Collins diffraction formula and the Wigner function in entangled state representation

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Abstract

Based on the correspondence between Collins diffraction formula (optical Fresnel transform) and the transformation matrix element of a three-parameters two-mode squeezing operator in the entangled state representation (Opt. Lett. 31 (2006) 2622) we further explore the relationship between output field intensity determined by the Collins formula and the input field's probability distribution along an infinitely thin phase space strip both in spacial domain and frequency domain. The entangled Wigner function is introduced for recapitulating the result.

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In a preceding Letter [1] we have reported that the Collins diffraction formula in cylindrical coordinates is just the transformation matrix element of a three-parameter (k and t are complex and satisfy the unimodularity condition $kk^*-tt^*=1$) two-mode squeezing operator [2, 3]

$$F^{(t,k)}=\exp\left(\frac{t}{k^*}a_1^\dagger a_2^\dagger\right)\exp\left[\left(a_1^\dagger a_1+a_2^\dagger a_2+1\right)\ln\left(k^*\right)^{-1}\right]\exp\left(-\frac{t^*}{k^*}a_1a_2\right), \tag{1}$$

in the deduced entangled state representation $\langle s,r'|$,

$$\begin{aligned}\phi_s\left(r'\right) &\equiv \langle s,r'|\,\phi\rangle=\langle s,r'|\,F^{(t,k)}\,|\psi\rangle\\ &= \frac{\mathtt{i}^s}{2\mathtt{i}B}\int_0^\infty \mathtt{d}\left(r^2\right)\exp\left[\frac{\mathtt{i}}{2B}\left(Ar^2+Dr'^2\right)\right]J_s\left(-\frac{rr'}{B}\right)\psi_s\left(r\right),\end{aligned} \tag{2}$$

where ψ_s and ϕ_s denote the incoming and output light, respectively, $[a_i,a_j^\dagger]=\delta_{i,j}$,

$$k=\frac{1}{2}\left[A+D-\mathtt{i}\left(B-C\right)\right],\quad t=\frac{1}{2}\left[A-D+\mathtt{i}\left(B+C\right)\right], \tag{3}$$

we see that the relation $kk^*-tt^*=1$ becomes $AD-BC=1$, J_s is the sth Bessel function, and

$$\langle s,r'|=\frac{1}{2\pi}\int_0^{2\pi}\mathtt{d}\theta e^{\mathtt{i}s\theta}\left\langle\eta=r'e^{\mathtt{i}\theta}\right|, \tag{4}$$

here $|\eta\rangle$ is the entangled states in two-mode Fock space [4, 5, 6, 7] named after Einstein-Podolsky-Rosen (EPR)'s [8] concept of quantum entanglement,

$$|\eta\rangle=\exp\{-\frac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^*a_2^\dagger+a_1^\dagger a_2^\dagger\}\,|00\rangle\,. \tag{5}$$

Thus $\langle s, r'| F^{(t,k)} |\psi\rangle$ is the quantum optics version of the Collins formula (generalized Hankel transformation). In [9] we have also found

$$\mathcal{K}^{(t,k)}(\eta',\eta) = \frac{1}{\pi}\langle\eta'| F^{(t,k)} |\eta\rangle = \frac{1}{2\mathtt{i}B\pi}\exp\left\{\frac{\mathtt{i}}{2B}\left[A|\eta|^2 - (\eta\eta'^* + \eta^*\eta') + D|\eta'|^2\right]\right\}. \tag{6}$$

Comparing with the integral kernel of usual Fresnel transform which describes how a general beam $\psi(x')$, propagating through an $(ABCD)$ optical paraxial system, becomes output field $\phi(x)$[10, 11]

$$\phi(x) = \int_{-\infty}^{\infty}\mathcal{K}(x,x')\,\psi(x')\,\mathtt{d}x', \tag{7}$$

where $AD - BC = 1$,

$$\mathcal{K}(x,x') = \frac{1}{\sqrt{2\pi\mathtt{i}B}}\exp\left[\frac{\mathtt{i}}{2B}\left(Ax'^2 - 2x'x + Dx^2\right)\right], \tag{8}$$

