# Travelling wave solutions to nonlinear Schrödinger equation with self-steepening and self-frequency shift

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We investigate exact travelling wave solutions of higher order nonlinear Schrödinger equation in the absence of third order dispersion. We show that this system possesses a rich solution space with a nontrivial self phase modulation. It is found that localized solutions to this system can be identified with separatrix of a nonlinear ordinary differential equation. Interestingly, hydrodynamic equation governing the intensity dynamics turn out to be KdV and modified KdV equations, which are true hydrodynamical equations governing swallow water waves.

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## I. INTRODUCTION

The higher order nonlinear Schrödinger equation (HNLSE) governs the pulse dynamics in the femtosecond domain, where third order dispersion, self-steepening of pulse due to dependence of the slowly varying part of the nonlinear polarization on time and self-frequency shift arising from delayed Raman response become important and can not be neglected [1]. Third order dispersion, in particular, becomes important for femtosecond pulses, when group velocity dispersion (GVD) is close to zero. It can be neglected for optical pulses, whose width is of the order of 100 femtoseconds or more, having power of the order of 1 Watt and GVD far away from zero. In this case, the problem of pulse propagation can be analytically ascertained to a great extent. In our earlier study, localized wave packets, with nontrivial chirping, have emerged as exact solutions of this equation [2]. We had shown that, chirping has a kinematic component, determined through initial conditions and a dynamical component, owing its origin to self-steepening of pulse and to delayed Raman response.

These two chirpings have inverse characteristics and can be used for chirp control. Further, periodic solutions, with sinusoidal functions in fractional linear form, have also been identified. Parameter domain, where superimposed solutions were found to exist in this nonlinear system, have also been identified.

In this paper, we explore the structure of the solution space of HNLSE, in absence of third order dispersion, and connect it through a conformal transformation to elliptic function equation. The earlier obtained solutions emerge as limiting solutions of the general case. Using a pseudo-potential picture, we show that the localized solitons of this system can be mapped to separatix solutions of the elliptic function equation. In this picture, both singular solutions and periodic solutions are identified, with unbounded and bounded solutions, separated by the separatrix. Very interestingly, it is found that the dynamics of travelling wave solutions can be identified in its generality, either by modified KdV equation or KdV equation, which are genuine hydrodynamical equations and appear in dynamics of shallow water waves.

Nonlinear Schrödinger equation:

$$i\psi_x + a_1\psi_{tt} + a_2|\psi|^2\psi = 0,$$

with Kerr nonlinearity is the equation that governs picosecond pulse propagation in optical fibers [1]. Here the slowly varying envelope of electric field is $\psi$, $a_1$ is proportional to GVD and $a_2$ specifies the strength of Kerr nonlinearity, and subscripts $x$ and $t$ denote partial derivatives, with respect to space and time coordinates, respectively. It was first predicted by Hasegawa and Tappert [3] and later observed by Mollenauer *et al.* [4], that this system supports robust soliton solutions. Such solutions exist due to complete integrability of this dynamical system [5]. The higher order effects like third order dispersion, self-steepening of pulse due to dependence of the slowly varying part of nonlinear polarization on time and self frequency shift arising from delayed Raman response become important, while studying propagation of the pulses with width of the order of 10 femtoseconds. Inorder to account for them, Kodama [6] and Kodama & Hasegawa [7] proposed higher order nonlinear Schrödinger equation (HNLSE) as a generalization of NLSE:

$$\begin{aligned} &i\psi_x + a_1\psi_{tt} + a_2|\psi|^2\psi \\ &+i\left[a_3\psi_{ttt} + a_4(|\psi|^2\psi)_t + a_5\psi(|\psi|^2)_t\right] = 0, \end{aligned} \tag{1}$$

where a third order dispersion term, with coefficient $a_3$, self-steepening term with coefficient $a_4$ and self-frequency shift effect with coefficient $a_5$ have been added. This model, unlike NLSE, is not integrable in general. A few integrable cases have been identified; these are known as (i) Sasa-Satsuma case ($a_3$:$a_4$:$(a_4+a_5)$ = 1:6:3) [8], (ii) Hirota case ($a_3$:$a_4$:$(a_4+a_5)$ = 1:6:0) [9] and (iii) derivative NLSE of type I and type II [10]. A few restrictive special solutions of bright and dark type have been obtained for this system [11, 12, 13]. The effects of these higher order terms on pulse propagation have been studied numerically extensively [1, 14], and some special solutions to this system are also known [15]. Third order dispersion becomes significant only for femtosecond pulses, when GVD is close to zero, but can be neglected for optical pulses whose width is of the order of 100 femtoseconds or more, having power of the order of 1 Watt and GVD far away from zero.

