Any l-state improved quasi-exact analytical solutions of the spatially dependent mass Klein-Gordon equation for the scalar and vector Hulthén potentials

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Abstract

We present a new approximation scheme for the centrifugal term to obtain a quasi-exact analytical bound state solutions within the framework of the position-dependent effective mass radial Klein-Gordon equation with the scalar and vector Hulthén potentials in any arbitrary D dimension and orbital angular momentum quantum numbers l. The Nikiforov-Uvarov (NU) method is used in the calculations. The relativistic real energy levels and corresponding eigenfunctions for the bound states with different screening parameters have been given in a closed form. It is found that the solutions in the case of constant mass and in the case of s-wave ($l = 0$) are identical with the ones obtained in literature.

Keywords: Bound states, approximation schemes, Hulthén potential, Klein-Goron equation, position- dependent mass distributions, NU method

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I. INTRODUCTION

The bound and scattering states of the s- and l-waves for any interaction system have raised a great interest in non-relativistic as well as in relativistic quantum mechanics [1-3]. The exact solution of the wave equation is very important since the wavefunction contains all the necessary information regarding the quantum system under consideration. A number of methods have been used to solve the wave equations exactly or quasi-exactly for non-zero angular momentum quantum number ($l \neq 0$) by means of a given potential. The bound state eigenvalues were solved numerically [4,5] and quasi-analytically using variational [4,6], perturbation [7], shifted $1/N$ expansion [8,9], NU [10,11], SUSYQM [12-14] and AIM [15] methods.

The Hulthén potential [10,12,13,15,16] is one of the important short–range potentials in physics and it has been applied to a number of areas such as nuclear and particle physics [17], atomic physics [18,19], molecular physics [20,21] and chemical physics [22]. Therefore, it would be interesting and important to solve the relativistic equation for this potential for $l \neq 0$ case since it has been extensively used to describe the bound and continuum states of the interaction systems. Recently, the exact solutions for the bound and scattering states of the s-wave Schrödinger [16,23], Klein-Gordon [1-3] and Dirac equation [24,25] with the scalar and vector Hulthén potentials are investigated.

Relativistic effects with the scalar plus vector Hulthén-type potential [1,2] in three- and D dimensions and harmonic oscillator-type potential [26,27] have been also discussed in the literature. The bound-states of the Dirac and Klein-Gordon equations with the Coulomb-like scalar plus vector potentials have been studied in arbitrary dimension [28-32]. Furthermore, the exact results for the scattering states of the Klein-Gordon equation with Coulomb-like scalar plus vector potentials have been investigated in an arbitrary dimension [33]. This equation has been exactly solved for a larger class of linear, exponential and linear plus Coulomb potentials to determine the bound state energy spectrum using two semiclassical methods with the following relationship between the scalar and vector potentials: $V(r) = V_0 + \beta S(r)$, $S(r) > V(r)$ where V_0 and β being arbitrary constants [34]. In particular, inserting the constants $V_0 = 0$ and $\beta = \pm 1$ provides the equal scalar and vector potential case $V(r) = \pm S(r)$.

Also, the position-dependent mass solutions of the nonrelativistic and relativistic systems have received much attention recently. Many authors have used different methods to study the partially exactly solvable and exactly solvable Schrödinger, Klein-Gordon and Dirac equations in the presence of variable mass having a suitable mass distributions function in $1D, 3D$ and/or any dimension D cases for different potentials, such as the exponential-type potentials [35], the Coulomb potential [36], the Lorentz scalar interactions [37], the hyperbolic-type potentials [38], the Morse potential [39], the Pöschl-Teller potential [40], the Coulomb and harmonic potentials [41], the modified Kratzer-type, rotationally corrected Morse potentials [42], Mie-type and pseudoharmonic potentials [43]. Recently, the point canonical transformation (PCT) has also been employed to solve the D-dimensional position-dependent effective mass Schrödinger equation for some molecular potentials to get the exact bound state solutions including the energy spectrum and corresponding wave functions [41-43].

