground field k in the Krichever construction by an arbitrary scheme S, see [24].

Using the Krichever correspondence Φ_1 one can construct the integral varieties in $X = \mathrm{Gr}(V)/k[[z]]^*$ for the vector fields T_n (see section 1). Let us fix C, P, z. Then the image $\Phi_1(C, P, z, \mathcal{F}, e_P)$ does not depend on e_P in X and will run through the generalized jacobian $\mathrm{Jac}(C)$ of the curve C when we vary the invertible sheaf $\mathcal{F}$.

To show this fact we consider the commutative diagram

$$\begin{array}{ccccccccc} & & & & k[[z]] & = & k[[z]] & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & A_W & \rightarrow & K & \stackrel{\alpha}{\rightarrow} & Hom(W, V/W) & = & T_{Gr,W} \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & A_W & \rightarrow & K/k[[z]] & \rightarrow & T_{X,W} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Here α is the action of K on V by multiplications, $W \in Gr(V)$ is the second space from the image $\Phi_1(C, P, z, \mathcal{F}, e_P)$ and $A_W = \{f \in K : fW \subset W\}$.

The diagram explains what happens with tangent spaces when we go from the Grassmanian manifold to it's quotient X by the group $k[[z]]^*$. The space $k[[z]]$ is the Lie algebra of the last group.

For the case of invertible sheaf $\mathcal{F}$ we know that $A_W = A$ (see [26][n 6]). Thus we get the exact sequence

$$A + k[[z]] \rightarrow k((z)) \rightarrow T_{X,W},$$

where $k((z))/A + k[[z]] = H^1(C, \mathcal{O}_C)$ is the tangent space to the generalized jacobian of the curve C. From the sequence we conclude that all vector fields T_n belong to the image of $H^1(C, \mathcal{O}_C)$ in $T_{X,W}$ and consequently they are tangent to the image of generalized jacobian $\mathrm{Jac}(C)$ of the curve C under the map Φ_1 (see details in [17]).

This result works for the component of $Gr(V)$ ($V = k((z))$) containing the subspace W such that $W \oplus k[[z]] = V$. The image of the Sato correspondence belongs to another component containing the space W_0 and the big cell $Gr_+(V)$ (see (5), section 1).

The multiplication by z transforms one component onto another preserving the vector fields T_n. Thus we get the integral varieties for the KP flow which are abelian varieties for smooth curves C. If $k = \mathbb{C}$ they are topological toruses with the movement along the straight lines.

This explain why the KP system can be considered to some extent as an integrable one. But we must have in mind that there is a lot of points W in $Gr(V)$ where the dynamical behavior is quite different. Take, for example, the points with $A_W = k$.

If C is a projective line with a double point then $\mathrm{Jac}(C) = k^*$ and we get a 1-soliton solution of the KP equation.

We see that local fields enters into this picture in essential way. The ring E is a subring of 2-dimensional local skew-field $P = k((x))((\partial^{-1}))$ (see [22]). The fields $k((z))$ and K_P are 1-dimensional local fields. The complex (1)(section 1) from definition of the Grassmanian manifold intimately connected with adelic complex on an algebraic curve (property i) from Theorem 2).

We suggest that analogous constructions should exist for higher dimensions as well (concerning the rings of PDO see [11]). Here we consider a generalization of the Krichever map Φ_1 to the case of algebraic surfaces.

3 Adelic complexes in dimension 2

Let X be a projective irreducible algebraic surface over a field k, $C \subset X$ be an irreducible projective curve, and $P \in C$ be a smooth point on both C and X. Let $\mathcal{F}$ be a torsioh free coherent sheaf on X.

We remind the definition of the standard rings attached to a pair $x \in D \subset X$ on the surface X. Here the D is an irreducible divisor. It correspond to an ideal $\wp \subset \mathcal{O}_x$ in the local ring of the point x. First we apply localization by $\wp$ to $\hat{\mathcal{O}}_x$ and then take a completion by the ideal $\wp$. We get a ring $\mathcal{O}_{x,D}$ which is a complete discrete valuation ring if the point x is smooth on both D and X. The local field $K_{x,D}$ is a quotient ring of the $\mathcal{O}_{x,D}$. It is really a field in the smooth case. In this case we have $\mathcal{O}_{x,D} = k((u))[[t]],\ K_{x,D} = k((u))((t))$ if $\hat{\mathcal{O}}_x = k[[u,t]]$ and $\wp = (t)$. The field $K_{x,D}$ is an example of 2-dimensional local field.

There are some rings attached to the point x and the divisor D. Denote by K the field of rational functions on the X. Let $K_x K\hat{\mathcal{O}}_x \subset$ a quotient ring of the local ring $\hat{\mathcal{O}}_x$. If $\hat{\mathcal{O}}_D$ is a local ring of the divisor D then let K_D be it's quotient ring. In the smooth case $\hat{\mathcal{O}}_D = k(D)[[t]],\ K_D = k(D)((t))$.

Definition 3. Let $x \in C$. We let

$$B_x(\mathcal{F}) = \bigcap_{D \neq C} ((\hat{\mathcal{F}}_x \otimes K_x) \cap (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D})),$$

where the intersection is done inside the group $\hat{\mathcal{F}}_x \otimes K_x$,

$$B_C(\mathcal{F}) = (\hat{\mathcal{F}}_C \otimes K_C) \cap (\bigcap_{x \neq P} B_x),$$

where the intersection is done inside $\hat{\mathcal{F}}_x \otimes K_{x,C}$,

$$A_C(\mathcal{F}) = B_C(\mathcal{F}) \cap \hat{\mathcal{F}}_C,$$

$$A(\mathcal{F}) = \hat{\mathcal{F}}_\eta \cap (\bigcap_{x \in X - C} \hat{\mathcal{F}}_x).$$

We will freely use the following shortcuts:

$$\begin{aligned} K\hat{\mathcal{F}}_x &= \hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_x} K_x, \\ K\hat{\mathcal{F}}_D &= \hat{\mathcal{F}}_D \otimes_{\hat{\mathcal{O}}_D} K_D, \\ \hat{\mathcal{F}}_{x,D} &= \hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_x} \mathcal{O}_{x,D}, \\ K\hat{\mathcal{F}}_{x,D} &= \hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_x} K_{x,D}. \end{aligned}$$

Next, we need two lemmas connecting the adelic complexes on X and C. They are the versions of the relative exact sequences, see [19], [7]. The curve C defines the following ideals:

$$K_{x,C} \supset \hat{\mathcal{O}}_{x,C} \ldots \supset \wp^n_{x,C} \supset \ldots,$$

$$K_C \supset \hat{\mathcal{O}}_C \ldots \supset \wp^n_C \supset \wp^{n+1}_C \supset \ldots,$$

$$K_x \supset \hat{\mathcal{O}}_x \ldots \supset \wp^n_x \ldots,$$

and $\wp_x = \hat{\mathcal{O}}_x \cap \wp_{x,C}$.

