

# Experimental implementation of a four-player quantum game

Christian Schmid<sup>1,2</sup>, Adrian P. Flitney<sup>3</sup>, Witlef Wieczorek<sup>1,2</sup>, Nikolai

Kiesel<sup>1,2</sup>, Harald Weinfurter<sup>1,2</sup>, and Lloyd C. L. Hollenberg<sup>3,4</sup>

<sup>1</sup>*Sektion Physik, Ludwig-Maximilians-Universität, D-80797 München, Germany*

<sup>2</sup>*Max-Planck-Institut für Quantenoptik, D-85748 Garching, Germany*

<sup>3</sup>*School of Physics, The University of Melbourne, Melbourne, Victoria 3010, Australia and*

<sup>4</sup>*Centre for Quantum Computing, University of Melbourne, Melbourne, Victoria 3010, Australia*

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Game theory is central to the understanding of competitive interactions arising in many fields, from the social and physical sciences to economics. Recently, as the definition of information is generalized to include entangled quantum systems, quantum game theory has emerged as a framework for understanding the competitive flow of quantum information. Up till now only two-player quantum games have been demonstrated. Here we report the first experiment that implements a four-player quantum Minority game over tunable four-partite entangled states encoded in the polarization of single photons. Experimental application of appropriate quantum player strategies give equilibrium payoff values well above those achievable in the classical game. These results are in excellent quantitative agreement with our theoretical analysis of the symmetric Pareto optimal strategies. Our result demonstrate for the first time how non-trivial equilibria can arise in a competitive situation involving quantum agents and pave the way for a range of quantum transaction applications.

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## INTRODUCTION

Originally developed for economics, the impact of game theory in this field is now pervasive, and has spread to many other disciplines as diverse as evolutionary biology [1], the social sciences [2] and international conflict [3]. Game theory is characterized by the solution concepts of the *Nash equilibrium* (NE), the result from which no player can improve their payoff by a unilateral change in strategy, and the *Pareto optimal* (PO) outcome, the one from which no player can improve their payoff without another player being worse off. The former can be considered as the equilibrium that is best for each player individually, while the latter is generally the best result for the players as a group.

In the search for new applications of quantum information processing based on competitive interactions between agents the appropriate language is quantum game theory. Applications can be based on exchanges between several agents who share entanglement, in contrast to direct classical communication. The sharing of an entangled resource permits competitive von Neumann-type games, with applications such as quantum auctions [4], quantum voting [5], and quantum communication. This is distinct from cooperative games that arise where agents are permitted to communicate, either directly or through a third party. In quantum games, entanglement is used as a resource that can enhance the payoffs of well known game-theoretic equilibria [6].

While there have been implementations of two-player quantum games with limited strategic spaces [7, 8], here we report for the first time the implementation of a genuine multiplayer quantum game driven by a four particle entangled state. Generally, two-player quantum games do not see the enhancement of the pure strategy equilibrium payoffs [9, 10] that can arise in multiplayer quantum games. The practical exploration of quantum games allows us to learn about the nature of quantum information in a real situation. Figure 1 compares the classical and quantum versions of a four-player game.

We choose to implement a four-player Minority game. The Minority game arises as a simple multi-agent model for studying strategic decision making within a group of agents [11]. Its classical version has been used as an iterative model of buying and selling in a stockmarket [12, 13, 14]. In each play, the agents independently select between one of two options, labeled ‘0’ and ‘1’ (‘buy’ and ‘sell’). Those that choose the minority are awarded a payoff of one unit, while the others receive no payoff. Players can utilize knowledge of past successful choices to optimize their strategy. In the quantum situation we can represent each player’s binary choice by the polarization of a photon as in Fig. 2. Quantum versions of a one-shot Minority game have generated some theoretical interest [6, 15, 16, 17, 18]. In this paper we report on the first experimental implementation of the quantum Minority game (QMG).

We begin with a theoretical exploration of the QMG. With the particular experiment in mind, we choose the initial state to belong to a continuous set of four-partite entangled states involving a mixture of the GHZ state and products of EPR pairs. We calculate the NE and PO strategy profiles as a function of the initial state. Our method is a general