A new method to obtain the exactly solvable PT-symmetric potential potentials within the framework of the variable mass Dirac equation with the vector potential coupling scheme in $(1+1)$ dimensions [38]. Three PT-symmetric potentials are produced which are PT-symmetric harmonic oscillator-like potential, PT-symmetric of linear plus inversely linear potential and PT-symmetric kink-like potential. The SUSYQM formalism and function analysis method are use to obtain the real energy levels and corresponding spinor components for the bound states. Further, the position-dependent effective mass Dirac equation with the PT-symmetric hyperbolic cosecant potential can be mapped into the Schrödinger-like equation with the exactly solvable modified Pöschl-Teller potential [38]. The real relativistic energy levels and corresponding spinor wavefunctions for the bound states have been given in a closed form.

The Nikiforov-Uvarov (NU) method [44] and other methods have also been used to solve the D-dimensional Schrödinger equation [45] and relativistic D-dimensional Klein-Gordon [46], Dirac [47] and spinless Salpeter equations [48].

In strong coupling cases, it is crucial to understand relativistic effects on a moving particle in a potential field. In a non-relativistic case, Schrödinger equation with the Hulthén potential [10,12,13,15] was solved using the usual existing approximation, $\frac{1}{r^2} \approx \alpha^2 \frac{e^{\alpha r}}{(e^{\alpha r}-1)^2}$ for the centrifugal potential which was found to be consistent with the results of other methods [4,8,13,15]. Unfortunately, this approximation is valid only for small values of the screening parameter α, but the agreement becomes poor in the high-screening region [10, 15].

Hence, it is of sufficient need to improve the analytical results for the Schrödinger equation with the Hulthén potential by means of a new approximation scheme. Recently, Haouat and Chetouani [49] have solved the Klein-Gordon and Dirac equations in the presence of the Hulthén potential, where the energy spectrum and the scattering wavefunctions are obtained for spin-0 and spin-$\frac{1}{2}$ particles, using a more general approximation scheme, $\frac{1}{r^2} \approx \alpha^2 \frac{e^{-\gamma\alpha r}}{(1-e^{-\alpha r})^2}$ where γ is a dimensionless parameter ($\gamma = 0, 1$ and 2) for the centrifugal potential. They found that the good approximation, however, when the screening parameter α and the dimensionless parameter γ are taken as $\alpha = 0.1$ and γ =1, respectively, which is simply the case of the usual approximation [10,12,13,15]. Also, Jia and collaborators [50] have recently proposed an alternative approximation scheme, $\frac{1}{r^2} \approx \alpha^2 \left(\frac{\omega}{e^{\alpha r}-1} + \frac{1}{(e^{\alpha r}-1)^2} \right)$ where ω is a dimensionless parameter ($\omega = 1.030$), for the centrifugal potential to solve the Schrödinger equation with the Hulthén potential. Taking $\omega = 1$, their approximation can be reduced into the usual approximation [10,12,13,15]. However, the accuracy of their numerical results [50] is found to be in poor agreement with the other numerical methods like integration and variational methods [4,5]. In order to improve the accuracy of the used approximation, we propose and apply an alternative shifted approximation scheme to approximate the centrifugal term given by [51,52]

$$\frac{1}{r^2} = \lim_{\alpha \to 0} \alpha^2 \left[c_0 + \frac{e^{\alpha r}}{(e^{\alpha r}-1)^2} \right], \tag{1}$$

where c_0 is a shifting dimensionless parameter. The approximation scheme (1) emerged as a quite successful formalism to study the Schrödinger equation with the Manning-Rosen, hyperbolic and Hulthén potentials in calculating the energy eigenvalues within the framework of the NU method [51-53]. The accuracy of the results are significantly improved over all other existing literature approximation schemes and analytical methods [13,15,50]. With extremely high accuracy, we have obtained the numerical energy eigenvalues as with those obtained by the numerical integration [4,5,53], variational [4] methods and also by a MATHEMATICA package programmed by Lucha and Schöberl [54].

The purpose of this work is to employ the approximation scheme given in (1) to solve the position-dependent mass radial Klein-Gordon equation with any orbital angular quantum number l for the scalar and vector Hulthén potentials in D-dimensions. This offers a simple, accurate and efficient scheme for the exponential-type potential models in quantum mechanics.

Our paper is organized as follows. In section 2, we review the NU method. In section 3, we present a brief a derivation to find the shifting parameter c_0. Then, the analytical solution of the position-dependent mass Klein-Gordon equation with the scalar and vector Hulthén potentials is obtained for any l-state by means of the N-U method. Section 4 contains the summary and conclusions.