Lemma 3 . *We assume that the curve C is a locally complete intersection. Let $N_{X/C}$ be the normal sheaf for the curve C in X. For all $n \in \mathbb{Z}$ the maps*

$$\prod_{x \in C} \wp^n_{x,C} \hat{\mathcal{F}}_{x,C} / \wp^{n+1}_{x,C} \hat{\mathcal{F}}_{x,C} \longrightarrow \mathbb{A}_{C,01}(\mathcal{F} \otimes \check{N}^{\otimes n}_{X/C}),$$

$$\prod_{x,C} \wp^n_x \hat{\mathcal{F}}_x / \wp^{n+1}_x \hat{\mathcal{F}}_x \longrightarrow \mathbb{A}_{C,1}(\mathcal{F} \otimes \check{N}^{\otimes n}_{X/C}),$$

$$\wp^n_C \hat{\mathcal{F}}_C / \wp^{n+1}_C \hat{\mathcal{F}}_C \longrightarrow \mathbb{A}_{C,0}(\mathcal{F} \otimes \check{N}^{\otimes n}_{X/C}),$$

are bijective.

In general, we have an exact sequence

$$0 \longrightarrow \mathcal{J}^{n+1} \longrightarrow \mathcal{J}^n \longrightarrow \mathcal{J}^n|_C \longrightarrow 0$$

where $\mathcal{J} \subset \mathcal{O}_X$ is an ideal defining the curve C. In our case $\mathcal{J} = \mathcal{O}_X(-C)$ and $N_{X/C} = \mathcal{O}_X(C)|_C$. Thus the isomorphisms from the lemma are coming from the exact relative sequence

$$0 \longrightarrow \mathbb{A}_X(\mathcal{F}(-(n+1)C)) \longrightarrow \mathbb{A}_X(\mathcal{F}(-nC)) \longrightarrow \mathbb{A}_C(\mathcal{F}(-nC)|_C) \longrightarrow 0.$$

Lemma 4 . *Let $P \in C$. For all $n \in \mathbf{Z}$ the complex*

$$\wp^n_C \hat{\mathcal{F}}_C / \wp^{n+1}_C \hat{\mathcal{F}}_C \oplus \prod_{x \in C} \wp^n_x \hat{\mathcal{F}}_x / \wp^{n+1}_x \hat{\mathcal{F}}_x \longrightarrow \prod_{x \in C} \wp^n_{x,C} \hat{\mathcal{F}}_{x,C} / \wp^{n+1}_{x,C} \hat{\mathcal{F}}_{x,C}$$

is quasi-isomorphic to the complex

$$(A_C(\mathcal{F}) \cap \wp^n_C \hat{\mathcal{F}}_C)/(A_C(\mathcal{F}) \cap \wp^{n+1}_C \hat{\mathcal{F}}_C) \oplus \wp^n_P \hat{\mathcal{F}}_P / \wp^{n+1}_P \hat{\mathcal{F}}_P \longrightarrow \wp^n_{P,C} \hat{\mathcal{F}}_{P,C} / \wp^{n+1}_{P,C} \hat{\mathcal{F}}_{P,C}.$$

This lemma is an extension of the proposition 1 above. The proves of the both lemmas are straightforward and we will skip them.

Theorem 3 . *Let X be a projective irreducible algebraic surface over a field k, $C \subset X$ be an irreducible projective curve, and $P \in C$ be a smooth point on both C and X. Let $\mathcal{F}$ be a torsioh free coherent sheaf on X.*

Assume that the the surface $X - C$ is affine. Then the following complexes are quasi-isomorphic:

i) the adelic complex

$$\hat{\mathcal{F}}_\eta \oplus \prod_D \hat{\mathcal{F}}_D \oplus \prod_x \hat{\mathcal{F}}_x \longrightarrow \prod_D (\hat{\mathcal{F}}_D \otimes K_D) \oplus \prod_x (\hat{\mathcal{F}}_x \otimes K_x) \oplus \prod_{x \in D} (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D}) \longrightarrow$$

$$\longrightarrow \prod_{x \in D} (\hat{\mathcal{F}}_x \otimes K_{x,D})$$

for the sheaf $\mathcal{F}$ and

ii) the complex

$$A(\mathcal{F}) \oplus A_C(\mathcal{F}) \oplus \hat{\mathcal{F}}_P \longrightarrow B_C(\mathcal{F}) \oplus B_P(\mathcal{F}) \oplus (\hat{\mathcal{F}}_P \otimes \hat{\mathcal{O}}_{P,C}) \longrightarrow \hat{\mathcal{F}}_P \otimes K_{P,C}$$

Proof will be divided into several steps. We will subsequently transform the adelic complex checking that every time we get a quasi-isomorphic complex.

Step I. Consider the diagram

$$\begin{array}{ccccccccccc}
 & & \prod\limits_{D \neq C} \hat{\mathcal{F}}_D & \oplus & \prod\limits_{x \in U} \hat{\mathcal{F}}_x & \longrightarrow & \prod\limits_{D \neq C} \hat{\mathcal{F}}_D & \oplus & \prod\limits_{x \in U} \hat{\mathcal{F}}_x & \oplus \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\hat{\mathcal{F}}_\eta & \oplus & \prod\limits_D \hat{\mathcal{F}}_D & \oplus & \prod\limits_x \hat{\mathcal{F}}_x & \longrightarrow & \prod_D K\hat{\mathcal{F}}_D & \oplus & \prod_x K\hat{\mathcal{F}}_x & \oplus \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\hat{\mathcal{F}}_\eta & \oplus & \hat{\mathcal{F}}_C & \oplus & \prod\limits_{x \in C} \hat{\mathcal{F}}_x & \longrightarrow & (\prod_{D \neq C} K\hat{\mathcal{F}}_D / \hat{\mathcal{F}}_D \oplus K\hat{\mathcal{F}}_C) & \oplus & (\prod_{x \in U} K\hat{\mathcal{F}}_x / \hat{\mathcal{F}}_x \oplus \prod_{x \in C} K\hat{\mathcal{F}}_x) & \oplus
\end{array}$$

$$\begin{array}{cccc}
\oplus & \prod_{x \in D \neq C} \hat{\mathcal{F}}_{x,D} & \longrightarrow & \prod_{x \in D \neq C} \hat{\mathcal{F}}_{x,D} \\
 & \downarrow & & \downarrow \\
\oplus & \prod_{x \in D} \hat{\mathcal{F}}_{x,D} & \longrightarrow & \prod_{x \in D} K\hat{\mathcal{F}}_{x,D} \\
 & \downarrow & & \downarrow \\
\oplus & \prod_{x \in C} \hat{\mathcal{F}}_{x,C} & \longrightarrow & \prod_{x \in D \neq C} K\hat{\mathcal{F}}_{x,D} / \hat{\mathcal{F}}_{x,D} \ \oplus \ \prod_{x,C} K\hat{\mathcal{F}}_{x,C}
\end{array}$$

where $U = X - C$. The middle row is the full adelic complex and the first row is an exact subcomplex. The commutativity of the upper squares is obvious.