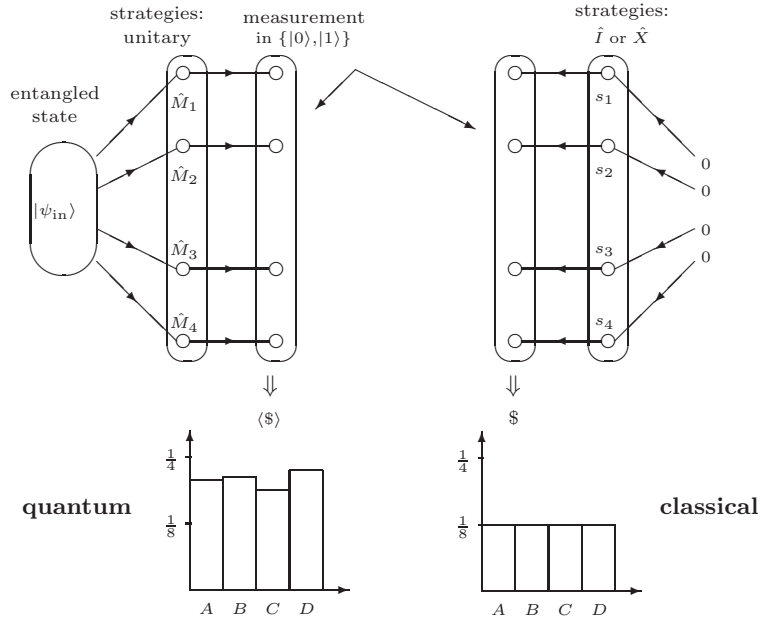


FIG. 1: Schematic showing the flow of information in a four-player quantum game (left) and a four-player classical game (right). In the quantum version the players utilize an entangled state as a resource, however in both versions there is no communication between the players. For the particular case of the Minority game the equilibrium payoffs are indicated in the bar graph, with those of the quantum case being the values achieved experimentally with our setup.

one that can be applied to any quantum game that has a similar protocol. Our quantization of the Minority game is described as follows and is shown schematically in Fig. 1. Each of four players is given one qubit from a known four-partite entangled state. This state is an element of the subspace spanned by the four-qubit GHZ state and a product state of two Bell (or EPR) pairs. The players are permitted to act on their qubit with any local unitary operation. The choice of such an operator is the player's strategy. During this stage coherence is maintained and no communication between the players is permitted. After the player actions, a referee measures the qubits in the computational basis and awards payoffs using the classical payoff scheme. In the dominant protocol of quantum games due to Eisert *et al.* [6, 19], an entangling gate is used to produce a GHZ state from an initial state of  $|00\dots 0\rangle$ . After the players actions the inverse operator is applied to the multi-partite state. For the Minority game this last operator only has the effect of interchanging between states where the same player(s) win and so can be omitted without changing the expected payoffs. Our arrangement is consistent with the generalized quantum game formalisms of Lee and Johnson [20] and Gutoski and Watrous [21], and is particularly suited for an implementation using entangled photon states and linear optics quantum logic.

## THEORY

A four player QMG was first examined by Benjamin and Hayden [6], and later generalized to multiple players [15] and to the consideration of decoherence [16]. Formally the QMG is played by computing the state

$$\rho_{\text{final}} = (\hat{A} \otimes \hat{B} \otimes \hat{C} \otimes \hat{D}) \rho_{\text{in}} (\hat{A} \otimes \hat{B} \otimes \hat{C} \otimes \hat{D})^\dagger, \quad (1)$$

where  $\hat{A}, \hat{B}, \hat{C}, \hat{D}$  are the operators representing the strategies of the four players and  $\rho_{\text{in}}$  is some four qubit input state. The qubits in  $\rho_{\text{final}}$  are subsequently measured in the computation basis and payoffs are awarded using the classical payoff matrix. Work to date has concentrated on the consideration of an initial GHZ state. In this paper we instead consider an the initial state that is a superposition of the GHZ state with products of EPR pairs:

$$\begin{aligned} |\Psi(\alpha)\rangle &= \frac{\alpha}{\sqrt{2}} (|0000\rangle + |1111\rangle) + \frac{\sqrt{1-\alpha^2}}{2} (|01\rangle + |10\rangle) \otimes (|01\rangle + |10\rangle) \\ &\equiv \frac{\alpha}{\sqrt{2}} (|HHHH\rangle + |VVVV\rangle) \\ &\quad + \frac{\sqrt{1-\alpha^2}}{2} (|HVHV\rangle + |HV VH\rangle + |VHHV\rangle + |VHVH\rangle), \end{aligned} \quad (2)$$