The paper is organized as follows. In the following section, we obtain a class of exact solutions to (1). In Sec. III, we show how these exact solutions can be idenfied with phase space of a classical system. For certain parameter values, the procedure delineated in Sec. II is not valid, so in Sec. IV, we address this possibility and show a procedure to find exact solutions in this regime. Section V deals with conclusion and future directions of work.

## II. EXACT SOLUTIONS TO HNLSE

In this case, a travelling wave packet solutions, modulo an overall phase term, can be generally be written as:

$$\psi(x,t) = \rho(\xi)e^{i\chi(\xi)}, \tag{2}$$

where the travelling coordinate $\xi = \alpha(t - ux)$, and $\rho$ and $\chi$ are real functions of $\xi$. Here, $\alpha$ is the scale parameter and $u = 1/v$ where $v$ is group velocity of the ansatz solution. Substituting (2) in (1), and equating real and imaginary parts yields two coupled equations:

$$\begin{aligned} &-\alpha u\rho' + 2\alpha^2 a_1\chi'\rho' + \alpha^2 a_1\chi''\rho \\ &+ 3\alpha a_4\rho^2\rho' + 2\alpha a_5\rho^2\rho' = 0, \quad \text{and} \end{aligned} \tag{3}$$

$$\begin{aligned} &\alpha u\chi'\rho + \alpha^2 a_1\rho'' - \alpha^2 a_1{\chi'}^2\rho \\ &+ a_2\rho^3 - \alpha a_4\chi'\rho^3 = 0. \end{aligned} \tag{4}$$

Equation (3) can be exactly integrated to yield:

$$\chi' = \frac{u}{2\alpha a_1} + \frac{c}{\alpha a_1 \rho^2} - \frac{(3a_4 + 2a_5)}{4\alpha a_1}\rho^2, \tag{5}$$

where $c$ is to be determined by initial conditions. Notice that the phase has a nontrivial form and has two intensity dependent chirping terms, apart from kinematic first term which is of usual $e^{i(kx-wt)}$ type. As is evident, the second term is of kinematic origin and is infact present in linear Schrödinger equation. The last term is due to higher order nonlinearities, which has a dynamical origin, and leads to chirping that is exactly inverse to that of the former. This is a novel form of self-phase modulation.

Using above expression, equation (4) can be written as:

$$\theta_1 \rho'' + \theta_2 \rho + \theta_3 \rho^3 + \theta_4 \rho^5 = \frac{c^2}{\rho^3}, \tag{6}$$

with $\theta_1 = \alpha^2 {a_1}^2$, $\theta_2 = \frac{(u^2+2a_4c+4a_5c)}{4}$, $\theta_3 = \frac{(2a_1a_2-ua_4)}{2}$ and $\theta_4 = \frac{(a_4-2a_5)(3a_4+2a_5)}{16}$.

Note that, in parameter regime, where $\theta_4 = 0$, the amplitude dynamics will be like NLSE case, albeit with dressed parameters. Phase dynamics however will not be same as NLSE, and will depend whether $(a_4 - 2a_5) = 0$ or $(3a_4 + 2a_5) = 0$. In latter case only it will be exactly like NLSE, devoid of dynamical chirping. As shown in Ref. [2], localized solutions of the above equation, in these parameter domains, show directionality and hence are chiral, and the propagation direction for these solitons is decided by sign of $a_4$. Because, these solutions satisfy NLSE with dressed parameters, many of the parameteric restrictions on solution space are relaxed, for example both dark and bright solitons are present in both anomalous and normal dispersion regimes.

