Self-dual Hopfions

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Abstract

We construct static and time-dependent exact soliton solutions with non-trivial Hopf topological charge for a field theory in $3+1$ dimensions with the target space being the two dimensional sphere S^2. The model considered is a reduction of the so-called extended Skyrme-Faddeev theory by the removal of the quadratic term in derivatives of the fields. The solutions are constructed using an ansatz based on the conformal and target space symmetries. The solutions are said self-dual because they solve first order differential equations which together with some conditions on the coupling constants, imply the second order equations of motion. The solutions belong to a sub-sector of the theory with an infinite number of local conserved currents. The equation for the profile function of the ansatz corresponds to the Bogomolny equation for the sine-Gordon model.

1 Introduction

Two important aspects in the study of non-linear phenomena are the identification of the symmetries and the relevant degrees of freedom in a given regime of the dynamics. In many cases solitons connect those two aspects in a very special way. On one hand, the appearance of solitons is related to a high degree of symmetries and so of conservation laws. On the other hand, solitons are in many cases the proper relevant degrees of freedom. In low dimensional phenomena the soliton theory is quite well developed, with the prototype being the sine-Gordon model. That is an integrable field theory which in the weak coupling regime is described by the elementary excitations of a scalar field, and at strong coupling the natural degrees of freedom become the solitons with their dynamics being governed by the massive Thirring model [1]. The fact that a unique quantum field theory can be described by two different classical Lagrangians is a remarkable fact, and it is believed that similar phenomena occur in higher dimensions, with the electromagnetic duality in gauge theories being the most important example [2].

Soliton theory in dimensions higher than two is far from being well understood, even at the classical level. One way of addressing the problem is to look for the equivalent of the structures responsible for the conservation laws in two dimensional soliton theories, i.e. the zero curvature condition or Lax-Zakharov-Shabat equation [3]. The results obtained so far show that loop spaces play a crucial role in that approach. It has been shown that the generalization of the Lax-Zakharov-Shabat equation to higher dimensions can be best formulated as the zero curvature condition for a connection on generalized loop spaces [4, 5]. Conservation laws are then obtained in way very similar to that of two dimensions. Most of the known soliton theories fit into that scheme. However, it is not clear how the full loop space structure works to render them solvable, i.e. to provide a method for constructing exact solutions.

There are two main contexts in which solitons appear in field theories. As the solutions of the classical equations of motion, like the instantons and magnetic monopoles in non-abelian gauge theories, or then as solutions of low energy effective actions like the skyrmions and hopfions [6, 7, 8]. The purpose of this paper is to study solitons belonging to that second class. We consider a field theory described by the Lagrangian

$$\mathcal{L} = -\frac{1}{e^2}\, H_{\mu\nu}^2 + \frac{\beta}{2}\, \left(\partial_\mu \vec{n} \cdot \partial^\mu \vec{n}\right)^2 \tag{1.1}$$

where $\vec{n}$ is a triplet of scalar fields taking values on the two dimensional sphere S^2, i.e. $\vec{n}^2 = 1$, $H_{\mu\nu}$ is the pull back of the area form on S^2, e^2 and β are dimensionless coupling constants. We shall consider the theory (1.1) in four dimensional Minkowski space-time and also in three and four dimensional Euclidean spaces. Notice that the second term in

(1.1) is quartic in time derivatives and so it breaks the positivity of the energy. However, as we explain below, we shall be considering sectors of the theory (1.1) where the energy is indeed positive definite.

Using the stereographic projection one can parametrize S^2 with a complex scalar field u, related to $\vec{n}$ as

$$\vec{n} = \frac{1}{1 + \mid u \mid^2} \left(u + u^*, -i \left(u - u^* \right), \mid u \mid^2 - 1 \right) \tag{1.2}$$

In addition, we shall parametrize u as

$$u = \sqrt{\frac{1-g}{g}}\, e^{i\,\theta} \tag{1.3}$$

where g and θ are real scalar fields, with $0 \leq g \leq 1$, and $0 \leq \theta \leq 2\pi$, playing the role of Darboux variables for the two-form $H_{\mu\nu}$. We then have

$$H_{\mu\nu} \equiv \vec{n} \cdot (\partial_\mu \vec{n} \wedge \partial_\nu \vec{n}) = -2i \frac{(\partial_\mu u \partial_\nu u^* - \partial_\nu u \partial_\mu u^*)}{(1 + \mid u \mid^2)^2} = 2 \left[\partial_\mu g\, \partial_\nu \theta - \partial_\nu g\, \partial_\mu \theta \right] \tag{1.4}$$

and

$$(\partial_\mu \vec{n} \cdot \partial^\mu \vec{n}) = 4 \frac{\partial_\mu u\, \partial^\mu u^*}{(1 + \mid u \mid^2)^2} = \frac{(\partial_\mu g)^2}{g\,(1-g)} + 4\, g\,(1-g)\,\left(\partial_\mu \theta\right)^2 \tag{1.5}$$

The Lagrangian (1.1) is then written as

$$\mathcal{L} = \frac{8}{e^2} \left[\frac{(\partial_\mu u)^2 \, (\partial_\nu u^*)^2}{(1 + \mid u \mid^2)^4} + \left(\beta\, e^2 - 1 \right) \frac{(\partial_\mu u\, \partial^\mu u^*)^2}{(1 + \mid u \mid^2)^4} \right] \tag{1.6}$$

The theory (1.1) is invariant under the global $SO(3)$ rotations on the target space S^2, and also invariant under the conformal groups $SO(4,2)$ or $SO(5,1)$, depending if we consider Minkowski or Euclidean four dimensional space-time respectively. Those two symmetries play a crucial role in this paper since the soliton solutions will be constructed, following [9], with an ansatz invariant under the combined action of the $U(1)$ subgroup of $SO(3)$ responsible for the phase transfomation $u \rightarrow e^{i\alpha}\, u$, and three special commuting $U(1)$ subgroups of the conformal group. Similar ansätze were used in [10, 11, 12].

The Lagrangian (1.1) is part of the Wilsonian low energy effective Lagrangian for the pure (without matter) $SU(2)$ Yang-Mills theory [13], if one assumes that the Cho-Faddeev-Niemi decomposition of the gauge potentials [14] holds true at low energies. The missing term is the quadratic operator, $M^2\, \partial_\mu \vec{n} \cdot \partial^\mu \vec{n}$, with M being a mass parameter generated by the so-called dimensional transmutation. Even though that term breaks the conformal symmetry, exact vortex solutions have been constructed for such model [15]. The solutions belong to a special sector defined by particular relations among the coupling constants and a constraint that leads to an infinite number of conservation laws. In addition, static Hopf topological solitons have been constructed numerically for the theory with the M^2 term and outside that special sector [16]. For the case of the Faddeev-Niemi model [7] (without the second term in (1.1) but with the quadratic term mentioned above), numerical solutions have been calculated in [17, 18]. The solitons we calculate in this paper have a free parameter which allow to rescale their size and rate of time evolution. That is a consequence of the conformal invariance of the equations of motion and it may play a role in the study of the solutions of the theory with the M^2 term.

The special property of theory (1.1) is that it admits a Bogomolny type equation, i.e. a first order differential equation implying the second order equations of motion. That is given by [4]

$$(\partial_\mu u)^2 = 0 \tag{1.7}$$

together with the following relation among the coupling constants

$$\beta e^2 = 1 \tag{1.8}$$

An interesting point is that for the ansatz configurations we consider, the equation (1.7) becomes the Bogomolny equation for the sine-Gordon model. Therefore, the self-dual soliton solutions of (1.1) at the critical point (1.8), are built from the static one-soliton of the sine-Gordon model. The exact vortex solutions constructed in [15] belong to the sector where $\beta e^2 = 1$ and they satisfy (1.7).