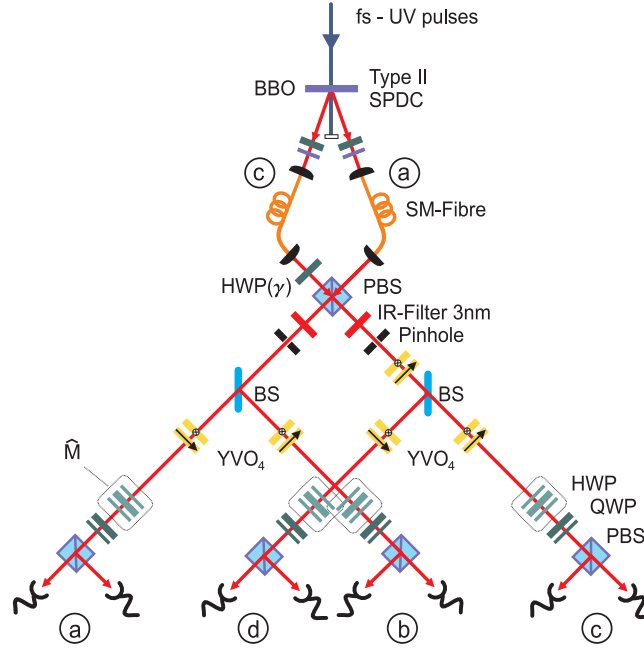


FIG. 2: Setup for the observation of  $|\Psi(\alpha)\rangle$ . Four photons, emitted in two spatial modes  $a$  and  $c$ , interfere at a polarizing beam splitter (PBS). Prior to the interference the polarization of the photons in mode  $c$  is rotated by a half-waveplate (HWP) set to an angle  $\gamma$ . Afterwards the photons are split into four output modes  $a, b, c, d$  by two beam splitters (BS). Under the condition of detecting one photon in each mode the desired state is observed. The player unitary strategies are executed by a combination of a quarter-, a half- and a second quarter-waveplate ( $\hat{M}$ ).

where  $\alpha \in [0, 1]$  is an adjustable parameter. The qubits are encoded in the polarization of single photons propagating in well-defined spatial modes. The computational basis states  $|0\rangle$  and  $|1\rangle$  are represented by the states  $|H\rangle_i$  and  $|V\rangle_i$ , denoting the state of a single photon in the spatial mode  $i$  with linear horizontal and linear vertical polarization, respectively. Most of the time the spatial mode will be evident from the context and hence the subscript be omitted.

To make allowances for some loss of fidelity in the preparation of this state the initial state will be written as the mixed state

$$\rho_{\text{in}} = f|\Psi\rangle\langle\Psi| + \frac{1-f}{16}\sum_{ijkl=0,1}|ijkl\rangle\langle ijkl|, \quad (3)$$

where  $f \in [0, 1]$  is a measure of the fidelity of production of the desired initial state. The second term in (3) represents completely random noise [22]. The players' strategies are single-qubit unitary operators that can be parametrized in the form

$$\hat{M}(\theta, \beta_1, \beta_2) = \begin{pmatrix} e^{i\beta_1} \cos(\theta/2) & ie^{i\beta_2} \sin(\theta/2) \\ ie^{-i\beta_2} \sin(\theta/2) & e^{-i\beta_1} \cos(\theta/2) \end{pmatrix}, \quad (4)$$

where  $\theta \in [0, \pi]$ ,  $\beta_1, \beta_2 \in [-\pi, \pi]$ . In our construction of the QMG only the difference in the phases is relevant to the expected payoff so it suffices to use a restricted set of operators where  $\beta \equiv \beta_1 = -\beta_2$ .

In the case where  $\alpha = 1$  it is known that a symmetric NE occurs when all players choose the strategy [6]

$$\hat{M}\left(\frac{\pi}{2}, \frac{\pi}{8}, \frac{-\pi}{8}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/8} & ie^{-i\pi/8} \\ ie^{i\pi/8} & e^{-i\pi/8} \end{pmatrix}. \quad (5)$$

Although the value  $\beta_1 = -\beta_2 = \pi/8$  is not the unique optimum, it is a focal point [23] that attracts the attention of the players since it is the simplest optimum value, and therefore there is no great difficulty in arriving at this NE. Given that at most one player of the four can be in the minority,  $\frac{1}{4}$  is the greatest average payoff that can be expected. This is realized with the above strategy when the initial state has maximum fidelity. As  $f \rightarrow 0$ , the payoff reduces to  $\frac{1}{8}$ , the optimal payoff in a one-shot classical Minority game, where the the players can do no better than choosing between the two options with equal probability. This is as expected since in the absence of entanglement the QMG cannot give any advantage over its classical counterpart.

We can search for a NE in the case of general  $\alpha$  as follows. If there exists  $\theta^*, \beta^*$  such that

$$\left\langle \$_D \left( \hat{M}(\theta^*, \beta^*, -\beta^*)^{\otimes 4} \right) \right\rangle \geq \left\langle \$_D \left( \hat{M}(\theta^*, \beta^*, -\beta^*)^{\otimes 3} \otimes \hat{M}(\theta, \beta, -\beta) \right) \right\rangle \quad \forall \theta, \beta, \quad (6)$$

where  $\$_D(\cdot)$  denotes the payoff to the fourth player (Debra) for the indicated strategy profile, then  $\hat{M}(\theta^*, \beta^*, -\beta^*)$  is a symmetric NE strategy. There is no in principle objections to asymmetric NE strategy profiles, where the players choose different strategies, however in practice these cannot be reliably achieved in the absence of communication between the players since it is otherwise impossible for the players to know which of the different strategies to select. Necessary, but not sufficient, conditions for the existence of a symmetric NE are