II. NU

The NU method is breifly outlined here and the details can be found in [44]. This method is proposed to solve the second-order differential equation of the hypergeometric type:

$$\psi_n''(z) + \frac{\widetilde{\tau}(z)}{\sigma(z)}\psi_n'(z) + \frac{\widetilde{\sigma}(z)}{\sigma^2(z)}\psi_n(z) = 0, \tag{2}$$

where $\sigma(z)$ and $\widetilde{\sigma}(z)$ are polynomials, at most, of second-degree, and $\widetilde{\tau}(s)$ is a first-degree polynomial. In order to find a particular solution for Eq. (2), let us decompose the wave-function $\psi_n(z)$ as follows:

$$\psi_n(z) = \phi_n(z)y_n(z). \tag{3}$$

We can reduce Eq. (2) into the form

$$\sigma(z)y_n''(z) + \tau(z)y_n'(z) + \lambda y_n(z) = 0, \tag{4}$$

with

$$\tau(z) = \widetilde{\tau}(z) + 2\pi(z), \;\; \tau'(z) < 0, \tag{5}$$

where $\tau'(z) = \frac{d\tau(z)}{dz}$ is the derivative. Also, λ is a constant given in the form

$$\lambda = \lambda_n = -n\tau'(z) - \frac{1}{2}n\left(n-1\right)\sigma''(z), \quad n = 0, 1, 2, \cdots, \tag{6}$$

where

$$\lambda = k + \pi'(z). \tag{7}$$

The $y_n(z)$ can be written in terms of the Rodrigues relation

$$y_n(z) = \frac{B_n}{\rho(z)}\frac{d^n}{dz^n}\left[\sigma^n(z)\rho(z)\right], \tag{8}$$

where B_n is the normalization constant and the weight function $\rho(z)$ satisfies

$$\sigma(z)\rho'(z) + \left(\sigma'(z) - \tau(z)\right)\rho(z) = 0. \tag{9}$$

The other wavefunction in the solution is defined by

$$\sigma(z)\phi'(z) - \pi(z)\phi(z) = 0. \tag{10}$$

Further, to find the weight function in Eq. (8) we need to obtain the following polynomial:

$$\pi(z) = \frac{1}{2}\left[\sigma'(z) - \widetilde{\tau}(z)\right] \pm \left\{\frac{1}{4}\left[\sigma'(z) - \widetilde{\tau}(z)\right]^2 - \widetilde{\sigma}(z) + k\sigma(z)\right\}^2. \tag{11}$$

The expression under the square root sign in Eq. (11) can be arranged as the square of a polynomial. This is possible only if its discriminant is zero. In this regard, an equation for k is being obtained. After solving such an equation, the determined values of k are included in the NU method.

III. BOUND-STATE SOLUTIONS

A. An Impoved Shifted Approximation Scheme

The approximation is based on the expansion of the centrifugal term in a series of exponentials depending on the intermolecular distance r. Therefore, instead of using the usual existing approximation in literature, let us, instead, take the following exponential-type potential to approximate the centrifugal potential,

$$\frac{1}{r^2} \approx \alpha^2\left[c_0 + v(r) + v^2(r)\right],\ v(r) = \frac{e^{\alpha r}}{e^{\alpha r} - 1},$$

$$\frac{1}{r^2} \approx \alpha^2\left[c_0 + \frac{1}{e^{\alpha r} - 1} + \frac{1}{\left(e^{\alpha r} - 1\right)^2}\right]. \tag{12}$$

In the low-screening region, $0.4 \leq \alpha r \leq 1.2$ [15] (i.e., small screening parameter α), Eq. (12) is a very well approximation to the centrifugal potential and the Schrödinger equation for such an approximation can be easily solved analytically. In Fig. 1, we give a plot of the variation of the centrifugal potential and its approximation given in Eq. (12) versus αr. It shows that the approximation (12) and $1/r^2$ are similar and coincide in both high-screening as well as in the low-screening regions.