The equation (1.7) also plays an important role in the integrability properties of the model. The theory (1.1) has three conserved currents associated to the $SO(3)$ target space symmetry. However, for the sub-model satisfying the constraint (1.7), in addition to the Euler-Lagrange equations associated to (1.1), the number of conserved currents become infinite. Indeed, the equations of motion for (1.1) are

$$\left(1+ \mid u \mid^2\right) \partial^\mu \mathcal{K}_\mu - 2\, u^* \, \mathcal{K}_\mu \, \partial^\mu u = 0 \tag{1.9}$$

and its complex conjugate, where

$$\mathcal{K}_\mu \equiv \frac{(\beta e^2 - 1)\, (\partial_\nu u \, \partial^\nu u^*)\, \partial_\mu u + (\partial_\nu u)^2 \, \partial_\mu u^*}{(1+ \mid u \mid^2)^2} \tag{1.10}$$

If one imposes (1.7) as a constraint it follows that the currents [4, 9]

$$J_\mu \equiv \frac{\delta G}{\delta u} \, \mathcal{K}^c_\mu - \frac{\delta G}{\delta u^*} \, \mathcal{K}^{c\,*}_\mu \tag{1.11}$$

are conserved for any functional G of u and u^*, but not of their derivatives, and where $\mathcal{K}^c_\mu$ is obtained from (1.10) by imposing (1.7), i.e.

$$\mathcal{K}^c_\mu \equiv \frac{(\beta\, e^2 - 1)\; (\partial_\nu u\, \partial^\nu u^*)\; \partial_\mu u}{(1+ \mid u \mid^2)^2} \tag{1.12}$$

The conservation of (1.11) follows from the fact that $\mathcal{K}_\mu \partial^\mu u^*$ is real, $\mathcal{K}^c_\mu \partial^\mu u = 0$, and from the equations of motion which now read $\partial^\mu \mathcal{K}^c_\mu = 0$. Notice that all the currents (1.11) vanish at the critical point (1.8).

The theory (1.1) possesses another sub-model with an infinite number of conserved currents defined by the constraint

$$\partial_\mu u\, \partial^\mu u^* = 0 \tag{1.13}$$

Consider the currents

$$\tilde{J}_\mu \equiv \frac{\delta \tilde{G}}{\delta u}\, \tilde{\mathcal{K}}^c_\mu - \frac{\delta \tilde{G}}{\delta u^*}\, \left(\tilde{\mathcal{K}}^c_\mu\right)^* \tag{1.14}$$

where $\tilde{G}$ is a functional of u and u^* but not of their derivatives, and $\tilde{\mathcal{K}}^c_\mu$ is obtained from (1.10) by imposing (1.13), i.e.

$$\tilde{\mathcal{K}}^c_\mu \equiv \frac{(\partial_\nu u)^2\, \partial_\mu u^*}{(1+ \mid u \mid^2)^2} \tag{1.15}$$

Similarly, the conservation of (1.14) follows from the fact that $\tilde{\mathcal{K}}^c_\mu \partial^\mu u^*$ is real, $\tilde{\mathcal{K}}^c_\mu \partial^\mu u = 0$, and from the equations of motion which now read $\partial^\mu \tilde{\mathcal{K}}^c_\mu = 0$.

The importance of the constraints (1.7) and (1.13) is twofold. Besides leading to an infinite number of conserved currents as we showed above, they also make the energy positive definite, as we now explain. Notice that, by using (1.4), (1.5) and (1.6), the Lagrangian (1.1) can be written in three different ways as

$$\mathcal{L} = \frac{(\beta\, e^2 - 1)}{e^2}\, \mathcal{L}_1 + \frac{1}{e^2}\, \mathcal{L}_2 = -\frac{1}{e^2}\, H^2_{\mu\nu} + \beta\, \mathcal{L}_1 = \frac{(\beta\, e^2 - 1)}{e^2}\, H^2_{\mu\nu} + \beta\, \mathcal{L}_2 \tag{1.16}$$

where $H_{\mu\nu}$ is given in (1.4), and where we have defined

$$\mathcal{L}_1 = 8\, \frac{(\partial_\mu u\, \partial^\mu u^*)^2}{(1+ \mid u \mid^2)^4} \qquad\qquad \mathcal{L}_2 = 8\, \frac{(\partial_\mu u)^2\; (\partial_\nu u^*)^2}{(1+ \mid u \mid^2)^4} \tag{1.17}$$

The Legendre transform of $\mathcal{L}_1$ is

$$\mathcal{H}_1 \equiv \dot{u}\, \frac{\delta \mathcal{L}_1}{\delta \dot{u}} + \dot{u}^*\, \frac{\delta \mathcal{L}_1}{\delta \dot{u}^*} - \mathcal{L}_1 = 24\, \frac{\left[\mid \dot{u} \mid^2 + \frac{1}{3}\, \vec{\nabla} u \cdot \vec{\nabla} u^*\right] \left[\mid \dot{u} \mid^2 - \vec{\nabla} u \cdot \vec{\nabla} u^*\right]}{(1+ \mid u \mid^2)^4} \tag{1.18}$$

where $\dot{u}$ denotes the x^0-derivative of u, and $\vec{\nabla} u$ its spatial gradient. Notice that $\mathcal{H}_1 = 0$ when (1.13) holds true. On the other hand the Legendre transform of $\mathcal{L}_2$ is

$$\mathcal{H}_2 \equiv 24 \, \frac{\left(\vec{\nabla} u\right)^2 \left(\vec{\nabla} u^*\right)^2}{\left(1+ \mid u \mid^2\right)^4} \, \left[F^2 - \left(\frac{2}{3}\right)^2\right] \tag{1.19}$$

where we have denoted

$$\frac{\dot{u}^2}{\left(\vec{\nabla} u\right)^2} \equiv \frac{1}{3} + F \, e^{i\,\Phi} \tag{1.20}$$

with $F > 0$ and $0 \leq \Phi \leq 2\pi$, being functions of the space-time coordinates. Therefore, $\mathcal{H}_2$ vanishes on a circle on the $\left[\frac{\dot{u}^2}{\left(\vec{\nabla} u\right)^2}\right]$-complex plane, with radius equal to $\frac{2}{3}$ and center at $\frac{\dot{u}^2}{\left(\vec{\nabla} u\right)^2} = \frac{1}{3}$. Notice that the constraint (1.7), for the Minkowski case, lies on that circle and corresponds to $\Phi = 0$ and $F = \frac{2}{3}$. In fact, (1.7) is the only Lorentz invariant condition that lies on that circle.

The Legendre transform of the term $\left[-H_{\mu\nu}^2\right]$ is always positive definite. Therefore, from (1.16), (1.18) and (1.19) one obtains the following result.

Theorem 1. *The Hamiltonian density $\mathcal{H}$ associated to (1.1) is positive definite when:*

- *The constraint (1.7) holds true and $\frac{\left(\beta \, e^2 - 1\right)}{e^2} < 0$.*
- *The constraint (1.13) holds true and $e^2 > 0$.*

A more careful analysis, exploring (1.18), (1.19) and the three ways of writing the Lagrangian as in (1.16), leads to the following result.