$$\left. \frac{d\langle \$_D \rangle}{d\theta} \right|_{\theta=\theta^*, \beta=\beta^*} = 0, \quad \left. \frac{d\langle \$_D \rangle}{d\beta} \right|_{\theta=\theta^*, \beta=\beta^*} = 0, \quad (7a)$$

$$\left. \frac{d^2\langle \$_D \rangle}{d\theta^2} \right|_{\theta=\theta^*, \beta=\beta^*} \leq 0, \quad \left. \frac{d^2\langle \$_D \rangle}{d\beta^2} \right|_{\theta=\theta^*, \beta=\beta^*} \leq 0, \quad (7b)$$

where  $\langle \$_D \rangle$  is here the payoff on the right hand side of (6). Inequalities in (7b) indicate a local maximum in the payoff to Debra, however this may not be a global maximum. An equality in (7b) may mean a local maximum, minimum or an inflection point in the payoff. We shall now enumerate all the symmetric NE strategies by considering Eqs. (7a–7b) over the range of  $\alpha \in [0, 1]$ .

If Alice, Bob and Charles use the strategy  $\hat{M}(\theta, \beta, -\beta)$  while Debra plays  $\hat{M}(\theta', \beta', -\beta')$  then

$$\begin{aligned} \left. \frac{d\langle \$_D \rangle}{d\theta'} \right|_{\theta'=\theta, \beta'=\beta} &= \frac{\sin 2\theta}{8} \left[ 2\alpha^2 + 2\alpha\sqrt{2-2\alpha^2} \cos 4\beta \right. \\ &\quad \left. + (2\alpha^2 - 2 - 2\alpha\sqrt{2-2\alpha^2} \cos 4\beta - \alpha^2 \cos^2 4\beta) \sin^2 \theta \right], \\ \left. \frac{d\langle \$_D \rangle}{d\beta'} \right|_{\theta'=\theta, \beta'=\beta} &= \frac{\alpha}{2} \sin 4\beta \sin^2 \theta \\ &\quad \times \left[ (\sqrt{2-2\alpha^2} + \alpha \cos 4\beta) \sin^2 \theta - 2\sqrt{2-2\alpha^2} \right]. \end{aligned} \quad (8)$$

We want to find  $\theta, \beta$  for which these derivatives are simultaneously zero. Apart from the known NE for  $\alpha = 1$  we find that the only other symmetric NE occurs for  $\alpha \leq \sqrt{\frac{2}{3}}$  when  $\cos 4\beta = 1$  and

$$\cos \theta = \sqrt{\frac{2-3\alpha^2}{2-\alpha^2+2\alpha\sqrt{2-2\alpha^2}}}. \quad (9)$$

The expected payoff to each player for this equilibrium is

$$\langle \$ \rangle = \frac{\alpha(2-3\alpha^2)(\alpha+\sqrt{2-2\alpha^2})}{4-2\alpha^2+4\alpha\sqrt{2-2\alpha^2}}, \quad (10)$$

which reaches a maximum value of  $(3+2\sqrt{3})/(18+10\sqrt{3}) \approx 0.183$  at  $\alpha = \sqrt{\frac{1}{6}(3-\sqrt{3})}$ . Figure 3 gives the value of  $\theta$  and the resulting payoff for this solution.

We now consider the PO strategy profile. Again we will only consider symmetric strategy profiles. That is, we are interested in  $\theta^*, \beta^*$  for which

$$\left\langle \$ \left( \hat{M}(\theta^*, \beta^*, -\beta^*)^{\otimes 4} \right) \right\rangle \geq \left\langle \$ \left( \hat{M}(\theta, \beta, -\beta)^{\otimes 4} \right) \right\rangle \quad \forall \theta, \beta, \quad (11)$$

where  $\$$  represents the payoff to any one of the four players for the indicated strategy profile. Suppose all players select the strategy  $\hat{M}(\theta, \beta, -\beta)$  for some  $\theta, \beta$  to be determined. We proceed as before by finding the stationary points of the payoff to each player as a function of the parameters  $\theta$  and  $\beta$ . We find that for  $\alpha > \sqrt{\frac{2}{3}}$ , where the initial state is dominated by the GHZ state, the optimal strategy is  $\hat{M}(\frac{\pi}{2}, \frac{\pi}{8}, -\frac{\pi}{8})$ , while for the EPR-dominated region,  $\alpha < \sqrt{\frac{2}{3}}$ ,