Changing the r coordinate to x by shifting the parameters as $x = (r - r_0)/r_0$ to avoid singularities [55], we obtains

$$\frac{1}{r_0^2}(1+x)^{-2} = \alpha^2\left[c_0 + \frac{1}{e^{\gamma(1+x)} - 1} + \frac{1}{\left(e^{\gamma(1+x)} - 1\right)^2}\right],\ \gamma = \alpha r_0, \tag{13}$$

and expanding Eq. (13) around $r = r_0$ $(x = 0)$, we obtain the following expansion:

$$1 - 2x + O(x^2) = \gamma^2 \left(c_0 + \frac{1}{e^\gamma - 1} + \frac{1}{(e^\gamma - 1)^2} \right)$$

$$- \gamma^3 \left(\frac{1}{e^\gamma - 1} + \frac{3}{(e^\gamma - 1)^2} + \frac{2}{(e^\gamma - 1)^3} \right) x + O(x^2), \tag{14}$$

and consequently

$$\gamma^2 \left[c_0 + \frac{1}{e^\gamma - 1} + \frac{1}{(e^\gamma - 1)^2} \right] = 1,$$

$$\gamma^3 \left(\frac{1}{e^\gamma - 1} + \frac{3}{(e^\gamma - 1)^2} + \frac{2}{(e^\gamma - 1)^3} \right) = 2. \tag{15}$$

By solving Eqs. (14) and (15) for the dimensionless parameter c_0, we obtain

$$c_0 = \frac{1}{\gamma^2} - \frac{1}{e^\gamma - 1} - \frac{1}{(e^\gamma - 1)^2} = 0.0823058167837972, \tag{16}$$

where $e = 2.718281828459045$ is the base of the natural logarithms and the parameter $\gamma = 0.4990429999$.

Therefore, the centrifugal potential takes the form

$$\lim_{\alpha \to 0} \alpha^2 \left[\frac{1}{\gamma^2} - \frac{1}{e^\gamma - 1} - \frac{1}{(e^\gamma - 1)^2} + \frac{e^{-\alpha r}}{1 - e^{-\alpha r}} + \left(\frac{e^{-\alpha r}}{1 - e^{-\alpha r}} \right)^2 \right] = \frac{1}{r^2}. \tag{17}$$

Let us remark at the end of this analysis that the approximation used in many papers in literature [10,12,13,15] is a special case of Eq. (12) if c_0 is set to zero.

B. A Quasi-Exactly Energy Eigenvalues and Eigenfunctions

The D-dimensional time-independent radial position-dependent mass Klein-Gordon equation with scalar and vector potentials $S(r)$ and $V(r)$, respectively, $r = |\mathbf{r}|$, and position-dependent mass $m(r)$ describing a spin-zero particle takes the general form [3,46]

$$\nabla_D^2 \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(\mathbf{x}) + \frac{1}{\hbar^2 c^2} \left\{ [E_{nl} - V(r)]^2 - \left[m(r) c^2 + S(r) \right]^2 \right\} \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(\mathbf{x}) = 0,$$

$$\nabla_D^2 = \sum_{j=1}^{D} \frac{\partial^2}{\partial x_j^2}, \; \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(\mathbf{x}) = R_l(r) Y_{l_1 \cdots l_{D-2}}^{(l)}(\theta_1, \theta_2, \cdots, \theta_{D-1}), \tag{18}$$

where E_{nl} denotes the Klein-Gordon energy and ∇_D^2 denotes the D-dimensional Laplacian. Further, $\mathbf{x}$ is a D-dimensional position vector. Let us decompose the radial wavefunction $R_l(r)$ as follows:

$$R_l(r) = r^{-(D-1)/2} g(r), \tag{19}$$

we, then, reduce Eq. (18) into the following D-dimensional radial position-dependent effective mass Schrödinger-like equation

$$\frac{d^2g(r)}{dr^2}+\frac{1}{\hbar^2c^2}\left\{[E_{nl}-V(r)]^2-\left[m(r)c^2+S(r)\right]^2-\frac{(D+2l-1)(D+2l-3)\hbar^2c^2}{4r^2}\right\}g(r)=0. \tag{20}$$