Theorem 2. *The Hamiltonian density $\mathcal{H}$ associated to (1.1) is positive definite when:*

- *$e^2 < 0$, $\beta < 0$ and $F \leq \frac{2}{3}$.*
- *$e^2 > 0$, $\beta < 0$ and $F \leq \frac{2}{3}$ or $\mid \dot{u} \mid^2 \leq \vec{\nabla} u \cdot \vec{\nabla} u^*$,*

with F defined in (1.20).

There are sectors where the Hamiltonian is positive definite for $\beta > 0$, but they are not physically interesting since they exclude the static configurations.

The papers is organized as follows. In section 2 we constructed the ansatz based on the conformal symmetry and the target space phase transformation $u \rightarrow e^{i\,\alpha} \, u$. Using that ansatz we construct the exact hopfion self-dual solutions. In sub-section 2.1 we construct the static version of those hopfions. In sections 3 and 4 we calculate the Hopf topological charges and energies of those solutions respectively. Our results are also valid on Euclidean four dimensional space-time and also for a version of the theory (1.1) which possesses positive definite energy. Those results are shown in section 5. A summary of the results as well as the conclusions are presented in section 6. The technical results are given in the appendices.

2 The ansatz

We now construct exact soliton solutions for the theory (1.1) in the integrable sub-sectors defined by the constraints (1.7) and (1.13), using a special ansatz based on the conformal and internal symmetries. In order to implement the ansatz we have to work with special coordinates. Three of them are chosen to parametrize curves generated by three commuting $U(1)$ subgroups of the conformal group $SO(4,2)$ of Minkowski space-time. The fourth one, denoted z in what follows, is taken to parametrize curves orthogonal to those three (see [9] for details). In Minkowski space-time they are related to the Cartesian coordinates x^{μ}, $\mu = 0, 1, 2, 3$, as[1]

$$\begin{aligned} x^0 &= \frac{r_0}{p}\sin\zeta \qquad & x^1 &= \frac{r_0}{p}\sqrt{z}\,\cos\varphi \\ x^3 &= \frac{r_0}{p}\sqrt{1-z}\,\sin\xi \qquad & x^2 &= \frac{r_0}{p}\sqrt{z}\,\sin\varphi \end{aligned} \tag{2.1}$$

where r_0 is a constant with dimension of length, and

$$p = \cos\zeta - \sqrt{1-z}\,\cos\xi \tag{2.2}$$

The range of those coordinates are: $0 \leq z \leq 1$, $0 \leq \varphi\,,\,\xi \leq 2\pi$, and $0 \leq \zeta \leq \pi$. Notice that (ζ, z, ξ, φ) and $(\zeta+\pi, z, \xi+\pi, \varphi+\pi)$, describe the same point in space, and that is why the range of ζ is restricted. The Minkowski metric (with signature $(+,-,-,-)$), in those coordinates, becomes

$$ds^2 = \left(\frac{r_0}{p}\right)^2 \left[d\zeta^2 - \frac{dz^2}{4\,z\,(1-z)} - (1-z)\,d\xi^2 - z\,d\varphi^2\right]$$

The ansatz corresponds to those field configurations where the scalar fields introduced in (1.3) are of the form

$$g \equiv g\,(z) \qquad\qquad \theta \equiv m_1\,\xi + m_2\,\varphi + m_3\,\zeta \tag{2.3}$$

[1]Notice they correspond to the coordinates used in [11] with the change $y \rightarrow \frac{1-z}{z}$, and to the toroidal coordinates used in [10] for $\zeta = 0$ and $z = \tanh^2\eta$.

where m_i, $i = 1, 2, 3$, are integers satisfying the condition $m_1 + m_2 + m_3 =$ even integer. That is required in order for the complex scalar field u to be single valued, since $(\zeta = 0, z, \xi, \varphi)$ and $(\zeta = \pi, z, \xi + \pi, \varphi + \pi)$, correspond to the same point in space.

Replacing the ansatz (2.3) (see (1.3)) into the equations of motion (1.9) associated to (1.1), one gets that the equation for θ is automatically satisfied and that the profile function $g(z)$ has to satisfy

$$\partial_z \left[\left[\left(\frac{z\,(1-z)}{g\,(1-g)}\,g'\right)^2 + \gamma\,\Lambda_M\right] g'\right] + \frac{(1-2g)\,g\,(1-g)}{2\,(z\,(1-z))^2}\left[\left(\frac{z\,(1-z)}{g\,(1-g)}\,g'\right)^4 - \Lambda_M^2\right] = 0 \qquad (2.4)$$

where $g' \equiv \partial_z g$, and

$$\Lambda_M \equiv z\,m_1^2 + (1-z)\;m_2^2 - z\,(1-z)\;m_3^2 \qquad (2.5)$$

and

$$\gamma = \left(1 - \frac{2}{\beta\,e^2}\right) \qquad (2.6)$$

Notice that the equation (2.4) is invariant under the following two $\mathbb{Z}_2$ transformations

$$g \leftrightarrow 1 - g \qquad (2.7)$$

and

$$z \leftrightarrow 1 - z \qquad\qquad m_1^2 \leftrightarrow m_2^2 \qquad (2.8)$$

Now replacing the ansatz (2.3) into the constraints (1.7) and (1.13) one gets that

$$(\partial_\mu\,u)^2 = 0 \qquad \leftrightarrow \qquad \left[\frac{z\,(1-z)}{g\,(1-g)}\,g'\right]^2 = \Lambda_M \qquad (2.9)$$

and

$$\partial_\mu\,u\,\partial^\mu\,u^* = 0 \qquad \leftrightarrow \qquad \left[\frac{z\,(1-z)}{g\,(1-g)}\,g'\right]^2 = -\,\Lambda_M \qquad (2.10)$$

The equation (2.9) obviously has acceptable solutions only if $\Lambda_M \geq 0$ on the interval $0 \leq z \leq 1$. Analogously, (2.10) has solutions only for $\Lambda_M \leq 0$. The results of appendix A show that, on the interval $0 \leq z \leq 1$,

$$\Lambda_M \geq 0 \qquad \text{if} \qquad m_1,\, m_2 \neq 0 \qquad \text{and} \qquad (\mid m_1 \mid + \mid m_2 \mid)^2 > m_3^2 \qquad (2.11)$$

and

$$\Lambda_M \leq 0 \qquad \text{if} \qquad m_1 = m_2 = 0 \qquad \text{and so} \qquad \Lambda_M = -z\,(1-z)\;m_3^2 \qquad (2.12)$$

If g satisfies (2.9) or (2.10) then (2.4) reduces to

$$(\gamma \pm 1)\;\partial_z\,[\Lambda_M\,g'] = 0 \qquad (2.13)$$

where the factor $(\gamma+1)$ corresponds to (2.9) and $(\gamma-1)$ to (2.10). For $\gamma \neq \pm 1$ the equation (2.13) implies that $g' \sim 1/\Lambda_M$. If $m_3 \neq 0$, Λ_M is quadratic in z and so of the form $\Lambda_M \sim (z-z_+)(z-z_-)$. Therefore, $g \sim c_1 \ln\left[\frac{z-z_+}{z-z_-}\right] + c_2$, and so it does not satisfy (2.9) or (2.10), even for $c_1 = 0$. If $m_3 = 0$ then Λ_M is linear in z, and so g is still a logarithm, and consequently not a solution of (2.9) or (2.10). The only remaining possibility is to have $\Lambda_M =$ constant, and so $m_3 = 0$ and $m_1^2 = m_2^2$. For the case of (2.10) we then have $\Lambda_M = 0$ due to (2.12), and so the only acceptable solution is $g =$ constant, which implies $\vec{n} =$ constant. For the case of (2.9) we then have $\Lambda_M = m_2^2$, and so (2.13) implies that g is linear in z. But (2.9) then requires that $g\,(1-g) \sim z\,(1-z)$. The only way for g to be linear in z and satisfy $g\,(1-g) = \alpha\, z\,(1-z)$, is for $\alpha = 1$, and then either $g = z$ or $g = 1-z$. In such case we need $\Lambda_M = 1$. Therefore, the only solutions of (2.9) and (2.13), for $\gamma \neq -1$, and so $\beta\, e^2 \neq 1$, are

$$m_1^2 = m_2^2 = 1; \qquad m_3 = 0 \qquad \text{and} \qquad g = z \qquad \text{or} \qquad g = 1-z \tag{2.14}$$