The exactness follows from the trivial

Lemma 5 . *Let $f_{1,2} : A_{1,2} \longrightarrow B$ be homomorphisms of abelian groups. The complex*

$$0 \longrightarrow A_1 \oplus A_2 \longrightarrow A_1 \oplus A_2 \oplus B \longrightarrow B \longrightarrow 0,$$

where $(a_1 \oplus a_2) \mapsto (a_1 \oplus -a_2 \oplus -f(a_1) + f(a_2)), (a_1 \oplus a_2 \oplus b) \mapsto (f(a_1) + f(a_2) + b)$, is exact.

The third row in the diagram is a quotient-complex by the subcomplex and we conclude that it is quasi-isomorphic to the adelic complex.

Step II. We can make the same step with the adelic complex for the sheaf $\mathcal{F}$ on the surface U. By assumption the surface U is affine and we get an *exact* complex

$$\hat{\mathcal{F}}_\eta/A \longrightarrow \prod_{D\neq C}(\hat{\mathcal{F}}_D \otimes K_D)/\hat{\mathcal{F}}_D \oplus \prod_{x\in U}(\hat{\mathcal{F}}_x \otimes K_x)/\hat{\mathcal{F}}_x \longrightarrow$$

$$\prod_{\substack{x \in U \\ x \in D \neq C}} (\hat{\mathcal{F}}_x \otimes K_{x,D})/(\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D}),$$

where $A = \Gamma(U, \mathcal{F})$.

Lemma 6 . *The complex*

$$0 \longrightarrow \prod_{x\in C}(\hat{\mathcal{F}}_x \otimes K_x)/B_x(\mathcal{F}) \longrightarrow \prod_{\substack{x \in C \\ x \in D \neq C}} (\hat{\mathcal{F}}_x \otimes K_{x,D})/(\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,D}) \longrightarrow 0$$

is exact.

Proof. The injectivity follows directly from the definition of the ring B_x. The surjectivity is the local strong approximation around the point $x \in C$ (see [19][§1],[7][ch.4]).

Step III. Take the sum of the two complexes from step II. Then we have a map of the complex we got in the step I to this complex

$$\begin{array}{ccccccccccc}
\hat{\mathcal{F}}_\eta & \oplus & \hat{\mathcal{F}}_C & \oplus & \prod_x \hat{\mathcal{F}}_x & \longrightarrow & (\prod_{D\neq C} K\hat{\mathcal{F}}_D/\hat{\mathcal{F}}_D \oplus K\hat{\mathcal{F}}_C) & \oplus & (\prod_{x\in U} K\hat{\mathcal{F}}_x/\hat{\mathcal{F}}_x \oplus \prod_{x\in C} K\hat{\mathcal{F}}_x) & \oplus \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\hat{\mathcal{F}}_\eta/A & \oplus & (0) & \oplus & (0) & \longrightarrow & \prod_{D\neq C} K\hat{\mathcal{F}}_D/\hat{\mathcal{F}}_D & \oplus & (\prod_{x\in U} K\hat{\mathcal{F}}_x/\hat{\mathcal{F}}_x \oplus \prod_{x\in C} K\hat{\mathcal{F}}_x/B_x) & \oplus
\end{array}$$

$$\begin{array}{ccccccc}
\oplus & \prod_{x\in C}\hat{\mathcal{F}}_{x,C} & \longrightarrow & \prod_{x\in D\neq C} K\hat{\mathcal{F}}_{x,D}/\hat{\mathcal{F}}_{x,D} & & \oplus & \prod_{x\in C} K\hat{\mathcal{F}}_{x,C} \\
 & \downarrow & & \downarrow & & & \downarrow \\
\oplus & (0) & \longrightarrow & \prod_{\substack{x\in U\\ x\in D\\ D\neq C}} K\hat{\mathcal{F}}_{x,D}/\hat{\mathcal{F}}_{x,D} \oplus \prod_{\substack{x\in C\\ x\in D\\ D\neq C}} K\hat{\mathcal{F}}_{x,D}/\hat{\mathcal{F}}_{x,D} & & \oplus & (0)
\end{array}$$

For this map all the components which do not have arrows are mapped to zero. The diagram is commutative and the kernel of the map is equal to

$$A\oplus\hat{\mathcal{F}}_C\oplus\prod_{x\in C}\hat{\mathcal{F}}_x \longrightarrow K\hat{\mathcal{F}}_C\oplus\prod_{x\in C}B_x(\mathcal{F})\oplus\prod_{x\in C}K\hat{\mathcal{F}}_x \longrightarrow \prod_{x\in C}K\hat{\mathcal{F}}_{x,C}.$$

We conclude that this complex is quasi-isomorphic to the adelic complex.

Step IV. Using the embedding $\hat{\mathcal{F}}_x \longrightarrow B_x(\mathcal{F})$ and lemma 5 we have an exact complex and it's embedding into the complex of the step III:

$$\begin{array}{ccccccccccc}
 & & & & \prod_{x\in C-P}\hat{\mathcal{F}}_x & \longrightarrow & & \prod_{x\in C-P}B_x(\mathcal{F}) & \oplus & \prod_{x\in C-P}\hat{\mathcal{F}}_x & \longrightarrow \\
 & & & & \downarrow & & & \downarrow & & \downarrow & \\
A & \oplus & \hat{\mathcal{F}}_C & \oplus & \prod_{x\in C}\hat{\mathcal{F}}_x & \longrightarrow & K\hat{\mathcal{F}}_C \oplus & \prod_{x\in C}B_x(\mathcal{F}) & \oplus & \prod_{x\in C}\hat{\mathcal{F}}_{x,C} & \longrightarrow
\end{array}$$

$$\begin{array}{cc}
\longrightarrow & \prod_{x\in C-P}B_x(\mathcal{F}) \\
 & \downarrow \\
\longrightarrow & \prod_{x\in C}K\hat{\mathcal{F}}_{x,C}
\end{array}$$