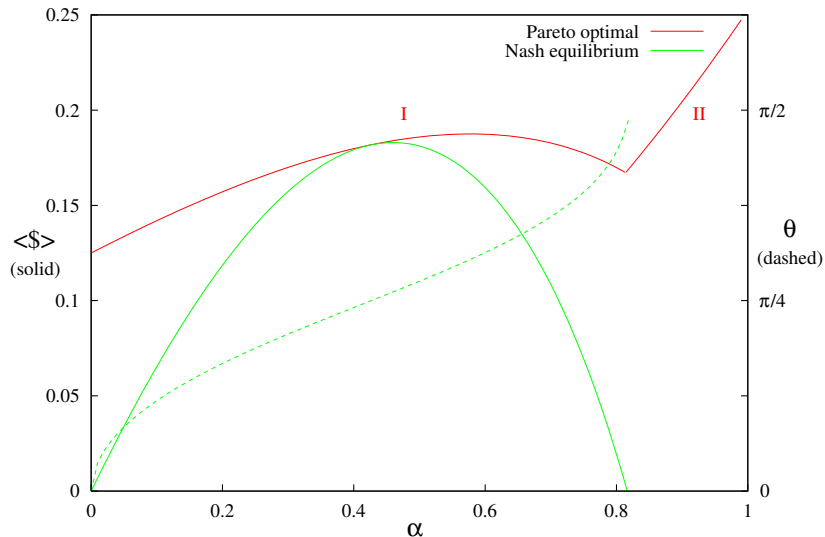


FIG. 3: *Left scale, solid lines:* The Pareto optimal (—•—) and Nash equilibrium (—•—) payoffs as a function of the initial state parameter  $\alpha$ . The Pareto optimal payoff curve is labeled by I or II for strategy  $\hat{M}(\frac{\pi}{4}, 0, 0)$  or  $\hat{M}(\frac{\pi}{2}, \frac{\pi}{8}, -\frac{\pi}{8})$ , respectively. *Right scale, dashed line:* The value of  $\theta$  given by (9) that, along with  $\beta = 0$ , gives a symmetric Nash equilibrium.

the optimal strategy is  $\hat{M}(\frac{\pi}{4}, 0, 0)$ . At  $\alpha = \sqrt{\frac{2}{3}}$  all components in the initial state are equally weighted and both strategies yield the same results. The payoffs for the two regions are

$$\langle \$ \rangle_{\text{I}} = \frac{1}{8} + \frac{f}{16} \alpha (2\sqrt{2 - 2\alpha^2} - \alpha) \quad \alpha \leq \sqrt{\frac{2}{3}} \quad (12)$$

$$\langle \$ \rangle_{\text{II}} = \frac{1}{8} + \frac{f}{8} \alpha (2\alpha^2 - 1) \quad \alpha \geq \sqrt{\frac{2}{3}} \quad (13)$$

Figure 3 shows the payoffs for these cases for a fidelity of  $f = 1$ .

There has been some recent interest in the correspondence between equilibria in quantum game theory and the violation of Bell inequalities [24, 25]. In our case we note that the curve for the symmetric PO payoff is the same as that for the maximal violation of the Mermin-Ardehali-Belinski-Klyshko (MABK) inequality [26].

## EXPERIMENT

### State observation:

The states are obtained with a new linear optics network which enables the observation of a whole family of states in a single setup by the tuning of one experimental parameter [27]. The setup relies on the interference of the second order emission of type II non-collinear spontaneous parametric down conversion (SPDC) and is depicted in Fig. 2. The down conversion emission yields four photons, two horizontally and two vertically polarized, in two spatial modes  $a$  and  $c$ . The photons are overlapped on a polarizing beam splitter (PBS) and afterwards symmetrically split up by two polarization independent beam splitters (BS) into the four spatial modes  $\{a, b, c, d\}$ . Prior to the interference at the PBS the polarization of the photons in mode  $c$  is rotated by a half-waveplate (HWP). Under the condition of detecting one photon in each spatial mode the desired state is observed. The weighting coefficient  $\alpha$  is thereby determined by the orientation of the principal axis  $\gamma$  of the HWP according to

$$\alpha(\gamma) = \frac{2\sqrt{2} \sin^2(2\gamma)}{\sqrt{5 - 4 \cos(4\gamma) + 3 \cos(8\gamma)}} \quad (14)$$

with  $\gamma \in [0, \frac{\pi}{8}]$ .

### Implementation of the game:

The implementation of the four-player QMG consists of acting with the strategy operator on each qubit and afterwards measuring the resulting output state in different bases. The unitary transformation corresponding to the players' choice of strategy is realized by a series of quarter-, half- and quarter-waveplate (see Fig. 2). In order to play strategy I and to act with  $\hat{M}_I \equiv \hat{M}(\frac{\pi}{2}, \frac{\pi}{8}, -\frac{\pi}{8})$  the angles of the waveplates are chosen as  $\{-\frac{\pi}{8}, \frac{5\pi}{16}, 0\}$ , and for strategy II and  $\hat{M}_{II} \equiv \hat{M}(\frac{\pi}{4}, 0, 0)$  as  $\{\frac{\pi}{2}, \frac{\pi}{16}, \frac{\pi}{2}\}$ .