In terms of the $\vec{n}$ field (see (1.2) and (1.3)) such solutions read

$$\vec{n} = \left(2\sqrt{z\,(1-z)}\,\cos\left(\varepsilon_1\,\xi + \varepsilon_2\,\varphi\right), 2\sqrt{z\,(1-z)}\,\sin\left(\varepsilon_1\,\xi + \varepsilon_2\,\varphi\right), \varepsilon_3\,\left(1-2\,z\right)\right) \tag{2.15}$$

with $\varepsilon_i = \pm 1$, $i = 1,2,3$, and $m_1 = \varepsilon_1$, $m_2 = \varepsilon_2$, $\varepsilon_3 = 1$ for $g = z$ and $\varepsilon_3 = -1$ for $g = 1-z$. Notice that although such solutions are ζ-independent they are time dependent.

From the above discussion we also have that the only solutions of (2.10) and (2.13), for $\gamma \neq 1$, and so finite $\beta\, e^2$, are those were the field $\vec{n}$ is constant.

Therefore, except for the solution (2.14)-(2.15), the only physically interesting solutions of (1.1), within the ansatz (1.3)-(2.3), are those at the critical points $\gamma = \pm 1$, which solve the constraints (2.9) and (2.10). In that sense the conditions (1.7) and (1.13) constitute Bogomolny type equations for the model (1.1) at the critical points $\beta\, e^2 = 1$ and $\beta\, e^2 \to \infty$ respectively.

The integration of the equations (2.9) and (2.10) is quite simple. We can write them as

$$\frac{d}{dz}\left[\ln \frac{g}{1-g}\right] = \varepsilon\, \frac{\sqrt{\mid \Lambda_M \mid}}{z\,(1-z)}\,, \qquad \varepsilon = \pm 1 \tag{2.16}$$

and so

$$g = \frac{e^{\varepsilon\, w}}{1+e^{\varepsilon\, w}} \tag{2.17}$$

with

$$w = \int_{\kappa}^{z} dz'\, \frac{\sqrt{\mid \Lambda_M \mid}}{z'\,(1-z')}\,, \qquad \text{with} \qquad 0 < \kappa < 1 \tag{2.18}$$

where the integration constant was encoded into κ. Notice that the integrand in (2.18) is positive, and so w is negative for $0 < z < \kappa$ and it is positive for $\kappa < z < 1$, and vanishes at $z = \kappa$. In addition, w is a monotonically increasing function of z.

We point out that the equations (2.9) and (2.10) correspond in fact to the static Bogomolny equation for the sine-Gordon model. Indeed, if one defines

$$g = \sin^2\left(\frac{\phi}{4}\right) , \qquad\qquad 0 \leq \phi \leq 2\,\pi \tag{2.19}$$

then the equation (2.16) becomes

$$\frac{d\,\phi}{d\,w} = \varepsilon\,\sin\frac{\phi}{2} \tag{2.20}$$

which implies the second order static sine-Gordon equation, $\frac{d^2\,\phi}{d\,w^2} = \frac{1}{4}\sin\phi$. Therefore, the solutions (2.17) correspond to the soliton ($\varepsilon = 1$) and anti-soliton ($\varepsilon = -1$) solutions of the sine-Gordon model.

For the solutions of (2.10) we have Λ_M given by (2.12) and so (2.18) gives

$$w_{(-)} = 2 \mid m_3 \mid \left[\mathrm{ArcTan}\sqrt{\frac{z}{1-z}} - \mathrm{ArcTan}\sqrt{\frac{\kappa}{1-\kappa}}\;\right] \tag{2.21}$$

where the index $(-)$ refer to the w solutions for the case (2.10) and (2.12). The field g is given by (2.17) and so using (1.2) and (1.3) one gets the corresponding solution for the $\vec{n}$ field as

$$\vec{n} = \left(\frac{\cos\left(m_3\,\zeta\right)}{\cosh\frac{w_{(-)}}{2}}, \frac{\sin\left(m_3\,\zeta\right)}{\cosh\frac{w_{(-)}}{2}}, -\varepsilon\,\tanh\frac{w_{(-)}}{2}\right) \tag{2.22}$$

Notice that as z varies from 0 to 1, $w_{(-)}$ varies on finite range and it never diverges. Therefore, the n_3 component never reaches the values 1 or -1.

That differs from the behavior of the solutions for the case (2.9) and (2.11). In that case m_1 and m_2 do not vanish, and from (2.5) we have

$$\frac{\sqrt{\mid \Lambda_M \mid}}{z\,(1-z)} \sim \begin{cases} \frac{\mid m_2 \mid}{z} & \text{for} \qquad z \sim 0 \\ \frac{\mid m_1 \mid}{1-z} & \text{for} \qquad z \sim 1 \end{cases} \tag{2.23}$$

Therefore, w, given in (2.18), is logarithmically divergent for $z \sim 0$ and $z \sim 1$, and so $w \rightarrow -\infty$ for $z \rightarrow 0$, and $w \rightarrow +\infty$ for $z \rightarrow 1$. Consequently, one observes from (2.17) that the profile function g satisfy the boundary conditions

$$\begin{array}{llll} g\,(0) = 0 & g\,(1) = 1 & \text{for} & \varepsilon = 1 \\ g\,(0) = 1 & g\,(1) = 0 & \text{for} & \varepsilon = -1 \end{array} \tag{2.24}$$

and $g\,(z=\kappa) = \frac{1}{2}$ in both cases. Clearly, g is a monotomic function of w and so of z. Therefore, we have

$$\vec{n}\,(z=0) = (0,0,\varepsilon) \qquad\qquad \vec{n}\,(z=1) = (0,0,-\varepsilon) \tag{2.25}$$

In fact, the solution for the $\vec{n}$ field in that case is

$$\vec{n} = \left(\frac{\cos\,(m_1\,\xi + m_2\,\varphi + m_3\,\zeta)}{\cosh\frac{w}{2}}, \frac{\sin\,(m_1\,\xi + m_2\,\varphi + m_3\,\zeta)}{\cosh\frac{w}{2}}, -\varepsilon\,\tanh\frac{w}{2}\right) \tag{2.26}$$

with

$$\beta\,e^2 = 1 \qquad\qquad m_1,\, m_2 \neq 0 \qquad \text{and} \qquad (\mid m_1 \mid + \mid m_2 \mid)^2 > m_3^2 \tag{2.27}$$

Notice that the solutions (2.22) and (2.26) depend on a free parameter κ, with $0 < \kappa < 1$. That parameter appeared in (2.18) as an integration constant. It determines where w vanishes, i.e. $w\,(z=\kappa) = 0$. Therefore, from (2.22) and (2.26) on observes that $z = \kappa$ is the point where the third component of $\vec{n}$ vanishes. Such parameter is associated to a symmetry of the equations (2.9) and (2.10). Indeed, they were written as (2.16), which in its turn can be cast as $\frac{d}{d\,w}\left[\ln\frac{g}{1-g}\right] = \varepsilon$. Therefore, it is invariant under the translations $w \rightarrow w + \text{const.}$, which amounts to the freedom encoded into κ.