Further, taking the vector and scalar potentials as the Hulthén potentials

$$V(r)=-\frac{V_0e^{-\alpha r}}{1-e^{-\alpha r}},\ S(r)=-\frac{S_0e^{-\alpha r}}{1-e^{-\alpha r}},\ \alpha=r_0^{-1}, \tag{21}$$

and choosing the following mass function

$$m(r)=m_0+\frac{m_1e^{-\alpha r}}{1-e^{-\alpha r}}, \tag{22}$$

we can rewrite Eq. (20) as

$$g''(r)+\frac{1}{\hbar^2c^2}\left\{\frac{2\left[m_0c^2\left(S_0-m_1c^2\right)+E_{nl}V_0\right]e^{-\alpha r}}{1-e^{-\alpha r}}\right.$$
$$\left.+\frac{\left[V_0^2-\left(S_0-m_1c^2\right)^2\right]e^{-2\alpha r}-\frac{\hbar^2c^2\alpha^2}{4}(D+2l-1)(D+2l-3)e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)^2}\right\}g(r)$$
$$=\frac{1}{\hbar^2c^2}\left[\left(m_0c^2\right)^2-E_{nl}^2+\Delta E_l\right]g(r),\ g(0)=0, \tag{23}$$

with the shift energy $\Delta E_l=\hbar^2c^2\alpha^2(D+2l-1)(D+2l-3)c_0/4$. On account of the wave function $g(r)$ satisfying the standard bound-state condition (real values), i.e., $g(r\rightarrow\infty)\rightarrow 0$. If we rewrite Eq. (23) by using a new variable of the form $z=e^{-\alpha r}$ ($r\in[0,\infty)$, $z\in[1,0]$), we get

$$\frac{d^2g(z)}{dz^2}+\frac{1-z}{z(1-z)}\frac{dg(z)}{dz}+\frac{1}{\left[z(1-z)\right]^2}$$
$$\times\left\{-\varepsilon_{nl}^2+(\beta_1-\beta_4-\gamma+2\varepsilon_{nl}^2)s-(\beta_1+\beta_2+\beta_3-\beta_4+\varepsilon_{nl}^2)s^2\right\}g(z)=0, \tag{24}$$

where the following definitions of parameters

$$\varepsilon_{nl}=\frac{\sqrt{\left(m_0c^2\right)^2-E_{nl}^2+\Delta E_l}}{Q},\ \ \beta_1=\frac{2\left(m_0c^2S_0+E_{nl}V_0\right)}{Q^2},\ \beta_2=\frac{S_0^2-V_0^2}{Q^2},$$

$$\beta_3=\frac{m_1c^2\left(m_1c^2-2S_0\right)}{Q^2},\ \beta_4=\frac{2m_0m_1c^4}{Q^2},\ \gamma=\frac{(D+2l-1)(D+2l-3)}{4},\ Q=\hbar c\alpha, \tag{25}$$

are used. For bound-state solutions, we require that $V_0 \leq (S_0 - m_1c^2)$ and $E_{nl} \leq \sqrt{(m_0c^2)^2 + \Delta E_l}$. In order to solve Eq. (24) by means of the N-U method, we should compare it with Eq. (2). The following values for parameters are found

$$\widetilde{\tau}(z) = 1-z,\ \sigma(z) = z-z^2,\ \widetilde{\sigma}(z) = -\varepsilon_{nl}^2+(\beta_1-\beta_4-\gamma+2\varepsilon_{nl}^2)s-(\beta_1+\beta_2+\beta_3-\beta_4+\varepsilon_{nl}^2)s^2. \quad (26)$$

If we insert these values of parameters into Eq. (11), with $\sigma'(z) = 1 - 2z$, the following linear function is obtained

$$\pi(z) = -\frac{z}{2}\pm\frac{1}{2}\sqrt{\left[1 + 4(\beta_1 + \beta_2 + \beta_3 - \beta_4 + \varepsilon_{nl}^2 - k)\right] z^2 + \left[4(k - \beta_1 + \beta_4 + \gamma - 2\varepsilon_{nl}^2)\right] z + 4\varepsilon_{nl}^2}. \quad (27)$$

The determinant of the square root must be set equal to zero, that is, $\Delta = (k - \beta_1 + \beta_4 + \gamma - 2\varepsilon_{nl}^2)^2 - \varepsilon_{nl}^2 \left[1 + 4(\beta_1 + \beta_2 + \beta_3 - \beta_4 + \varepsilon_{nl}^2 - k)\right] = 0$. Thus, the constant k found to be

$$k = \beta_1 - \beta_4 - \gamma \pm \varepsilon_{nl}\sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}. \quad (28)$$