we see that $\mathcal{K}^{(t,k)}(\eta',\eta)$ can be considered as the integration kernel of 2-dimensional entangled optical Fresnel transform,

$$\Psi(\eta') = \int\mathcal{K}^{(t,k)}(\eta',\eta)\,\Phi(\eta)\,\mathtt{d}^2\eta, \tag{9}$$

in this sense $F^{(t,k)}$ can be named entangled Fresnel operator (EFO), here $\Phi(\eta) = \langle\eta|\,\Phi\rangle$, $\Psi(\eta') = \langle\eta'|\,\Psi\rangle$, and we have used the completeness relation $\int\frac{\mathtt{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1$.

Clearly, if the $[ABCD]$ system is changed to $[D(-B)(-C)A]$ system, then Eq. (9) should read

$$\Psi(\eta') = \int\mathcal{K}_2^{(D,-B,-C)}(\eta',\eta)\,\Phi(\eta)\,\mathtt{d}^2\eta, \tag{10}$$

where $\mathcal{K}_2^{(D,-B,-C)}$ is

$$\mathcal{K}_2^{(D,-B,-C)}(\eta',\eta) = \frac{1}{-2\mathtt{i}B\pi}\exp\left\{\frac{\mathtt{i}}{-2B}\left[D|\eta|^2 - (\eta\eta'^* + \eta^*\eta') + A|\eta'|^2\right]\right\}. \tag{11}$$

On the other hand, signals or images in optical information theory may be described directly or indirectly by the Wigner distribution function (WDF) [12]. In one-dimensional (1D) case, the WDF of an optical signal field $\psi(x)$ is defined as

$$W_\psi(\nu,x) = \int_{-\infty}^{+\infty}\frac{\mathtt{d}u}{2\pi}e^{\mathtt{i}\nu u}\psi^*\left(x+\frac{u}{2}\right)\psi\left(x-\frac{u}{2}\right). \tag{12}$$

$W_\psi(\nu,x)$ involves both spatial distribution information and space-frequency distribution information of the signal, ν is named space frequency. Now, let us consider the entangled case. Like Eq. (12), it is natural to introduce the 2-D complex Wigner transform as

$$W(\sigma,\gamma) = \int\frac{\mathtt{d}^2\eta}{\pi^3}\psi(\sigma+\eta)\,\psi^*(\sigma-\eta)\,e^{\eta\gamma^*-\eta^*\gamma}, \tag{13}$$

where σ, γ, η are all complex variables. To see its physical meaning, using the integration formula of Dirac $\delta-$function, we perform the following integration,

$$\int\mathtt{d}^2\gamma W(\sigma,\gamma) = \int\frac{\mathtt{d}^2\eta}{\pi}\psi(\sigma+\eta)\,\psi^*(\sigma-\eta)\,\delta(\eta)\,\delta(\eta^*) = \frac{1}{\pi}|\psi(\sigma)|^2, \tag{14}$$

which is just the probability distribution of the complex function $\psi(\sigma)$. Further, let the ordinary Fourier transforms of $\psi(\sigma)$ be $j(\zeta)$,

$$\psi(\sigma) = \int\frac{\mathtt{d}^2\zeta}{2\pi}j(-\zeta)\,e^{(\zeta^*\sigma-\zeta\sigma^*)/2}, \tag{15}$$

then substituting (15) into (13) leads to

$$
\begin{aligned}
W(\sigma,\gamma) &= \int \frac{\mathrm{d}^2\eta}{\pi^3}\frac{\mathrm{d}^2\zeta}{2\pi}\frac{\mathrm{d}^2\zeta'}{2\pi} j(-\zeta)\, j^*\left(-\zeta'\right) e^{\frac{\left(\zeta^*-\zeta'^*\right)\sigma-\left(\zeta-\zeta'\right)\sigma^*}{2}} e^{\eta\left(\gamma^*+\frac{\zeta^*+\zeta'^*}{2}\right)-\eta^*\left(\gamma+\frac{\zeta+\zeta'}{2}\right)} \\
&= \int \frac{\mathrm{d}^2\zeta}{\pi^3} j(-\zeta)\, j^*(2\gamma+\zeta)\, e^{(\zeta^*+\gamma^*)\sigma-(\zeta+\gamma)\sigma^*} = \int \frac{\mathrm{d}^2\zeta}{\pi^3} j(\gamma-\zeta)\, j^*(\gamma+\zeta)\, e^{\zeta^*\sigma-\zeta\sigma^*} \qquad (16)
\end{aligned}
$$