Notice that, from (2.18) one has

$$w = \mid m \mid\,\ln\left[\frac{z}{1-z}\;\frac{1-\kappa}{\kappa}\right] \qquad \text{for} \qquad \mid m_1 \mid = \mid m_2 \mid \equiv \mid m \mid \quad m_3 = 0 \tag{2.28}$$

Therefore the solution (2.26) coincides with (2.15) for $m_1^2 = m_2^2 = 1$, $m_3 = 0$ and $\kappa = 1/2$.

2.1 Static solutions

The equations of motion (1.9), as well as the action associated to (1.1), are conformally invariant in a four dimensional space-time, i.e., in $3+1$ or $4+0$ dimensions. That is what allowed us to use the ansatz (1.3)-(2.3), since it is invariant under the direct product of commuting $U(1)$ subgroups of the conformal group with the internal $U(1)$ subgroup generated by the phase transformations $u \rightarrow e^{i\,\alpha}\,u$. The constraints (1.7) and (1.13) are conformally invariant in a space-time of any number of dimensions. Consequently, we can use an ansatz based on the conformal group to solve those constraints in any dimension. We shall do that to construct static solutions of the theory (1.1) in three spatial dimensions. Notice that the conditions

$$\beta\,e^2 = 1 \qquad\qquad \partial_0 u = 0 \qquad\qquad \vec{\nabla}u\cdot\vec{\nabla}u = 0 \tag{2.29}$$

or

$$\beta\, e^2 \to \infty \qquad\qquad \partial_0 u = 0 \qquad\qquad \vec{\nabla} u \cdot \vec{\nabla} u^* = 0 \tag{2.30}$$

where $\vec{\nabla} u$ is the spatial gradient of u, are sufficient conditions for the equations of motion (1.9) to be satisfied.

The existence of static solutions of the theory (1.1) seems to be against Derrick's theorem [19], since the energy scales as $E \to \frac{1}{\lambda}\, E$ as $x^i \to \lambda\, x^i$. However, for the integrable sub-sectors where the constraints (2.29) or (2.30) holds true the Derrick's theorem does not apply. The reason for that may be related to the fact that very probably there is not a Lagrangian such that the equation (1.9) and the constraint (1.7), or (1.13), can be derived as the corresponding Euler-Lagrange equations.

Therefore, we shall also calculate three dimensional static solutions using the toroidal coordinates given by

$$x^1 = \frac{r_0}{\tilde{p}}\, \sqrt{z}\, \cos\varphi \qquad x^2 = \frac{r_0}{\tilde{p}}\, \sqrt{z}\, \sin\varphi \qquad x^3 = \frac{r_0}{\tilde{p}}\, \sqrt{1-z}\, \sin\xi \tag{2.31}$$

with

$$\tilde{p} = 1 - \sqrt{1-z}\, \cos\xi \tag{2.32}$$

Notice they are obtained from the coordinates (2.1) by setting $\zeta = 0$. The metric is given by

$$ds^2 = \left(\frac{r_0}{\tilde{p}}\right)^2 \left[\frac{dz^2}{4\, z\, (1-z)} + (1-z)\; d\xi^2 + z\, d\varphi^2\right]$$

The ansatz in this case corresponds to (see (1.3))

$$g \equiv g\,(z) \qquad\qquad \theta \equiv m_1\, \xi + m_2\, \varphi \tag{2.33}$$

with m_1 and m_2 being integers. Replacing that into the constraints (2.29) or (2.30) one gets

$$\vec{\nabla} u \cdot \vec{\nabla} u = 0 \qquad \leftrightarrow \qquad \left[\frac{z\,(1-z)}{g\,(1-g)}\, g'\right]^2 = \Lambda_S \tag{2.34}$$

and

$$\vec{\nabla} u \cdot \vec{\nabla} u^* = 0 \qquad \leftrightarrow \qquad \left[\frac{z\,(1-z)}{g\,(1-g)}\, g'\right]^2 = -\Lambda_S \tag{2.35}$$

where

$$\Lambda_S \equiv z\, m_1^2 + (1-z)\; m_2^2 \tag{2.36}$$

Since, $\Lambda_S \geq 0$ on the interval $0 \leq z \leq 1$, for any m_1 and m_2, it is clear that there are no non-trivial solutions for (2.30) inside such ansatz.

The integration of (2.34) can be done following (2.16)-(2.18) to give

$$g = \frac{e^{\varepsilon\, w}}{1 + e^{\varepsilon\, w}} \tag{2.37}$$

with

$$w = \int_{\kappa}^{z} dz' \, \frac{\sqrt{\Lambda_S}}{z'\,(1 - z')} \; , \qquad \text{with} \qquad 0 < \kappa < 1 \tag{2.38}$$

The corresponding solutions for the $\vec{n}$ field are

$$\vec{n} = \left(\frac{\cos\left(m_1\,\xi + m_2\,\varphi\right)}{\cosh\frac{w}{2}}, \frac{\sin\left(m_1\,\xi + m_2\,\varphi\right)}{\cosh\frac{w}{2}}, -\varepsilon\,\tanh\frac{w}{2} \right) \qquad \beta\, e^2 = 1 \tag{2.39}$$

3 The Hopf topological charge

We have calculated time dependent and static solutions for the theory (1.1). At each instant of time the solutions define a mapping from the spatial $\mathbb{R}^3$ to the target space S^2. In fact, all the solutions we have calculated have finite energy and so the fields go to a constant at spatial infinity. Therefore, for topological considerations we can identify the points at infinity and consider the maps $S^3 \to S^2$, which are classified into homotopy classes labeled by the integer Hopf index Q_H [20, 10]. In order to calculate such index we first consider the mapping of the spatial $\mathbb{R}^3$ into a three dimensional sphere S^3_Z parametrized by two complex functions Z_α, $\alpha = 1, 2$, such that $\mid Z_1 \mid^2 + \mid Z_2 \mid^2 = 1$. We choose those complex functions, is terms of the ansatz functions given in (1.3) and (2.3), as

$$Z_1 = \sqrt{1 - g}\, e^{i\, m_1\, \xi} \qquad\qquad Z_2 = \sqrt{g}\, e^{-i\, m_2\, \varphi - i\, m_2\, \zeta} \tag{3.1}$$

which defines the map $\mathbb{R}^3 \to S^3_Z$. The map $S^3_Z \to S^2$ is then given by $u = Z_1/Z_2$, where u parametrizes the plane where the stereographic projection of S^2 has been performed according to (1.2). The Hopf index is then given by [20]

$$Q_H = \frac{1}{4\,\pi^2} \int d^3x \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, A_i\, \partial_j\, A_k \tag{3.2}$$

where

$$A_j = \frac{i}{2} \sum_{\alpha=1}^{2} \left[Z^*_\alpha\, \partial_j\, Z_\alpha - Z_\alpha\, \partial_j\, Z^*_\alpha \right] \tag{3.3}$$

The integral (3.2) is performed for a fixed value of time (i.e. x^0), and Z_α are functions of (ζ, z, ξ, φ). Therefore, in order to perform that integral we follow the procedure of the appendix B and eliminate the coordinate ζ in favor of the time. Then, ζ becomes a function of (x^0, z, ξ) given by (B.3). Notice that the product of the integrand by the volume element in (3.2) is invariant under change of coordinates. We then choose the coordinates (z,ξ,φ) to perform the calculation. The result is

$$d^3x \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, A_i\, \partial_j\, A_k = dz\, d\xi\, d\varphi\, \left(-m_1\, m_2\right)\, \partial_z g \tag{3.4}$$

and consequently

$$Q_H = m_1\, m_2\, \left[g\,(0) - g\,(1)\right] \tag{3.5}$$

4 The energy

According to the discussion in (1.16)-(1.20) the Hamiltonian density of the theory (1.1) is given by

$$\mathcal{H} = \frac{(\beta\, e^2 - 1)}{e^2}\, \mathcal{H}_1 + \frac{1}{e^2}\, \mathcal{H}_2 \tag{4.1}$$

with $\mathcal{H}_1$ and $\mathcal{H}_2$ being given in (1.18) and (1.19) respectively.