In this regard, we can find four possible functions for $\pi(z)$ as

$$\pi(s) = -\frac{z}{2} \pm \begin{cases} \varepsilon_{nl} \mp \left[\varepsilon_{nl} - \frac{1}{2}\sqrt{1 + 4b}\right] z & \text{for } k_1 = d + \varepsilon_{nl}\sqrt{1 + 4b}, \\ \varepsilon_{nl} \mp \left[\varepsilon_{nl} + \frac{1}{2}\sqrt{1 + 4b}\right] z & \text{for } k_2 = d - \varepsilon_{nl}\sqrt{1 + 4b}. \end{cases} \quad (29)$$

where $b = \beta_2 + \beta_3 + \gamma$ and $d = \beta_1 - \beta_4 - \gamma$. Thus, taking the following values

$$k = \beta_1 - \beta_4 - \gamma - \varepsilon_{nl}\sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}, \quad (30)$$

and

$$\pi(z) = -\frac{z}{2} + \varepsilon_{nl} - \left[\varepsilon_{nl} + \frac{1}{2}\sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right] z, \quad (31)$$

they give

$$\tau(z) = 1 + 2\varepsilon_{nl} - 2\left[1 + \varepsilon_{nl} + \frac{1}{2}\sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right] z,$$

$$\tau'(s) = -2\left[1 + \varepsilon_{nl} + \frac{1}{2}\sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right] < 0. \quad (32)$$

Eqs. (30)-(32) together with the assignments given in Eq. (26), the following expressions for λ are obtained

$$\lambda_n = \lambda = n^2 + \left[1 + 2\varepsilon_{nl} + \sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right] n,\ (n = 0, 1, 2, \cdots), \quad (33)$$

$$\lambda = \beta_1 - \beta_4 - \gamma - \frac{1}{2}(1 + 2\varepsilon_{nl})\left[1 + \sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right], \quad (34)$$

where n is the radial quantum number. By defining

$$\delta = \frac{1}{2}\left(1 + \sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right), \tag{35}$$

where $\beta_2 + \beta_3 = \delta^2 - \delta - \gamma$. With the aid of Eq. (35), we can easily obtain the energy eigenvalue equation of the Hulthén potential by solving Eqs. (33) and (34):

$$\begin{aligned}
\varepsilon_{nl}^{(D)} &= \frac{(\beta_1 - \beta_4 - \gamma - n^2) - (2n+1)\delta}{2(n+\delta)} \\
&= \frac{4\left[\beta_1 - \beta_4 - n^2 - (2n+1)\delta\right] - (D+2l-1)(D+2l-3)}{8(n+\delta)} \\
&= \frac{2\left[m_0 c^2 \widetilde{S}_0 + E_{nl}^{\pm} V_0\right] + \widetilde{S}_0^2 - V_0^2}{2Q^2(n+\delta)} - \frac{n+\delta}{2},\ (n = 0, 1, 2, \cdots),
\end{aligned} \tag{36}$$

where $\widetilde{S}_0 = S_0 - m_1 c^2$ is the modified scalar potential. Solving the last equation for the energy eigenvalues $E_{nl}^{\pm}$, we obtain

$$\begin{aligned}
E_{nl}^{\pm} &= \frac{V_0}{2}\left[1 - \frac{4\widetilde{S}_0\left(\widetilde{S}_0 + 2m_0c^2\right)}{4V_0^2 + \kappa_{nl}^2}\right] \pm \frac{\kappa_{nl}}{2}\sqrt{\xi - \frac{1}{4}\left[1 - \frac{4\widetilde{S}_0\left(\widetilde{S}_0 + 2m_0c^2\right)}{4V_0^2 + \kappa_{nl}^2}\right]^2}, \\
\xi &= \frac{(2m_0c^2)^2 + \hbar^2 c^2 \alpha^2 (D+2l-1)(D+2l-3)c_0}{4V_0^2 + \kappa_{nl}^2}, \\
\kappa_{nl} &= \hbar c\alpha\,(2n+1) + \sqrt{4\left(\widetilde{S}_0^2 - V_0^2\right) + (\hbar c\alpha)^2\,(D+2l-2)^2},
\end{aligned} \tag{37}$$

where $n = 0, 1, 2, \cdots$ and $l = 0, 1, 2, \cdots$ signify the usual radial and angular momentum quantum numbers, respectively, and