As a result we get the factor-complex

$$A\oplus\hat{\mathcal{F}}_C\oplus\hat{\mathcal{F}}_P \longrightarrow K\hat{\mathcal{F}}_C\oplus B_P(\mathcal{F})\oplus\prod_{x\in C-P}\hat{\mathcal{F}}_{x,C}/\hat{\mathcal{F}}_x\oplus\hat{\mathcal{F}}_{P,C}\longrightarrow$$

$$\longrightarrow\prod_{x\in C-P}K\hat{\mathcal{F}}_{x,C}/B_x(\mathcal{F})\oplus K\hat{\mathcal{F}}_{P,C}.$$

Step V. Now we need

Lemma 7 . *The complex*

$$0\longrightarrow(\hat{\mathcal{F}}_C\otimes K_C)/B_C(\mathcal{F})\longrightarrow\prod_{x\in C-P}(\hat{\mathcal{F}}_x\otimes K_{x,C})/B_x(\mathcal{F})\longrightarrow 0$$

is exact.

Proof. The injectivity is again the definition of the B_C and the surjectivity follows from the strong approximation on the curve C (see proof of proposition 3) and lemma 4 above.

As a corollary we have an isomorphism

$$\hat{\mathcal{F}}_C/A_C(\mathcal{F}) \xrightarrow{\cong} \prod_{x\in C-P} (\hat{\mathcal{F}}_x \otimes \hat{\mathcal{O}}_{x,C})/\hat{\mathcal{F}}_x,$$

where

$$A_C(\mathcal{F}) := B_C(\mathcal{F}) \cap \hat{\mathcal{F}}_C.$$

Combining the isomorphisms from the lemma and its corollary into a single complex of length 2, we get the diagram

$$\begin{array}{ccccccccccccc}
A & \oplus & \hat{\mathcal{F}}_C & \oplus & \hat{\mathcal{F}}_P & \longrightarrow & K\hat{\mathcal{F}}_C & \oplus & B_P(\mathcal{F}) & \oplus & (\prod_{x\in C-P} \hat{\mathcal{F}}_{x,C}/\hat{\mathcal{F}}_x \oplus \hat{\mathcal{F}}_{P,C}) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
(0) & \oplus & \hat{\mathcal{F}}_C/A_C & \oplus & (0) & \longrightarrow & K\hat{\mathcal{F}}_C/B_C & \oplus & (0) & \oplus & \prod_{x\in C-P} \hat{\mathcal{F}}_{x,C}/\hat{\mathcal{F}}_x & \longrightarrow
\end{array}$$

$$\begin{array}{cccc}
\longrightarrow & \prod_{x\in C-P} K\hat{\mathcal{F}}_{x,C}/B_x(\mathcal{F}) & \oplus & K\hat{\mathcal{F}}_{P,C} \\
& \downarrow & & \downarrow \\
\longrightarrow & \prod_{x\in C-P} K\hat{\mathcal{F}}_{x,C}/B_x(\mathcal{F}) & \oplus & (0)
\end{array}$$

The kernel of the map of the complexes is obviously equal to

$$A(\mathcal{F}) \oplus A_C(\mathcal{F}) \oplus \hat{\mathcal{F}}_P \longrightarrow B_C(\mathcal{F}) \oplus B_P(\mathcal{F}) \oplus (\hat{\mathcal{F}}_P \otimes \hat{\mathcal{O}}_{P,C}) \longrightarrow (\hat{\mathcal{F}}_P \otimes K_{P,C})$$

and we arrive to the conclusion of the theorem.

Remark 3. Sometimes we will call the complex from the theorem as the *restricted* adelic complex.

Lemma 8 *.Let X be a projective irreducible variety over a field k and $\mathcal{O}(1)$ be a very ample sheaf on X. Then*

1. *The following conditions are equivalent*

   i) X is a Cohen-Macaulay variety
   ii) for any locally free sheaf $\mathcal{F}$ on X and $i < dim(X)$ $H^i(X, \mathcal{F}(n)) = (0)$ for $n << 0$

2. *If X is normal of dimension > 1 then for any locally free sheaf $\mathcal{F}$ on X $H^1(X, \mathcal{F}(n)) = (0)$ for $n << 0$*

PROOF see in [9][ch. III, Thm. 7.6, Cor. 7.8]. The last statement is known as the lemma of Enriques-Severi-Zariski. For dimension 2 every normal variety is Cohen-Macaulay [16] and thus the second claim follows from the first one.

Proposition 4 . *Let $\mathcal{F}$ be a locally free coherent sheaf on the projective irreducible surface X.*

Assume that the local rings of the X are Cohen-Macaulay and the curve C is a locally complete intersection. Then, inside the field $K_{P,C}$, we have

$$B_C(\mathcal{F}) \cap B_P(\mathcal{F}) = A(\mathcal{F}).$$

PROOF will be done in several steps.

STEP 1. If we know the proposition for a sheaf $\mathcal{F}$ then it is true for the sheaf $\mathcal{F}(nC)$ for any $n \in \mathbb{Z}$. Thus taking a twist by $\mathcal{O}(n)$ we can assume that $\deg_C(\mathcal{F}) < 0$.

STEP 2. Now we show that $A_C(\mathcal{F}) \cap \hat{\mathcal{F}}_P = (0)$. The filtrations from lemma 3 gives the corresponding filtration of the group $A_C(\mathcal{F})$. Lemma 4 implies that

$$\frac{(A_C(\mathcal{F}) \cap \hat{\mathcal{F}}_P) \cap \wp^n \hat{\mathcal{F}}_P}{(A_C(\mathcal{F}) \cap \hat{\mathcal{F}}_P) \cap \wp^{n+1} \hat{\mathcal{F}}_P} \cong \Gamma(C, \mathcal{F} \otimes \check{N}_{X/C}^{\otimes n}).$$

Since $\deg_C(\mathcal{F}) < 0$, $N_{X/C} = \mathcal{O}_X(C)|_C$ and $\deg_C(N_{X/C}) > 0$ we get that the last group is trivial.