The time dependent solutions (2.26) and the static solutions (2.39) have vanishing energy since they both satisfy the constraint (1.7), which means that $\mathcal{H}_2$ vanishes. In addition they are solutions only at the critical point $\beta\, e^2 = 1$, which means that the first term in (4.1) vanishes.

The time dependent solutions (2.22) have ill defined energy. The reason is that they satisfy the constraint (1.13) and so $\mathcal{H}_1$ vanishes. However, they are solutions only in the limit $\beta\, e^2 \rightarrow \infty$. So, no matter what $\mathcal{H}_2$ is when evaluated on that solution, the first term in (4.1) is ill defined. In addition, such solution have vanishing Hopf index according to (3.5).

There remains to analyze the energy for the time dependent solutions (2.15). Since they satisfy (1.7) then $\mathcal{H}_2$ vanishes when evaluated on them. We have now to integrate the first term in (4.1) on the three dimensional space and show that the energies of such solutions are indeed time independent. We use the procedure of appendix B to eliminate the coordinate ζ in favor of the time x^0, and then pick up the surfaces of constant time.

The quantity $\mathcal{H}_1$ given in (1.18) can be written as

$$\mathcal{H}_1 = 32\, \frac{\mid \dot{u} \mid^2}{\left(1+ \mid u \mid^2\right)^2}\, \frac{\partial_\mu u\, \partial^\mu u^*}{\left(1+ \mid u \mid^2\right)^2} - 8\, \frac{\left(\partial_\mu u\, \partial^\mu u^*\right)^2}{\left(1+ \mid u \mid^2\right)^4} \tag{4.2}$$

Using the ansatz defined in (1.3) and (2.3), and the coordinates introduced in (2.1), one can check that for the solutions (2.15) one has

$$\frac{\partial_\mu u\, \partial^\mu u^*}{\left(1+ \mid u \mid^2\right)^2} = -2\, \left(\frac{p}{r_0}\right)^2 \tag{4.3}$$

with p and r_0 defined in (2.1). Using the derivatives w.r.t. time (B.5) of the coordinates introduced in (2.1), one finds that for the solutions (2.15) one has

$$\frac{\mid \dot{u} \mid^2}{(1+\mid u \mid^2)^2} = \left(\frac{p}{r_0}\right)^2 z\,\tau^2 \tag{4.4}$$

where τ is a dimensionless time introduced in (B.2). Therefore, the quantity $\mathcal{H}_1$ evaluated on the solutions (2.15) is given by

$$\mathcal{H}_1 = -32\,\left(\frac{p}{r_0}\right)^4 \left(1 + 2\,z\,\tau^2\right) \tag{4.5}$$

Using the expression for three dimensional volume element given in (B.4) one gets that the energies for the solutions (2.15) are

$$\begin{aligned} E &= \frac{(\beta\, e^2 - 1)}{e^2} \int d^3x\, \mathcal{H}_1 \\ &= -\frac{16}{r_0\,(1+\tau^2)}\,\frac{(\beta\, e^2 - 1)}{e^2} \int dz\, d\xi\, d\varphi\, \left(1 + 2\,z\,\tau^2\right) \left[1 - \frac{(\pm\,\sigma)}{\sqrt{1+\tau^2\,(1-\sigma^2)}}\right] \end{aligned}$$

with σ as in (B.3), i.e. $\sigma = \sqrt{1-z}\, \cos\xi$. Notice that the σ-dependent term in the integral is odd under the transformation $\xi \rightarrow \xi + \pi$. Therefore it vanishes when integrated in ξ from 0 to $2\,\pi$, and we then obtain that the energy for the solutions (2.15) are

$$E = -\frac{64\,\pi^2}{r_0}\,\frac{(\beta\, e^2 - 1)}{e^2} \tag{4.6}$$

and so it is positive for $\frac{(\beta\, e^2 - 1)}{e^2} < 0$, in agreement with Theorem 1 at the end of section 1. Notice that, similarly to solutions (2.26) and (2.39), the solutions (2.15) have vanishing energy at the critical point $\beta\, e^2 = 1$.

5 Other cases

The theory (1.1) on a four dimensional Euclidean space is invariant under the conformal group $SO(5,1)$ which also possesses, like $SO(4,2)$ in Minkowski space, three commuting $U(1)$ subgroups. Therefore, one can implement a similar ansatz invariant under the diagonal subgroups of the tensor product of those $U(1)$ subgroups with the internal $U(1)$ group of phase transformations $u \rightarrow e^{i\alpha}\, u$. The relevant coordinates in that case are

$$\begin{aligned} x^0 &= \frac{r_0}{q}\,\sinh y \qquad & x^1 = \frac{r_0}{q}\,\sqrt{z}\, \cos\varphi \\ x^3 &= \frac{r_0}{q}\,\sqrt{1-z}\, \sin\xi \qquad & x^2 = \frac{r_0}{q}\,\sqrt{z}\, \sin\varphi \end{aligned} \tag{5.1}$$

where

$$q = \cosh y - \sqrt{1-z}\; \cos\xi \tag{5.2}$$

The coordinates z, ξ and φ are the same as in (2.1), but ζ is replaced by y which can take any real value, $-\infty \leq y \leq \infty$. The Euclidean metric in such coordinates becomes

$$ds^2 = \left(\frac{r_0}{q}\right)^2 \left[dy^2 + \frac{dz^2}{4\,z\,(1-z)} + (1-z)\; d\xi^2 + z\, d\varphi^2\right]$$

The ansatz in the Euclidean case is very similar to the Minkowski one, and it is given by (1.3) with the functions given by

$$g \equiv g\,(z) \qquad\qquad \theta \equiv m_1\,\xi + m_2\,\varphi + \omega\, y \tag{5.3}$$

with ω being a real parameter (frequency).

If one replaces the ansatz (1.3) and (5.3) into the equations of motion (1.9) associated to (1.1), one gets that the equation for θ is automatically satisfied and that the profile function $g\,(z)$ has to satisfy (2.4) with γ being the same as in (2.6) and with Λ_M replaced by

$$\Lambda_E \equiv z\, m_1^2 + (1-z)\; m_2^2 + z\,(1-z)\; \omega^2 \tag{5.4}$$

In addition if one evaluates the constraints (1.7) and (1.13) one obtains the same relations as in (2.9) and (2.10) with Λ_M replaced by Λ_E. Since Λ_E is non-negative on the interval $0 \leq z \leq 1$ it follows that there no non-trivial solutions for the constraint (1.13) within the ansatz (1.3) and (5.3).