$$(\hbar c\alpha)^2 (D+2l-2)^2 + 4\widetilde{S}_0^2 \geq 4V_0^2,\ 4\xi \geq \left[1 - \frac{4\widetilde{S}_0\left(\widetilde{S}_0 + 2m_0c^2\right)}{4V_0^2 + \kappa_{nl}^2}\right]^2, \tag{38}$$

are constraints over the strength of the potential coupling parameters. In the above equation, let us remark that it is not difficult to conclude that all bound-states appear in pairs, two energy solutions are valid for the particle $E^p = E_{nl}^{+}$ and the second one corresponds to the anti-particle energy $E^a = E_{nl}^{-}$ in the Hulthén field. When we take the scalar and vector potentials as $\widetilde{S}_0 = 0$ (i.e., $S_0 = m_1 c^2$) and $V_0 \neq 0$, the energy equation (37) becomes

$$E_{nl}^{\pm} = \frac{V_0}{2} \pm \frac{\kappa_{nl}}{2}\sqrt{\frac{(2m_0c^2)^2 + \hbar^2 c^2 \alpha^2 (D+2l-1)(D+2l-3)c_0}{4V_0^2 + \kappa_{nl}^2} - \frac{1}{4}},$$

$$(4m_0c^2)^2 + 4\hbar^2c^2\alpha^2(D+2l-1)(D+2l-3)c_0 \geq 4V_0^2 + \kappa_{nl}^2,$$

$$\kappa_{nl} = \hbar c\alpha(2n+1) + \sqrt{(\hbar c\alpha)^2 (D+2l-2)^2 - 4V_0^2},\ D \geq 1, \tag{39}$$

with the following constraints on the coupling parameter of the vector potential:

$$(\hbar c\alpha)^2 (D+2l-2)^2 \geq 4V_0^2, \tag{40}$$

must be fulfilled for real eigenvalues.

Therefore, having solved the D-dimensional position-dependent mass Klein-Gordon equation for scalar and vector usual Hulthén potentials, we should make some useful remarks.

(i) For s-wave ($l = 0$), the exact energy eigenvalues of the $1D$ Klein-Gordon equation becomes

$$E_n^{\pm} = \frac{V_0}{2}\left(1 - \frac{4\widetilde{S}_0\left(\widetilde{S}_0 + 2m_0c^2\right)}{4V_0^2 + \kappa_n^2}\right) \pm \kappa_n \sqrt{\frac{m_0^2c^4}{4V_0^2+\kappa_n^2} - \frac{1}{16}\left(1 - \frac{4\widetilde{S}_0\left(\widetilde{S}_0 + 2m_0c^2\right)}{4V_0^2 + \kappa_n^2}\right)^2},$$

$$\kappa_n = \hbar c\alpha\,(2n+1) + \sqrt{(\hbar c\alpha)^2 + 4(\widetilde{S}_0^2 - V_0^2)}, \tag{41}$$

In order that at least one level might exist, it is necessary that the inequalities

$$\hbar^2c^2\alpha^2 + 4\widetilde{S}_0^2 \geq 4V_0^2,\ \frac{16m_0^2c^4}{4V_0^2+\kappa_n^2} \geq \left(1 - \frac{4\widetilde{S}_0\left(\widetilde{S}_0 + 2m_0c^2\right)}{4V_0^2 + \kappa_n^2}\right)^2, \tag{42}$$

are fulfilled. In the case $\widetilde{S}_0 = 0$, $V_0 \neq 0$, the energy spectrum (in units where $\hbar = c = 1$):

$$E_n^{\pm} = \frac{V_0}{2} \pm \left[\alpha\,(2n+1) + \sqrt{\alpha^2 - 4V_0^2}\right]\sqrt{\frac{m_0^2}{4V_0^2 + \left[\alpha\,(2n+1) + \sqrt{\alpha^2 - 4V_0^2}\right]^2} - \frac{1}{16}}, \tag{43}$$

with the following constraints on the potential coupling constant:

$$16m_0^2 \geq 4V_0^2 + \left[\sqrt{\alpha^2 - 4V_0^2} + (2n+1)\right]^2,\ \alpha \geq 2V_0, \tag{44}$$

are fulfilled for bound state solutions. We notice that the result given in Eq. (43) is identical to Eq. (31) of Ref. [56]. As can be seen from Eq. (43), there are only two lower-lying states ($n = 0, 1$) for a Klein-Gordon particle of rest mass $m_0 = 1$ and screening parameter $\alpha = 1$ with vector coupling strength $V_0 \leq 1/2$. As an example, one may calculate the ground state energy for the vector coupling strength $V_0 = \alpha/2$ as

$$E_0^{\pm} = \frac{V_0}{2}\left[1 \pm \sqrt{\frac{2m_0^2}{V_0^2} - 1}\right]. \tag{45}$$