STEP 3. The next step is to prove the equality:

$$B_C(\mathcal{F}(-D)) \cap B_P(\mathcal{F}(-D)) = A(\mathcal{F}(-D)),$$

where D is an sufficiently ample divisor on X distinct from the curve C. By theorem 3 the cohomology of $\mathcal{F}_X(-D)$ can be computed from the complex

$$A(\mathcal{F}(-D)) \oplus A_C(\mathcal{F}(-D)) \oplus \hat{\mathcal{F}}_P(-D) \longrightarrow B_C(\mathcal{F}(-D)) \oplus B_P(\mathcal{F}(-D)) \oplus \hat{\mathcal{F}}_{P,C}(-D)$$

$$\longrightarrow K\mathcal{F}_{P,C}.$$

Now take $a_{01} \in B_C(\mathcal{F}(-D)), a_{02} \in B_P(\mathcal{F}(-D))$ such that $a_{01}+a_{02} = 0$. They define an element $(a_{01} \oplus a_{02} \oplus 0)$ in the middle component of the complex. By our condition for D and lemma 8 we have $H^1(X, \mathcal{F}_X(-D)) = (0)$ and thus there exist $a_0 \in A(\mathcal{F}(-D)), a_1 \in A_C\mathcal{F}(-D), a_2 \in \hat{\mathcal{F}}_P(-D)$ such that $a_{01} = a_0 - a_1, a_{02} = a_2 - a_0, 0a_1 - a_2$.

By the second step $a_1 = a_2 \in (A_C(\mathcal{F}(-D)) \cap \hat{\mathcal{F}}_P(-D)) \subset A_C(\mathcal{F}) \cap \hat{\mathcal{F}}_P = (0)$ and, consequently, we have $a_{01}(= -a_{02}) \in A(\mathcal{F}(-D))$.

STEP 4. The last step is to take two distinct divisors D, D' such that $D \cap D' \subset C$. Since C is a hyperplane section we can choose for D, D' two hyperplane sections whose intersection belongs to C. Therefore their ideals in the ring $A(\mathcal{F})$ are relatively prime and

$$A(\mathcal{F}) = A(\mathcal{F}(-D)) + A(\mathcal{F}(-D')) \ni 1 = a + a', a \in A(\mathcal{F}(-D)), a' \in A(\mathcal{F}(-D')).$$

If now $b \in B_C(\mathcal{F}) \cap B_P(\mathcal{F})$, then $b = ba + ba'$, where $ba \in B_C(\mathcal{F}(-D)) \cap B_P(\mathcal{F}(-D)), ba' \in B_C(\mathcal{F}(-D')) \cap B_P(\mathcal{F}(-D'))$. We see that $b \in A(\mathcal{F})$ by the previous step.

REMARK 4. The method we have used cannot be applied if our variety is not Cohen-Macaulay (by lemma 8 above). It would be interesting to know how to extend the result to the arbitrary surfaces X and the sheaves $\mathcal{F}$ such that $\mathcal{F}$ are locally free outside C. The last condition is really necessary. We also note that any normal surface is Cohen-Macaulay [16][§17].

REMARK 5. This proposition is a version for the reduced adelic complex of the corresponding result for the full complex. Namely, $\mathbb{A}_{X,01} \cap \mathbb{A}_{X,02} = \mathbb{A}_{X,0}$, see [7, ch.IV]. This should be generalized to arbitrary dimension n in the following way.

Let $I, J \subset [0, 1, \ldots, n]$ and

$$\mathbb{A}_{X,I}(\mathcal{F}) = (\prod_{\{codim \eta_0, codim \eta_1, \ldots\} \in I} K_{\eta_0, \eta_1, \ldots}) \otimes \mathcal{F}_{\eta_0}) \bigcap \mathbb{A}_X(\mathcal{F}).$$

Then we have

$$\mathbb{A}_{X,I}(\mathcal{F}) \cap \mathbb{A}_{X,J}(\mathcal{F}) = \mathbb{A}_{X,I \cap J}(\mathcal{F})$$

for a locally free $\mathcal{F}$ and a Cohen-Macaulay X.

EXAMPLE. Let $X = \mathbf{P}_2 \supset C = \mathbf{P}_1 \ni P$. We introduce homogenous coordinates $(x_0 : x_1 : x_2)$ such that $C = (x_0 = 0); P(x_0 = x_1 = 0$ and $U = X - C = Spec k[x, y]$ with $x = x_1/x_0, y = x_2/x_0$. Then $k(C) = k(y/x), x^{-1}$ is the last parameter for any two-dimensional local field $K_{Q,C}$ with $Q \neq P$. For local field $K_{P,C}$ we have

$$K_{P,C} = k((u))((t)), u = xy^{-1}, t = y^{-1}.$$

Then we can easily compute all the rings from the complex of theorem 1 for the sheaf $\mathcal{O}_X$.

$$\begin{array}{rcccl} & & B_P & = & k[[u]]((t)) \\ & & B_C & = & k[u^{-1}]((u^{-1}t)) \\ & & \hat{\mathcal{O}}_{P,C} & = & k((u))[[t]] \\ A & = & \Gamma(U, \mathcal{O}_X) & = & k[ut^{-1}, t^{-1}] \\ & & A_C & = & k[u^{-1}][[u^{-1}t]] \\ & & \hat{\mathcal{O}}_P & = & k[[u,t]] \end{array}$$

We can draw the subspaces as some subsets of the plane according to the supports of the elements of the subspaces (on the plane with coordinates (i,j) for elements $u^i t^j \in K_{P,C}$. Then the first three subspaces $B_P, B_C, \hat{\mathcal{O}}_{P,C}$ will correspond to some halfplanes and the subspaces $A, A_C, \hat{\mathcal{O}}_P$ to the intersections of them.

4 Krichever correspondence for dimension 2

We need the following well known result.

Lemma 9 . *Let X be an projective variety, $\mathcal{F}$ be a coherent sheaf on and C be an ample divisor on X. If*

$$S = \oplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nC)), \ F = \oplus_{n \geq 0} \Gamma(X, \mathcal{F}(nC)),$$

then

$$X \cong Proj(S), \ \mathcal{F} \cong Proj(F).$$

PROOF. Let mC be a very ample divisor,$S = \oplus_{n \geq 0} S_n$ and $S' := \oplus_{n \geq 0} S_{nm}$. Then by [8][prop. 2.4.7]

$$Proj(S') \cong Proj(S).$$

The divisor mC defines an embedding $i : X \longrightarrow \mathbf{P}$ to a projective space such that $i^* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{O}_X(mC)$. Let $\mathcal{J}_X \subset \mathcal{O}_{\mathbf{P}}$ be an ideal defined by X. If

$$I := \oplus_{n \geq 0} \Gamma(\mathbf{P}, \mathcal{J}_X(n)),$$

$$A := \oplus_{n \geq 0} \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(n)),$$

then $I \subset A$ and by [8][prop. 2.9.2]

$$Proj(A/I) \cong X.$$

We have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{J}_X(n) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{O}_X(n) \longrightarrow 0,$$

which implies the sequence

$$0 \longrightarrow \bigoplus_{n\geq 0}\Gamma(\mathcal{J}_X(n)) \longrightarrow \bigoplus_{n\geq 0}\Gamma(\mathcal{O}_{\mathbf{P}}(n)) \longrightarrow \bigoplus_{n\geq 0}\Gamma(\mathbf{P},\mathcal{O}_X(n)) \longrightarrow \bigoplus_{n\geq 0}H^1(\mathbf{P},\mathcal{J}_X(n)).$$

Here the last term is trivial for sufficiently large n. The first three terms are equal correspondingly to I, A and S'. It means that the homogenous components of A/I and $S' \supset A/I$ are equal for sufficiently big degrees.