If g satisfies the constraint (1.7), i.e. (2.9) with Λ_M replaced by Λ_E, then (2.4) in this case reduces to

$$(\gamma+1)\,\partial_z\left[\Lambda_E\,\partial_z g\right] = 0 \tag{5.5}$$

Performing an analysis similar to that below (2.13) one concludes that (2.15) is a solution in the Euclidean case with $\varepsilon_i = \pm 1$, $i = 1, 2, 3$, $m_1 = \varepsilon_1$, $m_2 = \varepsilon_2$, $\omega = 0$, and $\varepsilon_3 = 1$ for $g = z$ and $\varepsilon_3 = -1$ for $g = 1 - z$.

If $\beta\, e^2 = 1$, i.e. $\gamma = -1$, then g has to satisfy (2.9) with Λ_M replaced by Λ_E. Performing and analysis similar to that leading to (2.26) one concludes that (2.26) is a solution in the Euclidean case with m_3 replaced by ω, the coordinate ζ replaced by y, and w being given by (2.18) with Λ_M replaced by Λ_E. In addition, since Λ_E is non-negative on the interval $0 \leq z \leq 1$, for any values of m_1, m_2 and ω, one does not have restrictions similar to those in (2.27) involving m_i, $i = 1, 2, 3$.

One can also consider, in Minkowski space-time, a modification of the theory (1.1) given by

$$\tilde{\mathcal{L}} = -\frac{1}{e^2}\, H_{\mu\nu}^2 + \frac{\beta}{2}\;\mid (\partial_\mu \vec{n})^2 \mid\; (\partial_\nu \vec{n})^2 \tag{5.6}$$

It has the same target space $SO(3)$ and conformal $SO(4,2)$ symmetries as (1.1), and so the ansatz (1.3) and (2.3) also works here. It has the advantage of having a positive definite Hamiltonian for $\beta > 0$ and $e^2 > 0$ [21].

If one replaces the ansatz (1.3) and (2.3) into the Euler-Lagrange equations associated to (5.6), one obtains that the equation for the field θ is automatically satisfied and the equation for g leads to (2.4) with Λ_M being given by (2.5), but with the parameter γ being now given by

$$\gamma = \left(1 + \frac{2}{\beta\, e^2}\right) \tag{5.7}$$

Therefore all the solutions we have constructed for (1.1) are also solutions of (5.6). In fact, the only point we have to pay attention is that the critical point now corresponds to $\beta\, e^2 = -1$, instead of $\beta\, e^2 = 1$. Therefore, (2.15) are solutions of (5.6) for any value of β and e^2. In addition (2.26) as well as the static configurations (2.39) are solutions of (5.6) for $\beta\, e^2 = -1$. The configurations (2.22) are also solutions of (5.6) for $\beta\, e^2 \rightarrow \infty$.

6 Summary and conclusions

We have constructed exact solutions for the theory (1.1) exploring its conformal symmetry $SO(4,2)$ and the target space symmetry $u \rightarrow e^{i\,\alpha}\, u$. Using those symmetries we built an ansatz invariant under the diagonal subgroup of the direct product of the $U(1)$ target space group of phase transformations with three commuting $U(1)$ subgroups of $SO(4,2)$. The ansatz is given by (1.3) and (2.3) and is based on the toroidal like coordinates (2.1). The crucial point of our construction is that our solutions are in fact solutions of Bogomolny type equations given by the (1.7) and (1.13). Those two equations define submodels of the theory (1.1) possessing an infinite number of local conserved currents given by (1.11) and (1.14). In addition there exist critical points, defined by special values of the coupling constants, where the theory (1.1) becomes even more integrable. The first order differential equation (1.7) together with the condition $\beta\, e^2 = 1$, imply the second order equations of motion of the theory (1.1). The same is true for the equation (1.13) and the condition $\beta\, e^2 \rightarrow \infty$. The exact solutions we have constructed are the following:

1. The configurations (2.15) are solutions of (1.7) as well as of the equations of motion of (1.1), for any value of the coupling constants β and e^2. Their Hopf topological charge, according to (3.5), are given by $Q_H = -\varepsilon_1\, \varepsilon_2\, \varepsilon_3$, with $\varepsilon_i = \pm 1$, $i = 1, 2, 3$. All those eight solutions have the same energy and given by (4.6).

2. The configurations (2.26) are solutions of (1.7), and for $\beta\, e^2 = 1$ are also solutions

of the equations of motion of (1.1). They all have vanishing energies, and their Hopf charges are $Q_H = -\varepsilon\, m_1\, m_2$, with $\varepsilon = \pm 1$, and m_i, $i = 1, 2$, any integers (see (3.5)). Notice that such solutions have a free parameter κ inside the definition of w given in (2.18), that allows the deformation of the surfaces of constant n_3, i.e. the third component of the triplet $\vec{n}$.

3. The static configurations (2.39) are solutions of (1.7), and for $\beta\, e^2 = 1$ are also solutions of the equations of motion of (1.1). They all have vanishing energies, and their Hopf charges are $Q_H = -\varepsilon\, m_1\, m_2$, with $\varepsilon = \pm 1$, and m_i, $i = 1, 2$, any integers. They also carry the same free parameter κ and are the static version of the solutions (2.26).

4. The configurations (2.22) are solutions of (1.13), and for $\beta\, e^2 \to \infty$ are also solutions of the equations of motion of (1.1). Such solutions are not very interesting because they have vanishing Hopf topological charge, and their energies are ill defined. In this sense, the condition (1.13) does not lead to relevant solutions within the ansatz (1.3) and (2.3).

If one wants to find solutions of the theory (1.1) outside the integrable sub-sector defined by the condition (1.7) (or (1.13)), one has to solve the ordinary differential equation (2.4), for the profile function $g\,(z)$, using numerical methods. Even though that is quite feasible we do not pursue it in this paper. Perhaps the main difficulties facing the numerical integration of (2.4) are related to its critical points at $z = 0$ and $z = 1$. The best strategy would perhaps be to use the shooting method starting from both ends at $z = 0$ and $z = 1$ and try to adjust the initial conditions to match the two branches around $z = 1/2$.

The theory (1.1) is closed related to the so-called extended Skyrme-Faddeev model defined by the Lagrangian

$$\mathcal{L}_{\rm ESF} = M^2\, \partial_\mu \vec{n} \cdot \partial^\mu \vec{n} - \frac{1}{e^2}\, \left(\partial_\mu \vec{n} \wedge \partial_\nu \vec{n}\right)^2 + \frac{\beta}{2}\, \left(\partial_\mu \vec{n} \cdot \partial^\mu \vec{n}\right)^2 \tag{6.1}$$

In [15] it was constructed exact vortex solutions for (6.1). Those vortices (static and time-dependent) are solutions of the constraint (1.7) and solve the equations of motion only on the critical point $\beta\, e^2 = 1$. In that sense they are self-dual vortices. In [16] numerical solutions for hopfions were constructed for the theory (6.1). Those solutions have some interesting properties. They are close to satisfy the constraint (1.7), and they become solutions of it only in the limit $\beta\, e^2 \to 1$ (the point $\beta\, e^2 = 1$ was not accessible numerically). In addition, as $\beta\, e^2$ approaches unity, the quantity $a^2 = -e^2\, r_0^2\, M^2$ approaches zero. That can be interpreted by the fact that the size of the solution (determined by

r_0) approaches zero and so the solution collapses. However, one can connect that fact to the solutions constructed in this paper. As a^2 approaches zero one can think of M^2 going to zero and the theory (6.1) reducing to (1.1). Therefore, the numerical solutions of [16] should reduce to the self-dual hopfions constructed here, in the limit $\beta\, e^2 \to 1$.