### Results:

We start the measurement with  $\alpha = 1$  as in this case a NE solution is expected. The expected state after the application of strategy I, which is here the optimal one, is of the form

$$\begin{aligned} (\hat{M}_I^{\otimes 4}|GHZ\rangle) &= -\frac{1}{2\sqrt{2}}\left(|HHHV\rangle + |HHVH\rangle + |HVHH\rangle + |VHHH\rangle\right. \\ &\quad \left. - |VVVH\rangle - |VVHV\rangle - |VHVV\rangle - |HVVV\rangle\right) \\ &= \frac{i}{\sqrt{2}}(|RRRR\rangle - |LLLL\rangle), \end{aligned} \quad (15)$$

with  $|R/L\rangle = \frac{1}{\sqrt{2}}(|H\rangle \pm i|V\rangle)$  being the right and left circular polarization states. The transformed state is also of GHZ-type, and thus the fidelity  $F = \langle GHZ | M_I^{\dagger \otimes 4} \rho_{\text{exp}}^{\alpha=1} \hat{M}_I^{\otimes 4} | GHZ \rangle$  of the experimental state  $\rho_{\text{exp}}^{\alpha=1}$  equals the average expectation value of the GHZ stabilizer operators [28]. A measurement of the respective correlations is hence sufficient to evaluate the state fidelity. In our case, this requires 9 measurement settings and we obtain a value of  $F = 0.746 \pm 0.019$ . For comparison, the fidelity of the untransformed initial GHZ state obtained with our setup is  $F = 0.745 \pm 0.022$ .

The payoff awarded to each player can be evaluated from the correlation measurement in the computational basis,  $\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$ . Averaged over all four players the payoff obtained for  $\alpha = 1$  is  $\langle \$I \rangle_{\alpha=1} = 0.206 \pm 0.009$  which is well above the classical limit of  $\frac{1}{8}$ .

As was shown earlier, the maximal achievable payoff depends on  $\alpha$  and, moreover, there is an interesting change in the Pareto optimal strategy at  $\alpha = \sqrt{2/3}$ . In order to test this experimentally we chose to perform measurements for other distinguished states,  $|\Psi(0)\rangle$ ,  $|\Psi(\sqrt{\frac{2}{11}})\rangle$ , and  $|\Psi(\sqrt{\frac{2}{3}})\rangle$ . The results obtained are summarized in Fig. 4 which gives the average PO payoffs for the four values of  $\alpha$  for strategies I and II, along with best fit curves and theoretical curves for perfect fidelity. As can be seen in the boxes at some example points, the payoffs differ slightly for each player but are generally comparable within typical measurement error of around 9–12%. The only exception occurs for player C and player A for  $\alpha = \sqrt{2/11}$  and  $\sqrt{2/3}$ , respectively. The cause of this could not be rigorously identified. The average payoffs follow the expected dependence for both strategies, once imperfect state quality is taken into account. To allow for loss in the states' fidelity the data in Fig. 4 is fitted assuming a mixed input state of the form (3). Although the assumptions of an admixture of white noise and constant state quality are only an approximations, the value of  $f = 0.71 \pm 0.03$  obtained from the fit is in good agreement with the measured GHZ fidelity, where  $F = (1 + 15f)/16 \approx 0.73$ . Let us discuss the results for each measured point in more detail.

For  $\alpha = 0$  a product of two Bell states is expected. As this state is not four-qubit entangled it should yield at best the classical payoff for strategy II. This is nicely reproduced in the experiment with an average payoff of  $\langle \$II \rangle_{\alpha=0} = 0.125 \pm 0.006$ . If the players chose to play strategy I zero payoff is expected for ideal, pure input states. However, for an increasing mixedness of the input states the payoff approaches the classical limit, with equality for  $f \rightarrow 0$ . This fact leads to a non-zero value for the experimentally measured average payoff for strategy I,  $\langle \$I \rangle_{\alpha=0} = 0.033 \pm 0.003$ .

Similar behavior is obtained for  $\alpha = \sqrt{2/11}$ , where the measured payoffs for strategy II,  $\langle \$II \rangle_{\alpha=\sqrt{2/11}} = 0.156 \pm 0.007$ , and strategy I,  $\langle \$I \rangle_{\alpha=\sqrt{2/11}} = 0.073 \pm 0.004$ , are lower and higher than expected, respectively. Never the less, the experimentally reached state quality is high enough to ensure for the proper strategy, a payoff which exceeds the one maximally achievable in the equivalent classical game.