Defining $k = \frac{\theta_1}{2}\rho'^2 + \frac{\theta_2}{2}\rho^2 + \frac{\theta_3}{4}\rho^4 + \frac{c^2}{2\rho^2}$, equation (6) can be written conveniently in terms of $\sigma$, where $\sigma = \rho^2$, to yield:

$$\frac{\theta_1}{2}\sigma'' + 2\theta_2\sigma + \frac{3\theta_3}{2}\sigma^2 + \frac{4\theta_4}{3}\sigma^3 = k, \tag{7}$$

where, $k$ is to determined via initial conditions. Above equation can be rewritten in a simpler form by going to $y$ variable, $y = \sigma + \frac{3\theta_3}{8\theta_4}$, given that $\theta_4 \neq 0$, to read:

$$y'' + py + qy^3 + r = 0, \tag{8}$$

where $p = \frac{2}{\theta_1}(2\theta_2 - \frac{9{\theta_3}^2}{8\theta_4} - \frac{3\theta_3}{2})$, $q = \frac{8\theta_4}{3\theta_1}$ and $r = \frac{2}{\theta_1}(\frac{3{\theta_3}^3}{16{\theta_4}^2}) - \frac{3\theta_2\theta_3}{4\theta_4} - k$. In case when $r = 0$, above equation coincides with the equation satisfied by Jacobi elliptic functions. Below we show that, for some real constants $A$, $B$, $C$ and $D$, the conformal transformation:

$$y = \frac{A + Bf}{C + Df}, \tag{9}$$

solves equation (8), with $f(\xi, m)$ being one of the twelve Jacobi elliptic functions with modulus parameter $m$ [16, 17]. These elliptic functions satisfy:

$$f'' = af + bf^3, \tag{10}$$

with appropriate real constants $a$ and $b$. For example, for $f = cn(\xi, m)$, $a = (2m-1)$ and $b = -2m$. The claim that expression (9) solves equation (8) can readily be seen, by defining the first integral $E_0 = \frac{f'^2}{2} - \frac{af^2}{2} - \frac{bf^4}{4}$, and substituting (9) in equation (8). Use of equation (10) leads to the following consistency conditions:

$$-4BCDE_0 + 4AD^2E_0 + pAC^2 + qA^3 + rC^3 = 0, \tag{11}$$

$$aBC^2 - aACD + pBC^2 + 2pACD + 3qA^2B + 3rC^2D = 0, \tag{12}$$

$$-aBCD + aAD^2 + 2pBCD + pAD^2 + 3qAB^2 + 3rCD^2 = 0, \tag{13}$$

$$bBC^2 - bACD + pBD^2 + qB^3 + rD^3 = 0. \tag{14}$$

Notice that these coupled algebraic equations are nonlinear in $A$, $B$, $C$ and $D$, but are linear in $p$, $q$, $r$ and $E_0$, and hence can be solved exactly to yield:

$$E_0 = -\frac{bAC^3 + aBC^2D + aACD^2}{4BD^3}, \tag{15}$$

$$p = \frac{-3bAC^2 - 2aBCD - aAD^2}{D(-BC + AD)}, \tag{16}$$

$$q = \frac{-bC^3 - aCD^2}{B(BC - AD)}, \tag{17}$$

$$r = -\frac{bABC^2 + bA^2CD + aB^2CD + aABD^2}{D^2(BC - AD)}. \tag{18}$$

Therefore, this shows that, provided above relations hold, equation (8) is indeed solved by (9). Notice the dependence of $E_0$ on $A$, $B$, $C$ and $D$, which simply shows that the initial values in equation (10) do play a role in determining $A$, $B$, $C$ and $D$, and hence they can not be fixed, given $p$, $q$ and $r$. Since, original equation parameters $a_i$ $(i = 1, ..., 5)$ can be expressed in terms of $p$, $q$ and $r$, one can determine the parameter regime for a given solution using these relations. The solutions presented in Ref. [2] form a subclass of the ones found above. For $m = 1$, one gets localized solutions and are often of experimental and technological interest. Setting $m = 0$, singles out periodic solutions, which were also reported in Ref.[2].

## III. SEPARATRIX CONNECTION

An alternative perspective to appreciate above mapping is as follows. Equation (8) can be thought of as equation of motion for a classical particle of unit mass, with displacement given by $y$, moving under influence of a nonlinear force $-(py+qy^3+r)$. The same also holds for equation (10), albeit the nonlinear force does not have a constant term. Also notice that both these dynamical systems are of one degree of freedom and their phase spaces are two dimensional, only position $y$ and momentum $y'$ is required to describe the dynamics completely. Looked in this setting, mapping (9) simply says how the two phase spaces are related; more precisely it shows how phase space for equation (8) can be generated by knowledge of phase space for equation (10). Since, the mapping is conformal, the singularity structure of both the phase spaces are identical, modulo movable poles. Infact, this simply shows that equation (8) is integrable in the sense of Painlevé, since equation (10) is integrable [18, 19]. The soliton solutions of the parent equation (1), in this setting, can be identified with the separatrix solutions of equation (10) via the conformal mapping [20]. Similarly, singular solutions of equation (10) can be seen to be related to unbounded solutions lying outside separatrix, and periodic solutions can be seen to be related to bounded solutions lying inside separatrix.