It then follows from (16) that

$$
\begin{aligned}
\int \mathrm{d}^2\sigma W(\sigma,\gamma) &= \int \frac{\mathrm{d}^2\zeta}{\pi^3} j(\gamma-\zeta)\, j^*(\zeta+\gamma) \int \mathrm{d}^2\sigma e^{\zeta^*\sigma-\zeta\sigma^*} \\
&= \int \frac{\mathrm{d}^2\zeta}{\pi} j(\gamma-\zeta)\, j^*(\zeta+\gamma)\, \delta(\zeta)\, \delta(\zeta^*) = \frac{1}{\pi}\left|j(\gamma)\right|^2, \qquad (17)
\end{aligned}
$$

which is the probability distribution of the complex function $j(\gamma)$. Thus our definition in (13) leads to two marginal distributions in σ and γ phase space, respectively. Hence $W(\sigma,\gamma)$ is indeed the correct complex 2-D Wigner function (Wigner transform) of complex function $\psi(\sigma)$ or $j(\gamma)$. If one wants to reconstruct the Wigner function by using various probability distributions, obviously the "position density" $\left|\langle\sigma|\,\psi\rangle\right|^2$ and the space-frequency density $\left|\langle\gamma|\,\psi\rangle\right|^2$ are not enough, so we extend $\delta(\eta)\,\delta(\eta^*) \equiv \delta(\eta_1)\,\delta(\eta_2)$ to $\delta(\eta_1 - D\sigma_1 - B\gamma_2)\,\delta(\eta_2 - D\sigma_2 + B\gamma_1)$ and generalize (14) to,

$$
R_2(\eta_1,\eta_2) \equiv \pi \int \delta(\eta_1 - D\sigma_1 - B\gamma_2)\,\delta(\eta_2 - D\sigma_2 + B\gamma_1)\, W(\sigma,\gamma)\, \mathrm{d}^2\sigma \mathrm{d}^2\gamma, \qquad (18)
$$

$R_2(\eta_1,\eta_2)$ is also a probability distribution along an infinitely thin phase space strip denoted by the real parameters B, D, which is a generalized entangled Radon transform [13, 14] of the two-mode Wigner function (in the entangled form) [15, 16],

Then an interesting question naturally arises: what is the relation between the generalized Fresnel transform and the WDF in entangled state representation?

We begin with rewriting the 2-D WF (13) as

$$
\begin{aligned}
W(\sigma,\gamma) &= \int \mathrm{d}^2\sigma' \mathrm{d}^2\sigma'' \int \frac{\mathrm{d}^2\eta}{\pi^3} \psi(\sigma')\, \psi^*(\sigma'')\, \delta^{(2)}(\sigma'-\sigma-\eta)\, \delta^{(2)}(\sigma-\eta-\sigma'')\, e^{\eta\gamma^*-\eta^*\gamma} \\
&= \int \frac{\mathrm{d}^2\sigma' \mathrm{d}^2\sigma''}{\pi^3} \psi(\sigma')\, \psi^*(\sigma'')\, \delta^{(2)}(2\sigma-\sigma'-\sigma'')\, e^{(\sigma'-\sigma)\gamma^*-(\sigma'-\sigma)^*\gamma}. \qquad (19)
\end{aligned}
$$