As mentioned before,  $\alpha = \sqrt{2/3}$  is special as both strategies lead to the same payoff. It is the point about which the players have to switch between the different strategies. The corresponding state,  $|\Psi(\sqrt{2/3})\rangle \equiv |\Psi_4\rangle$  is well known from literature as it can be directly obtained from SPDC [22, 29, 30] and has several interesting applications

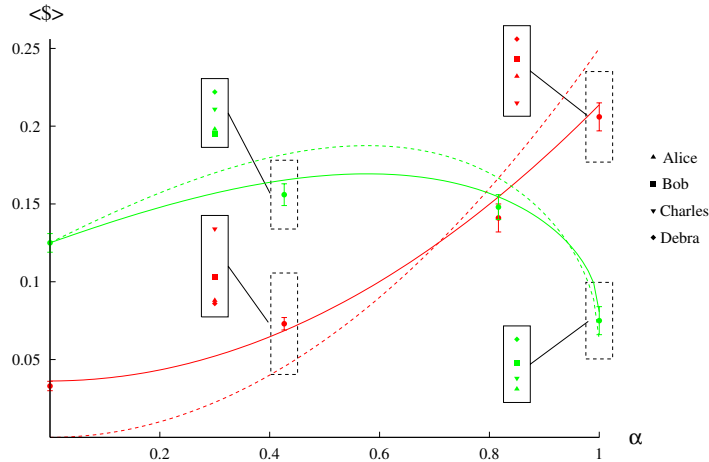


FIG. 4: Measured payoff  $\langle \$_{i,II} \rangle$  averaged over all four players as a function of  $\alpha$  for strategy I (-●-) and strategy II (-○-). The data is fitted assuming a mixed input state of the form  $f |\Psi(\alpha)\rangle\langle\Psi(\alpha)| + (1-f)/16 \mathbb{1}^{\otimes 4}$  [see (3)]. Dashed lines correspond to the ideal case  $f = 1$ . The boxes show the individual values for the four players for some sample points. Typical errors for individual measurements are 9–12 %. The maximum equilibrium payoff in a classical game is  $\frac{1}{8}$ .

in quantum information (see, for example, Refs. [31, 32]). The point’s feature of being a quantum fulcrum is nicely reproduced in the experiment. Within the measurement errors the average payoffs,  $\langle \$_I \rangle_{\alpha=\sqrt{2/3}} = 0.141 \pm 0.009$  and  $\langle \$_{II} \rangle_{\alpha=\sqrt{2/3}} = 0.148 \pm 0.008$  are the same for strategy I and II (see Fig. 4).

Finally, the values obtained in the GHZ case fit well in the dependence prescribed by the previous points. Like for the other states, due to imperfect state quality the average payoff is slightly lower (higher) than the ideal value of  $\frac{1}{4}$  ( $\frac{1}{16}$ ) for strategy I (II).

In summary, we can state that for all values of  $\alpha$  the players are awarded payoffs above the classical NE value if they choose the appropriate strategy. The values are consistent with a fidelity of 0.73 for the initial production of four-partite entangled state.

In our realization of the Minority game we do not use an unentangling gate. Thus a measurement in the computational basis alone does not prove that the higher than classical payoff values have their seeds in the four-qubit entanglement of  $|\Psi(\alpha)\rangle$ . In principle they could be caused by an admixture of the separable state

$$\rho_{\text{sep}} = \frac{1}{8} \left( |HHHV\rangle\langle HHHV| + \text{permutations} \right. \\ \left. + |VVVH\rangle\langle VVVH| + \text{permutations} \right), \quad (16)$$

which yields a maximal “payoff” of  $\frac{1}{4}$  when measured in the computational basis. However, measured in other bases, the state  $\rho_{\text{sep}}$  will not give a payoff above the classical limit.

In contrast, the state  $|\Psi(\alpha)\rangle$  has the extraordinary property that it exhibits the same term structure in the bases  $\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$  and  $\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$  when transformed by  $\hat{M}_I^{\otimes 4}$  and in the bases  $\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$  and  $\sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y$  when transformed by  $\hat{M}_{II}^{\otimes 4}$ . Consequently, if the payoff is evaluated analogously in these bases, the same dependence on  $\alpha$  is expected. In order to prove this we have performed the same measurements as before in the bases  $\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$  and  $\sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y$ . The result is shown in Fig. 5.

For each basis and appropriate strategy we find very similar curves to those for the computational basis. We applied the same fitting procedure as before. The resulting values for  $f$  are slightly different for both bases ( $f_x = 0.736 \pm 0.019$  and  $f_y = 0.706 \pm 0.053$ ) but comparable with the one found for the computational basis. The small asymmetry between the bases indicates that, contrary to our assumption, the experimental noise is not purely white noise.