It is very interesting to note that solutions to equation (8) are actually solutions to modified KdV equation: $v_{xxx}+p_1v^2v_x+p_2v_t=0$. This can be simply seen by going to the travelling variable $\zeta=x-vt$, integrating out the equation once, and identifying $p_1=3q$, $p_2=-\frac{p}{v}$ and $r$ as the constant of integration. So, at the travelling variable level, which restricts one to only one soliton solution, one can say that the modified KdV solitons can be simulated by considering these higher nonlinearities in optical fibres.

## IV. SOLUTIONS WHEN $\theta_4=0$

In case, when $\theta_4=0$, the above analysis is not valid since the degree of the equation (6) changes, and can be written as:

$$\sigma''+p'\sigma+q'\sigma^2+r'=0, \tag{19}$$

with $p' = \frac{4\theta_4}{\theta_1}$, $q' = \frac{3\theta_3}{\theta_1}$ and $r' = -\frac{2k}{\theta_1}$. The above equation can be mapped to equation (10) via a transformation:

$$\sigma = \frac{A + Bf^2}{C + Df^2}, \tag{20}$$

which is along the same lines as (9). Following the same procedure as the former case, using the first integral $E_0 = \frac{f'^2}{2} - \frac{af^2}{2} - \frac{bf^4}{4}$, we find the consistency conditions as:

$$4BC^2E_0 - 4ACDE_0 + p'AC^2 + q'A^2C + r'C^3 = 0, \tag{21}$$

$$\begin{aligned} &4aBC^2 - 4aACD - 12BCDE_0 + 12AD^2E_0 + p'BC^2 \\ &+ 2p'ACD + 2q'ABC + q'A^2D + 3r'C^2D = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &3bBC^2 - 3bACD - 4aBCD + 4aAD^2 + 2p'BCD + p'AD^2 \\ &+ q'B^2C + 2q'ABD + 3r'CD^2 = 0, \end{aligned} \tag{23}$$

$$- b'BCD + b'AD^2 + p'BD^2 + q'B^2D + r'D^3 = 0. \tag{24}$$

Again we observe that, the above equations are linear in $p'$, $q'$, $r'$ and $E_0$, and hence can be solved to give:

$$E_0 = -\frac{bC^2 - 2aCD}{4D^2}, \tag{25}$$

$$p' = -\frac{2(-3bBC^2 - 3bACD + 4aBCD + 2aAD^2)}{D(AD - BC)}, \tag{26}$$

$$q' = \frac{6(-bC^2 + aCD)}{AD - BC}, \tag{27}$$

$$r' = -\frac{-bB^2C^2 - 4bABCD + 2aB^2CD - bA^2D^2 + 4aABD^2}{D^2(BC - AD)}. \tag{28}$$

This shows that, the equation (19) is an integrable equation, in the sense of Painlevé, and the explicit solution can be expressed in terms of Jacobi elliptic functions. By choice, of appropriate values of $A$, $B$, $C$ and $D$ one can find out the parameter regime in which the given solution is valid. Further, solutions to equation (19) actually satisfy KdV equation: $v_{xxx} + p_1vv_x + p_2v_t = 0$, in travelling variable $\xi = x - vt$, with $p' = -v\beta$, $2q' = \alpha$ and $r'$ being an integration constant. It is very interesting to note that the condition $\theta_4 = 0$, can be fulfilled if $a_4 = 2a_5$ or $3a_4 = -2a_5$. In the case when the former is true then, the intensity profile of these solutions is exactly like NLSE solutions except with a non-trivial phase chirping. In the latter case, the solutions do not have this non trivial chirping and the solutions are indistinguishable from NLSE solutions with appropriate coefficients.

## V. CONCLUSION

We have shown that nonlinear Schrödinger equation in the presence of self-steepening and self-frequency shift, possesses a rich travelling wave dynamics with a non-trivial chirping. These exact solutions are found to be connected to elliptic functions via a conformal mapping. The field intensity is seen to obey modified KdV or KdV equation, which provides an indication that HNLSE in absence of third order dispersion, may possibily be integrable. Also, we have shown that the localized solutions can be identified with separatrix of a nonlinear potential problem. This strengthens further the hope that this system may be integrable, since separatrix is not known to exist in a system with chaotic dynamics.

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