By [8][prop. 2.9.1]

$$Proj(A/I) \cong Proj(S'),$$

and combining everything together we get the statement of the lemma. The statement concerning the sheaf $\mathcal{F}$ can be proved along the same line.

Now we move to the case of algebraic surfaces. The corresponding data has the following

Definition 4.

$\mathcal{M}_2$	:=	$\{X, C, P, (z_1, z_2), \mathcal{F}, e_P\}$
X		projective irreducible surface $/k$
$C \subset X$		projective irreducible curve $/k$
$P \in C$		a smooth point on X and C
z_1, z_2		formal local parameter at P such that $(z_2 = 0) = C$ near P
$\mathcal{F}$		torsion free rank r sheaf on X
e_P		a trivialization of $\mathcal{F}$ at P

Then we have

$$\hat{\mathcal{O}}_{X,P} = k[[z_1, z_2]],\ K_{P,C} = k((z_1))((z_2)),$$

$$\hat{\mathcal{F}}_P = \hat{\mathcal{O}}_P e_P = \hat{\mathcal{O}}_P^{\oplus r}.$$

For the field $K = k((z_1))((z_2))$ we have the following filtrations and subspaces:

$$\begin{array}{rcl} K_{02} & = & k[[z_1]]((z_2)), \\ K_{12} & = & k((z_1))[[z_2]], \\ K(n) & = & z_2^n K_{12}. \end{array}$$

Taking the direct sums we introduce the subspaces $V_{02}, V_{12}, V(n)$ of the space $V = K^{\oplus r}$.

Theorem 4 . *Let C be a hyperplane section on the surface X. Then there exists a canonical map*

$$\Phi_2 : \mathcal{M}_2 \longrightarrow \{\textit{vector subspaces } B \subset K, W \subset V\}$$

such that

i) for all n the complexes

$$\frac{B \cap K(n)}{B \cap K(n+1)} \oplus \frac{K_{02} \cap K(n)}{K_{02} \cap K(n+1)} \longrightarrow \frac{K(n)}{K(n+1)}$$

$$\frac{W \cap V(n)}{W \cap V(n+1)} \oplus \frac{V_{02} \cap V(n)}{V_{02} \cap V(n+1)} \longrightarrow \frac{V(n)}{V(n+1)}$$

are Fredholm of index $\chi(C, \mathcal{O}_C) + nC.C$ and $\chi(C, \mathcal{F}|_C) + nC.C$, respectively

ii) the cohomology of complexes

$$(B \cap K_{02}) \oplus (B \cap K_{12}) \oplus (K_{02}) \cap K_{12}) \longrightarrow B \oplus K_{02} \oplus K_{12} \longrightarrow K$$

$$(W \cap V_{02}) \oplus (W \cap V_{12}) \oplus (V_{02}) \cap V_{12}) \longrightarrow W \oplus V_{02} \oplus V_{12} \longrightarrow V$$

are isomorphic to $H^{\cdot}(X, \mathcal{O}_X)$ and $H^{\cdot}(X, \mathcal{F})$, respectively

iii)if $(B, W) \in Im\ \Phi_2$ then $B \cdot B \subset B, B \cdot W \subset W$

iv) for all n the map

$$(C, P, z_1|_C, \mathcal{F}(nC)|_C, e_P(n)|_C) \mapsto$$

$$\mapsto (\frac{B \cap K(n)}{B \cap K(n+1)} \subset \frac{K(n)}{K(n+1)} k((z_1)),$$

$$\frac{W \cap V(n)}{W \cap V(n+1)} \subset \frac{V(n)}{V(n+1)} k((z_1))^{\oplus r}\ )$$

coincides with the map Φ_1.

v) let the sheaf $\mathcal{F}$ be locally free and the surface X be Cohen-Macaulay. If $m, m' \in \mathcal{M}_1$ and $\Phi_2(m) = \Phi_2(m')$ then m is isomorphic to m'

Proof. If $m = (X, C, P, (z_1, z_2), \mathcal{F}, e_P) \in \mathcal{M}_2$ then to define the map Φ_2 we put

$$\begin{array}{rcl} B & = & B_C(\mathcal{O}_X), \\ W & = & B_C(\mathcal{F}), \\ \Phi_2(m) & = & (B, W). \end{array}$$

Since we have the local coordinates $z_{1,2}$ and the trivialization e_P the subspaces B and W will belong to the space $k((z_1))((z_2))$ exactly as in the case of dimension 1 considered above.

We note that our condition on the curve C implies that C ia a Cartier divisor and the surface $X - C$ is affine.

The property i) follows from lemma 4, the property ii) follows from theorem 3 and the general theorem of adelic theory: the cohomologies of the adelic complex of a sheaf $\mathcal{F}$ are equal to the cohomologies $H^{\cdot}(X, \mathcal{F})$ of the sheaf $\mathcal{F}$ [19, 3, 10, 7]. The property iii) is trivial again, to get iv) one needs again to apply lemma 4 and to get v) it is enough to use proposition 4 and lemma 9. They show that given a point $(B, W) \in \mathcal{M}_2$ such that $(B, W) = \Phi_2(m)$ we can reconstruct the data m up to an isomorphism.

Remark 6. The property v) of the theorem cannot be extended to the arbitrary torsion free sheaves on X. We certainly cannot reconstruct such sheaf if it is not locally free outside C. Indeed, if $\mathcal{F}, \mathcal{F}'$ are two sheaves and there is a monomorphism $\mathcal{F}' \longrightarrow \mathcal{F}$ such that $\mathcal{F}/\mathcal{F}'$ has support in $X - C$ then the restricted adelic complexes for the sheaves $\mathcal{F}, \mathcal{F}'$ are isomorphic.

Remark 7. A definition of the map Φ_n for all n was suggested in [18]. It has the properties that correspond to the properties i) - v) of the theorem. Also the proofs in [18] has the advantage: they are direct and don't use the general adelic machinery.

Appendix 1

Here we show how to deduce from the Lax form of the KP hierarchy for pseudo-differential operators L the classical KP and KdV equations.