A Analysis of Λ_M

Notice that for $m_3 = 0$, we have that Λ_M given in (2.5) is non-negative on the interval $0 \leq z \leq 1$. In addition, for $m_1 = m_2 = 0$, we have that Λ_M is non-positive on the same interval. We also have that

$$\Lambda_M \mid_{z=0} = m_2^2 \geq 0 \; , \qquad \Lambda_M \mid_{z=1} = m_1^2 \geq 0 \tag{A.1}$$

For $m_3 \neq 0$ we write

$$\Lambda_M = m_3^2 \left(z - z_+\right)\left(z - z_-\right) \; , \qquad z_\pm = \frac{1}{2}\left(-b \pm \sqrt{\Delta}\right) \tag{A.2}$$

with

$$b = r_+ \, r_- - 1 \; , \qquad \Delta = \left(r_+^2 - 1\right)\left(r_-^2 - 1\right) \; , \qquad r_\pm = \frac{m_1 \pm m_2}{m_3} \tag{A.3}$$

For $r_+^2 > 1$ and $r_-^2 < 1$ or $r_+^2 < 1$ and $r_-^2 > 1$, we have $\Delta < 0$ and so Λ_M does not have real zeros, and so $\Lambda_M \geq 0$ in that case.

Notice we can write b as

$$b = \frac{1}{2}\left[\left(r_+ - 1\right)\left(r_- + 1\right) + \left(r_+ + 1\right)\left(r_- - 1\right)\right] \tag{A.4}$$

Therefore, for $r_\pm > 1$ or $r_\pm < -1$ we can write

$$z_\pm = -\frac{1}{4}\left[\sqrt{\left(r_+ - 1\right)\left(r_- + 1\right)} \mp \sqrt{\left(r_+ + 1\right)\left(r_- - 1\right)}\right]^2 \tag{A.5}$$

Consequently, $z_- < 0$, and $z_+ \leq 0$ (with $z_+ = 0$ for $r_+ = r_-$, or $m_2 = 0$), and Λ_M has no zeros on the interval $0 < z \leq 1$, and so $\Lambda_M \geq 0$ on that interval.

For $r_+ > 1$ and $r_- < -1$, we write

$$\begin{aligned} z_\pm &= \frac{1}{4}\left[\sqrt{\left(r_+ + 1\right)\left(1 - r_-\right)} \pm \sqrt{\left(1 - r_+\right)\left(r_- + 1\right)}\right]^2 \\ &= \frac{1}{4}\left[\sqrt{4 + 2\, s_+ + 2\, s_- + s_+\, s_-} \pm \sqrt{s_+\, s_-}\right]^2 \end{aligned} \tag{A.6}$$

with $s_+ = r_+ - 1 > 0$, and $s_- = -\left(1 + r_-\right) > 0$. Consequently, $z_+ > 1$, and $z_- \geq 1$ (with $z_- = 1$ for $r_+ = -r_-$, or $m_1 = 0$), and Λ_M has no zeros on the interval $0 \leq z < 1$, and so $\Lambda_M \geq 0$ on that interval. For $r_+ < -1$ and $r_- > 1$, we use the same reasoning as in (A.6), with the replacement $r_+ \leftrightarrow r_-$, and the fact the b and Δ are invariant under such interchange.

For $-1 \leq r_\pm \leq 1$ we write

$$z_\pm = \frac{1}{4}\left[\sqrt{\left(r_+ + 1\right)\left(1 - r_-\right)} \pm \sqrt{\left(1 - r_+\right)\left(r_- + 1\right)}\right]^2 = \left[\sqrt{\left(1 - t_+\right)\left(1 - t_-\right)} \pm \sqrt{t_+\, t_-}\right]^2$$

with $t_+ = (1 - r_+)/2$, and $t_- = (r_- + 1)/2$, and so $0 \leq t_\pm \leq 1$. Consequently we have $0 \leq z_\pm \leq 1$, and so in that case, Λ_M does change sign on the interval $0 \leq z \leq 1$.

Summarizing we have that, on the interval $0 \leq z \leq 1$,

$$\Lambda_M = \begin{cases} \text{has no definite sign for } (\mid m_1 \mid + \mid m_2 \mid)^2 \leq m_3^2 \\ \text{is non-positive for } m_1 = m_2 = 0 \\ \text{is non-negative otherwise} \end{cases} \tag{A.7}$$

Notice that Λ_E introduced in (5.4) is non-negative on the interval $0 \leq z \leq 1$, as long as ω is taken to be real. However, if one allows it to be pure imaginary, i.e. $\omega = i\,\tilde{\omega}$, then the analysis above applies equally well for Λ_E with the replacement $m_3 \leftrightarrow \tilde{\omega}$.

B The surfaces of constant time

In order to evaluate the Hopf charges and the energies of the solutions we need to evaluate integrals over the three dimensional space at a given fixed value of time. The coordinates introduced in (2.1) are such that none of them are parallel to time. Therefore we shall choose to eliminate the coordinate ζ in favor of time and go from (ζ, z, ξ, φ) to (x^0, z, ξ, φ), which is then a non-orthogonal system of coordinates. In order to express ζ in terms of the new coordinates we use the first relation in (2.1), which leads to

$$\tau^2\, p^2 = 1 - \cos^2 \zeta \tag{B.1}$$

where we have introduced the dimensionless time

$$\tau \equiv \frac{x^0}{a} = \frac{c\,t}{a} \tag{B.2}$$

That is a quadractic relation for $\cos\zeta$. Solving it one has

$$\cos\zeta_\pm = \frac{\tau^2\,\sigma \pm \sqrt{1 + \tau^2\,(1 - \sigma^2)}}{1 + \tau^2} \qquad \text{with} \qquad \sigma \equiv \sqrt{1 - z}\,\cos\xi \tag{B.3}$$

with both signs giving equivalent relations among ζ, τ, z and ξ. In fact, $\cos\zeta_+(\sigma) = -\cos\zeta_-(-\sigma)$.

The three dimensional spatial volume element is then given by

$$dx^1\, dx^2\, dx^3 = \left(\frac{r_0}{p}\right)^4 \frac{1}{2\,r_0}\, \frac{1}{1 + \tau^2} \left[1 - \frac{(\pm\,\sigma)}{\sqrt{1 + \tau^2\,(1 - \sigma^2)}}\right] dz\, d\xi\, d\varphi \tag{B.4}$$

with the signs in front of σ being the same as those in (B.3). In addition, the derivatives of the coordinates with respect to time, useful in the calculation of the energy, are given by

$$
\begin{aligned}
\frac{\partial \zeta}{\partial x^0} &= \pm \left(\frac{p}{r_0}\right) \sqrt{1+\tau^2 \left(1-\sigma^2\right)} \\
\frac{\partial z}{\partial x^0} &= \left(\frac{p}{r_0}\right) 2\, z \sqrt{1-z}\, \cos \xi \;\; \tau \\
\frac{\partial \xi}{\partial x^0} &= \left(\frac{p}{r_0}\right) \frac{\sin \xi}{\sqrt{1-z}} \;\; \tau \\
\frac{\partial \varphi}{\partial x^0} &= 0
\end{aligned} \tag{B.5}
$$

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