Substituting (19) into (18) we rewrite the Radon transform of $W(\sigma,\gamma)$ as ($\mathrm{d}^2\sigma = \mathrm{d}\sigma_1\mathrm{d}\sigma_2$, $\mathrm{d}^2\gamma = \mathrm{d}\gamma_1\mathrm{d}\gamma_2$)

$$
\begin{aligned}
R_2(\eta_1,\eta_2) &= \int \frac{\mathrm{d}^2\sigma' \mathrm{d}^2\sigma''}{\pi^2} \psi(\sigma')\, \psi^*(\sigma'') \int \mathrm{d}^2\sigma \mathrm{d}^2\gamma \delta(\eta_2 - D\sigma_2 + B\gamma_1) \\
&\quad \times \delta(\eta_1 - D\sigma_1 - B\gamma_2)\, \delta^{(2)}(2\sigma-\sigma'-\sigma'')\, e^{(\sigma'-\sigma)\gamma^*-(\sigma'-\sigma)^*\gamma} \\
&= \int \frac{\mathrm{d}^2\sigma' \mathrm{d}^2\sigma''}{4\pi^2} \psi(\sigma')\, \psi^*(\sigma'') \int \mathrm{d}^2\gamma \delta\left(\eta_2 - D\frac{\sigma_2'+\sigma_2''}{2} + B\gamma_1\right) \\
&\quad \times \delta\left(\eta_1 - D\frac{\sigma_1'+\sigma_1''}{2} - B\gamma_2\right) \exp\left\{i\left[(\sigma_2'-\sigma_2'')\,\gamma_1 - (\sigma_1'-\sigma_1'')\,\gamma_2\right]\right\} \\
&= \int \frac{\mathrm{d}^2\sigma' \mathrm{d}^2\sigma''}{4B^2\pi^2} \psi(\sigma')\, \psi^*(\sigma'') \\
&\quad \times \exp\left\{\frac{i}{B}\left[(\sigma_2'-\sigma_2'')\left(-\eta_2 + D\frac{\sigma_2'+\sigma_2''}{2}\right) - (\sigma_1'-\sigma_1'')\left(\eta_1 - D\frac{\sigma_1'+\sigma_1''}{2}\right)\right]\right\} \\
&= \int \frac{\mathrm{d}^2\sigma' \mathrm{d}^2\sigma''}{4B^2\pi^2} \psi(\sigma')\, \psi^*(\sigma'') \exp\left\{\frac{i}{2B}\left[D\left(\left|\sigma'\right|^2 - \left|\sigma''\right|^2\right) - 2\eta_1(\sigma_1'-\sigma_1'') - 2\eta_2(\sigma_2'-\sigma_2'')\right]\right\} \qquad (20)
\end{aligned}
$$

On the other hand, when the beam $\Phi(\eta)$ propagates through the $[D(-B)(-C)A]$ optical system, according to the Fresnel integration (10)-(11), we have

$$
\begin{aligned}
|\Psi(\eta')|^2 &= \int \frac{\mathtt{d}^2\eta}{\pi} \mathcal{K}_2^{(D,-B,-C)}(\eta',\eta)\Phi(\eta) \int \frac{\mathtt{d}^2\eta''}{\pi} \mathcal{K}_2^{*(D,-B,-C)}(\eta',\eta'')\Phi^*(\eta'') \\
&= \frac{1}{4B^2} \int \frac{\mathtt{d}^2\eta}{\pi} \exp\left\{ \frac{\mathtt{i}}{2B} \left[-D|\eta|^2 + (\eta\eta'^* + \eta^*\eta') - A|\eta'|^2 \right] \right\} \Phi(\eta) \\
&\quad \times \int \frac{\mathtt{d}^2\eta''}{\pi} \exp\left\{ \frac{\mathtt{i}}{2B} \left[D|\eta''|^2 - (\eta''^*\eta' + \eta''\eta'^*) + A|\eta'|^2 \right] \right\} \Phi^*(\eta'') \\
&= \frac{1}{4B^2\pi^2} \int \frac{\mathtt{d}^2\eta}{\pi} \Phi(\eta)\Phi^*(\eta'') \exp\left\{ \frac{\mathtt{i}}{2B} \left[D\left(|\eta''|^2 - |\eta|^2\right) - 2\eta_1'(\eta_1'' - \eta_1) - 2\eta_2'(\eta_2'' - \eta_2) \right] \right\}, \quad (21)
\end{aligned}
$$