Let $L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \ldots$, where $u_m = u_m(x, t_1, t_2, \ldots) \in k[[x, t_1, t_2, \ldots]]$ for $m \geq 1$. If we denote $\partial/\partial t_n$ as ∂_n then the KP hierarchy has the following form

$$\partial_n L = [(L^n)_+, L].$$

This gives us for any n an infinite sequence of differential equations for the functions u_m. Denote by $u', u'', \ldots$ the subsequent derivatives by x.

First for $n = 1$, we get

$$\partial_1 u_m = u_m' \text{ for all } m \geq 1$$

This means that we can take $t_1 = x$.

Now we write down the first two equations for $n = 2$ and the first equation for $n = 3$.

$$\partial_2 u_1 = u_1'' + 2u_2' \quad (1)$$

$$\partial_2 u_2 = u_2'' + 2u_3' + 2u_1 u_1' \quad (2)$$

$$\partial_3 u_1 = u_1''' + 3u_2'' + 3u_3' + 6u_1 u_1' \quad (3)$$

Let us introduce the new notations: $u = u_1(x, y, t)$ with $y = t_2, t = t_3$. Also we use the standard notations $u_t, u_y, u_{yy}, \ldots$ for derivatives.

We can eliminate u_3' from equations (2) and (3) and then we get

$$2u_t - 2u''' - 6uu' = 3(u_2'' + u_{2y}). \quad (4)$$

Differentiating this equation by x we have

$$(2u_t - 2u''' - 6uu')' = 3(u_2''' + u_{2y}') \quad (5).$$

¿From (1) we find

$$u_2''' = 1/2(u_y'' - u''''), \ u_{2y}' = 1/2(u_{yy} - u_y'')$$

and inserting these expressions into (5) we finally get the KP equation

$$(4u_t - u''' - 12uu')' = 3u_{yy}.$$

In the space E' there is an invariant submanifold defined by condition $(L^2)_- = 0$ [22][Lemma 2]. Let us look what this condition means for the operator L as above. We have

$$L^2 = \partial^2 + (2u_2 + u_1')\partial^{-1} + \ldots$$

and thus in our notations

$$2u_2 + u' = 0. \quad (6)$$

Combining (1) and (6) we see that $u_{2y} = 0$. Together with (4) this implies $3u_2'' = 2u_t - 2u''' - 6uu'$. Using once more (6) we get at last the KdV equation

$$4u_t - 7u''' - 12uu' = 0.$$

The numerical coefficients here are not essential since one can get all possible values by rescaling of x, t and u.

These computations and the choice of variables look quite artificial comparing with the previous constructions. It is an interesting independent problem how to deduce the KP and KdV equations from the Lax operator equations in a more conceptual way.

Now, we show how to do that for the classical KdV equation. Comparing with the previous deduction this one can be done in a purely formal way, without any tricks.

Again, we order that our equation will satisfy the constraint condition $(L^2)_- = 0$. For the operator L as above we have

$$L^2 = \partial^2 + (2u_2 + u_1')\partial^{-1} + (2u_3 + u_1^2 + u_2')\partial^2 + \ldots$$

and thus in our notations

$$2u_2 + u' = 0, 2u_3 + u_1^2 + u_2' = 0$$

and so on. This means that we can compute all u_m for $m > 1$ starting from u_1 and its derivatives. Under this condition the first non-trivial equation of the KP hierarchy is the following one

$$\partial_3 L = [(L^3)_+, L].$$

and taking the coefficient nearby ∂^{-1} we have

$$u_1''' + 3u_2'' + 3u_3' + 6u_1u_1' = 0.$$

Substituting the given above expressions for u_2 and u_3 we get at last the KdV equation

$$4u_t - 7u''' - 12uu' = 0.$$

Problem. Let us go to the case of dimension two. Then we have the constraint conditions of the type

$$(L^m M^n)_- = 0$$

and the particular components of hierarchy. Can we deduce some concrete equations using the same way as above ? Certainly, we have more possibilities. The first question is how many initial functions u_{mn} one has to use so to generate all other coefficients ? For dimension 1 only u_1 was enough !

Remark. For higher dimensions it is crucial to transform the whole system to avoid the asymmetry of the variables. It means that we have to consider not the ring P alone but at least the direct sum $P \oplus P'$ where P' has interchanged variables, namely y, x instead of x, y. As a goal one can hope to get in this way the equations of the plane hydrodynamics. It is known they have infinitely many conservation laws.

Appendix 2

Here, we present a well-known construction of semi-infinite monomes in an infinite-dimensional vector space. Let V be a vector space over a fied k and $V_n, n \in \mathbb{Z}$ be an exhausted increasing filtration in V. The example one has to have in mind is when $V = k((z)), V_n = z^{-n}k[[z]]$. If V would be a finite-dimensional space,the k-dimensional vector subspaces $W \subset V$ can be described by the elements $\bigwedge^k(W) \subset \bigwedge^k(V)$. In the infinite-dimensional case we assume that

$$\mathrm{dim} W \cap V_n = k + n \text{ for large } n.$$

In the case of $V = k((z))$ this k is exactly the index of subspace W from the Sato Grassmanian. In general case for such n, we have the diagram

$$\begin{array}{ccc} W \cap V_n & \longrightarrow & W \cap V_{n+1} \\ \downarrow & & \downarrow \\ V_n & \longrightarrow & V_{n+1} \end{array}$$

We denote the one-dimensional space $(W \cap V_{n+1})/(W \cap V_n)$ as 1_n. The diagram induces isomorphism $1_n \to V_{n+1}/V_n$.

For any exact sequence $0 \to V \to V' \to V'/V \to 0$ with one-dimensional space V'/V, there is a canonical map $\bigwedge^{n+1}(V') \to \bigwedge^n(V) \otimes (V'/V)$. If $v = v_1 \wedge v_2 \wedge \ldots v_{n+1} \in \bigwedge^{n+1}(V')$ then the image of v is zero if either all v_i belong to V or at least two v_i belong to $V' - V$. If exactly one v_i belong to $V' - V$ then the image is $(-1)^{n-i} v_1 \ldots \wedge v_{i-1} \wedge v_{i+1} \wedge \ldots v_{n+1} \otimes v_i \mathrm{mod} V$.