## CONCLUSIONS

We have studied the four-player quantum Minority game where the initial state is drawn from a continuous set of four-partite entangled states consisting of a superposition of the GHZ state and a product of EPR pairs. There is a well known Nash equilibrium when the initial state is a GHZ state. For other initial states in our set there

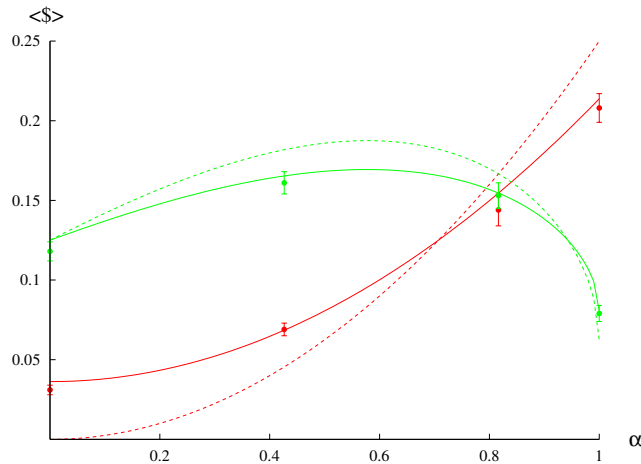


FIG. 5: Measured payoff  $\langle \$_{i,m} \rangle$  averaged over all four players as a function of  $\alpha$  in the basis  $\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$  for strategy I (-●-) and in the basis  $\sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y$  for strategy II (-●-). The data is fitted assuming a mixed input state of the form  $f |\Psi(\alpha)\rangle\langle\Psi(\alpha)| + (1-f)/16 \mathbb{1}^{\otimes 4}$  [see (3)]. Dashed lines correspond to the ideal case  $f = 1$ . The maximum average equilibrium payoff in a classical game is  $\frac{1}{8}$ .

are only symmetric Nash equilibria when the initial state is in the EPR-dominated region, though the payoff for this equilibrium is only sometimes higher than that achievable classically. There are two symmetric Pareto optimal strategies, one for the EPR- and one for the GHZ-dominated initial state. In both cases the payoff is equal to the classical optimal payoff of  $\frac{1}{8}$  plus a value proportional to the fidelity to which the initial entanglement is produced.

An experimental implementation of this game is carried out using four-photon entangled states generated from spontaneous parametric down conversion. Results for four different initial states were obtained. The results are consistent with the theory for fidelities in the range 0.71–0.73 depending on the initial state. For these fidelities the average payoffs obtained are above those that can be obtained in a classical Minority game (with an unentangled initial state) thus demonstrating the usefulness of entanglement. This is the first implementation of a multiplayer quantum game.

The study of quantum games is interesting in general as it provides another point of view to exploit pre-existing classical frameworks as a base for finding new ways of understanding and using entanglement in quantum systems. While game theory is the mathematical language of competitive (classical) interactions, quantum game theory is the natural extension to consider competitive situations in quantum information settings. For example, eavesdropping in quantum communication (see, for example, Refs. [33, 34, 35]) and optimal cloning [36] can be conceived as strategic games between two or more players. The particular game discussed here represents a different means of studying the entanglement structure of the used input states. As shown elsewhere [26], there even exists an interesting correspondence between the equilibria in the Minority game and the violation of Bell inequalities.

## METHODS

For operation of the SPDC a  $\beta$ -Barium borate (BBO) crystal is pumped by UV pulses having a central wavelength of 390 nm and an average power of 600 mW obtained from a frequency-doubled Ti:Sapphire oscillator (pulse length is 130 fs). Another HWP and a 1 mm thick BBO crystal compensate walk-off effects. The spatial modes  $a$  and  $c$  are defined by coupling the emitted photons in single mode fibres. Spectral filtering of each mode is achieved by interference filters centered around the degenerate wavelength of 780 nm with 3 nm full width at half maximum. The optical path lengths in modes  $a$  and  $c$  are aligned by a Hong–Ou–Mandel (HOM) interference measurement [37] to ensure temporal mode matching. Pinholes serve as mode filters to enhance the spatial indistinguishability of the photons at the PBS. Phases between horizontal and vertical polarization are set by pairs of perpendicularly oriented birefringent Yttrium-Vanadate (YVO<sub>4</sub>) crystals. Polarization analysis is performed in all of the four outputs by half- and quarter-waveplates followed by a PBS. The photons are detected by eight Silicon avalanche photo diodes (Si-APD) whose signals are fed into a multichannel coincidence unit which allows us to simultaneously register any possible coincidence detection between the inputs. The rates for each of the 16 characteristic fourfold coincidences are corrected for the difference in the efficiencies of the detectors. The errors on all quantities are deduced from



Poissonian counting statistics of the raw detection events and independently determined efficiencies.

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[Competing Interests] The authors declare that they have no competing financial interests.

[Correspondence] Correspondence and requests for materials should be addressed to CS (email: christian.schmid@mpq.mpg.de).