which is the same as $R_2(\eta_1,\eta_2)$ in (20). So combining (20), (10)-(11) and (21) we reach the conclusion

$$
\begin{aligned}
& \left| \frac{1}{-2\mathtt{i}B} \int \frac{\mathtt{d}^2\eta}{\pi} \exp\left\{ \frac{\mathtt{i}}{-2B} \left[D|\eta|^2 - (\eta\eta'^* + \eta^*\eta') + A|\eta'|^2 \right] \right\} \Phi(\eta) \right|^2 \\
&= \pi \int \delta(\eta_1' - D\sigma_1 - B\gamma_2)\,\delta(\eta_2' - D\sigma_2 + B\gamma_1)\, W(\sigma,\gamma)\, \mathtt{d}^2\sigma \mathtt{d}^2\gamma, \qquad (22)
\end{aligned}
$$

where $AD - BC = 1$. The physical meaning of Eq. (22) is: when an input field propagates through an optical $[D(-B)(-C)A]$ system, the energy density of the output field is equal to the Radon transform of the two-mode entangled Wigner function of the input field. So far as our knowledge is concerned, this conclusion seems new. Eq. (22) is the relationship between the input amplitude and output one in spatial-domain. Next we turn to the frequency domain.

If taking the matrix element of $F^{(t,k)}$ in the $|\xi\rangle$ representation which is conjugate to $|\eta\rangle$, where the overlap $\langle\eta|\,\xi\rangle$ is $\langle\eta|\,\xi\rangle = \frac{1}{2}\exp[(\xi\eta^* - \xi^*\eta)/2]$, we obtain the 2-dimensional GFT in its 'frequency domain', i.e.,

$$
\begin{aligned}
& \frac{1}{\pi}\langle\xi'|\,F^{(t,k)}\,|\xi\rangle = \int \frac{\mathtt{d}^2\eta \mathtt{d}^2\eta'}{\pi^3} \langle\xi'|\,\eta'\rangle \langle\eta'|\,F^{(t,k)}\,|\eta\rangle \langle\eta|\,\xi\rangle \\
&= \frac{1}{4}\int \frac{\mathtt{d}^2\eta \mathtt{d}^2\eta'}{\pi^2} \exp\left(\frac{\xi'^*\eta' - \xi'\eta'^* + \xi\eta^* - \xi^*\eta}{2} \right) \mathcal{K}^{(t,k)}(\eta',\eta) \\
&= \frac{1}{2\mathtt{i}(-C)\pi} \exp\left[\frac{\mathtt{i}}{2(-C)} \left(D|\xi|^2 + A|\xi'|^2 - \xi'^*\xi - \xi'\xi^* \right) \right] \equiv \mathcal{K}_2^N(\xi',\xi), \qquad (23)
\end{aligned}
$$

where the superscript N of $\mathcal{K}_2^N$ means that this transform kernel corresponds to the parameter matrix $N = [D,-C,-B,A]$. Thus if the $[D,-C,-B,A]$ system is changed to $\tilde{N} = [A,C,B,D]$ system, the GFT in its 'frequency domain' is given by

$$
\Psi(\xi') = \int \mathcal{K}_2^{\tilde{N}}(\xi',\xi)\Phi(\xi)\,\mathtt{d}^2\xi, \qquad (24)
$$

where $\mathcal{K}_2^{\tilde{N}}(\xi',\xi)$ is

$$
\mathcal{K}_2^N(\xi',\xi) = \frac{1}{2\mathtt{i}C\pi} \exp\left[\frac{\mathtt{i}}{2C} \left(A|\xi|^2 + D|\xi'|^2 - \xi'^*\xi - \xi'\xi^* \right) \right]. \qquad (25)
$$