We get the new diagram

$$\begin{array}{ccc}
\bigwedge^{k+n+1}(W \cap V_{n+1}) & \longrightarrow & \bigwedge^{k+n}(W \cap V_n) \otimes 1_n \\
\big\downarrow & & \big\downarrow \\
\bigwedge^{k+n+1}(V_{n+1}) & \longrightarrow & \bigwedge^{k+n}(V_n) \otimes V_{n+1}/V_n
\end{array}$$

For any two subspaces $A, B \subset V$, we say they are commesurable iff the intersection $A \cap B$ has finite codimension in both of them. In this case, we define

$$(A \mid B) = \det(A/A \cap B)^{-1} \otimes \det(B/A \cap B),$$

where $\det(A) = \bigwedge^{\dim(A)}(A)$ and $(A)^{-1}$ is the space dual to A. Note that all the spaces $(A \mid B)$ are one-dimensional. There are the canonical isomorphisms $(A \mid B) \cong (B \mid A)^{-1}$ and $(A \mid C) \cong (A \mid B) \otimes (B \mid C)$ which we will treat as equalities. These rules allows us to write

$$V_{n+1}/V_n = (V_n \mid V_{n+1}) = (V_n \mid V_0)(V_{n+1} \mid V_0)^{-1},$$

$$1_n = (V_n \mid V_{n+1}) = (V_n \mid V_0)(V_{n+1} \mid V_0)^{-1} \text{ for large } n.$$

Thus, we can rewrite the diagram as

$$\begin{array}{ccc}
\bigwedge^{k+n+1}(W \cap V_{n+1})(V_{n+1} \mid V_0)) & \longrightarrow & \bigwedge^{k+n}(W \cap V_n) \otimes (V_n \mid V_0) \\
\big\downarrow & & \big\downarrow \\
\bigwedge^{k+n+1}(V_{n+1})(V_{n+1} \mid V_0)) & \longrightarrow & \bigwedge^{k+n}(V_n) \otimes (V_n \mid V_0)
\end{array}$$

At last, the space of semi-infinite monomes (of index k) can be defined as projective limit respect these maps:

$$\bigwedge^{k+\frac{\infty}{2}}(V) = \lim_n \bigwedge^{k+n}(V_n) \otimes (V_n \mid V_0)$$

and we can attach to the subspace W the line

$$[W] = \lim_n \bigwedge^{k+n} (W \cap V_n) \otimes (V_n \mid V_0).$$

Using these constructions we can present the Sato Grassmanian as a disjoint union of connected components

$$Gr(V) = \coprod_k Gr_k(V)$$

and every component has a projective embedding

$$Gr_k(V) \to \mathbf{P}(\bigwedge^{k+\frac{\infty}{2}} (V))$$

where $W \mapsto [W]$. Note that the vector space $\wedge^{k+\frac{\infty}{2}}(V)$ does depend on the choice of the subspace V_0 from the filtration but it's projectivization doesn't.

References

[1] Álvarez Vásquez A., Munoz Porras J. M., Plaza Martín F. J., *The algebraic formalism of soliton equations over arbitrary base fields*, alg-geom/9606009

[2] Arbarello E., De Concini C., Kac V. G., Procesi C., *Moduli Spaces of Curves and Representation Theory*, Comm. Math. Phys., **117** (1988), 1-36

[3] Beilinson A.A., *Residues and adeles*, Funct. Analysis and its appl.**14**(1980), 34-35

[4] Beilinson A. A., Schechtman V. V., *Determinant bundles and Virasoro algebra*, Comm. Math. Physics, **118** (1988), 651-701

[5] Ben-Zvi D., Frenkel E., *Spectral curves, opers and integrable systems*, alg-geom/9802068

[6] Bourbaki N., *Algèbre commutative*, Hermann, Paris, 1961,1965

[7] Fimmel T., Parshin A. N., *Introduction to the Higher Adelic Theory* (book in preparation)

[8] Grothendieck A., Dieudonné J., *Eléments de Géométrie Algébrique II.*, Public. Mathem. IHES, **8** (1961)

[9] Hartshorne R., *Algebraic Geometry*, Springer-Verlag, Berlin, 1977

[10] Huber A., *On the Parshin-Beilinson Adeles for Schemes*, Abh. Math. Sem. Univ. Hamburg, **61** (1991), 249-273

[11] Kashiwara M., *The flag manifold of Kac-Moody Lie algebra*, in *Algebraic analysis, geometry and number theory*, Johns Hopkins University Press, Baltimore, 1989, 161-190

[12] Krichever I. M., *Methods of algebraic geometry in the theory of non-linear equations*, Russian Math. Surveys, **32** (1977), 185-214

[13] Kurke H., Osipov D., Zheglov A. *Formal punctured ribbons and two-dimensional local fields*, J. reine u. angew. Math., **629**(2009),133-170 (arXiv: 0708.0985)

[14] Kurke H., Osipov D., Zheglov A. *Formal groups arising from formal punctured ribbons*, Intern. J. of Math. (to appear)(arXiv:0901.1607)

[15] Li Y., Mulase M., *Category of morphisms of algebraic curves and a characterization of Prym varieties*, alg-geom/9203002

[16] Matsumura H., *Commutative Ring Theory*, Cambridge University Press, 1986

[17] Mulase M., *Category of vector bundles on algebraic curves and infinite dimensional Grassmanians*, Int. J. Math., **1** (1990), 293-342

[18] Osipov D. V., *Krichever correspondence for algebraic varieties*, Izvestija RAN, ser.math., **65** (2001), N 4; see also math/0003188

[19] Parshin A. N., *To the arithmetics of two-dimensional schemes I.*, Izvestija AN SSSR, ser. math., **40** (1976), 736-773

[20] Parshin A. N., *Higher Bruhat-Tits buildings and the Vector Bundles on an Algebraic Surface*, in *Algebra and Number Theory*, Berlin, de Gruyter, 1994, 165-192

[21] Parshin A. N., *Commentaries*, in Hilbert D., *Selected works*, vol. II, Moscow, Factorial, 1998, 535-538.

[22] Parshin A. N., *On a ring of formal pseudo-differential operators*, Proc. Steklov Math. Institute, **224** (1999), 266-280

[23] Pressley A., Segal G., *Loop groups*, Clarendon Press, Oxford, 1986

[24] Quandt I.,*On a relative version of the Krichever correspondence*, alg-geom/9606015

[25] Sato M., Sato Y., *Soliton Equations as Dynamical Systems on Infinite Dimensional Grassmann Manifold*, Lecture Notes in Num. Appl. Anal., **5** (1982), 259-271

[26] Segal G., Wilson G., *Loop groups and equations of the KdV type*, Publ. Math. IHES, **80** (1985), 301-342

[27] Zheglov A. B., Osipov D.V.,*On questions related to the Krichever correspondence*, Math. Notes, **81**(2007), 467-476 (arXiv:math/0610364)

Steklov Mathematical Institute
Gubkina str. 8
117966 Moscow GSP-1
Russia
e-mail: parshin@mi.ras.ru