It then follows from Eqs.(24) and (25) that

$$
\begin{aligned}
|\Psi(\xi')|^2 &= \int \mathcal{K}_2^{\tilde{N}}(\xi',\xi)\Phi(\xi)\,\mathtt{d}^2\xi \int \mathcal{K}_2^{*\tilde{N}}(\xi',\xi'')\Phi^*(\xi'')\,\mathtt{d}^2\xi'' \\
&= \frac{1}{4\pi^2C^2} \int \mathtt{d}^2\xi \mathtt{d}^2\xi'' \Phi(\xi)\Phi^*(\xi'') \\
&\quad \times \exp\left\{ \frac{\mathtt{i}}{2C} \left[A\left(|\xi|^2 - |\xi''|^2\right) + 2\xi_1'(\xi_1'' - \xi_1) + 2\xi_2'(\xi_2'' - \xi_2) \right] \right\}. \qquad (26)
\end{aligned}
$$

On the other hand, in similar to (18), we consider the integration transform,

$$R_2(\xi_1,\xi_2)=\pi\int\delta(\xi_1-A\sigma_1-C\gamma_2)\,\delta(\xi_2-A\sigma_2+C\gamma_1)\,W(\sigma,\gamma)\,\mathrm{d}^2\sigma\mathrm{d}^2\gamma, \tag{27}$$

$R_2(\xi_1,\xi_2)$ is also a probability distribution along an infinitely thin phase space strip denoted by the real parameters A,C. Substituting (19) into (27) yields

$$\begin{aligned}R_2(\xi_1,\xi_2) &= \int\frac{\mathrm{d}^2\sigma'\mathrm{d}^2\sigma''}{\pi^2}\psi(\sigma')\,\psi^*(\sigma'')\int\delta^{(2)}(2\sigma-\sigma'-\sigma'')\,\mathrm{d}^2\sigma\mathrm{d}^2\gamma\\ &\quad\times\delta(\xi_2-A\sigma_2+C\gamma_1)\,\delta(\xi_1-A\sigma_1-C\gamma_2)\,e^{(\sigma'-\sigma)\gamma^*-(\sigma'-\sigma)^*\gamma}\\ &= \int\frac{\mathrm{d}^2\sigma'\mathrm{d}^2\sigma''}{4\pi^2}\psi(\sigma')\,\psi^*(\sigma'')\int\mathrm{d}^2\gamma\delta\left(\xi_2-A\frac{\sigma_2'+\sigma_2''}{2}+C\gamma_1\right)\\ &\quad\times\delta\left(\xi_1-A\frac{\sigma_1'+\sigma_1''}{2}-C\gamma_2\right)\exp\left[\frac{\sigma'-\sigma''}{2}\gamma^*-\frac{\sigma'^*-\sigma''^*}{2}\gamma\right]\\ &= \int\frac{\mathrm{d}^2\sigma'\mathrm{d}^2\sigma''}{4\pi^2C^2}\psi(\sigma')\,\psi^*(\sigma'')\\ &\quad\times\exp\left\{\frac{\mathrm{i}}{2C}\left[A\left(|\sigma'|^2-|\sigma''|^2\right)+2\xi_1(\sigma_1''-\sigma_1')+2\xi_2(\sigma_2''-\sigma_2')\right]\right\},\end{aligned} \tag{28}$$

which is the same as $|\Psi(\xi')|^2$ in (26). So combining (28), (24)-(25), and (26) we can draw the conclusion

$$\begin{aligned}&\left|\frac{1}{2\mathrm{i}C\pi}\int\exp\left[\frac{\mathrm{i}}{2C}\left(A|\xi|^2+D|\xi'|^2-\xi'^*\xi-\xi'\xi^*\right)\right]\Phi(\xi)\,\mathrm{d}^2\xi\right|^2\\ &= \pi\int\delta(\xi_1-A\sigma_1-C\gamma_2)\,\delta(\xi_2-A\sigma_2+C\gamma_1)\,W(\sigma,\gamma)\,\mathrm{d}^2\sigma\mathrm{d}^2\gamma.\end{aligned} \tag{29}$$

This is the relationship between the output amplitude and input one's entangled Wigner function in 'frequency domain'.

In sum, based on the correspondence between Collins diffraction formula (optical Fresnel transform) and the transformation matrix element of a three-parameters two-mode squeezing operator in the entangled state representation, we have explored the relationship between output field intensity determined by the Collins formula and the input field's probability distribution along an infinitely thin phase space strip. The entangled Wigner function is introduced for recapitulating the result.

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