Further, in the case of pure scalar potential ($V_0 = 0, S_0 = m_1 c^2$), the energy spectrum

$$E_n^{\pm} = \pm\sqrt{m_0^2 c^4 - \frac{(\hbar c\alpha)^2 (n+1)^2}{4}}, \; 4m_0^2 c^4 \geq (\hbar c\alpha)^2 (n+1)^2 . \tag{46}$$

Since the Klein-Gordon equation is independent of the sign of E_n for scalar potentials, the wavefunctions become the same for both energy values. If the range parameter α is chosen to be $\alpha = 1/\lambda_c$, where $\lambda_c = \hbar/m_0 c = 1/m_0$ denotes the Compton wavelength of the Klein-Gordon particle. It can be seen easily that while $S_0 \to m_1 c^2$ in ground state ($n = 0$), all energy eigenvalues tend to the value $E_0 \approx 0.866\ m_0$.

(ii) For $D = 3$, the mixed scalar and vector Hulthén potentials, the energy eigenvalues for $l \neq 0$ are given by

$$E_{nl}^{\pm} = \frac{V_0}{2}\left(1 - \frac{4\widetilde{S}_0\left(\widetilde{S}_0 + 2m_0c^2\right)}{4V_0^2 + \widetilde{\kappa}_{nl}^2}\right) \pm \widetilde{\kappa}_{nl}\sqrt{\widetilde{\xi} - \frac{1}{4}\left(1 - \frac{4\widetilde{S}_0\left(\widetilde{S}_0 + 2m_0c^2\right)}{4V_0^2 + \widetilde{\kappa}_{nl}^2}\right)^2},$$

$$\widetilde{\xi} = \frac{(m_0c^2)^2 + \hbar^2 c^2 \alpha^2 l(l+1)c_0}{4V_0^2 + \widetilde{\kappa}_{nl}^2},$$

$$\widetilde{\kappa}_{nl} = \hbar c\alpha\,(2n+1) + \sqrt{(\hbar c\alpha)^2 (2l+1)^2 + 4\left(\widetilde{S}_0^2 - V_0^2\right)}. \tag{47}$$

Further, in order that at least one real eigenvalue might exist, it is necessary that the inequality

$$(\hbar c\alpha)^2 (2l+1)^2 + 4\widetilde{S}_0^2 \geq 4V_0^2, \; 4\widetilde{\xi} \geq \left(1 - \frac{4\widetilde{S}_0\left(\widetilde{S}_0 + 2m_0c^2\right)}{4V_0^2 + \widetilde{\kappa}_{nl}^2}\right)^2, \tag{48}$$

must be fulfilled. For the case where $\widetilde{S}_0 = 0$ in the spatial-dependent mass ($S_0 = 0$, in the constant mass case) [46], the energy eigenvalues turn out to be

$$E_{nl}^{\pm} = \frac{V_0}{2} \pm \eta_{nl}\sqrt{\frac{(m_0c^2)^2 + \hbar^2 c^2 \alpha^2 l(l+1)c_0}{4V_0^2 + \eta_{nl}^2} - \frac{1}{16}},$$

$$\eta_{nl} = \hbar c\alpha\,(2n+1) + \sqrt{(\hbar c\alpha)^2 (2l+1)^2 - 4V_0^2}, \; \hbar c\alpha(2l+1) \geq 2V_0, \tag{49}$$

with the following constraint over the potential parameters:

$$(4m_0c^2)^2 + 16\hbar^2 c^2 \alpha^2 l(l+1)c_0 \geq 4V_0^2 + \left[\hbar c\alpha(2n+1) + \sqrt{(\hbar c\alpha)^2 (2l+1)^2 - 4V_0^2}\right]^2 . \tag{50}$$

(iii) When $D=3$ and $l=0$, the centrifugal term $\frac{(D+2l-1)(D+2l-3)}{4r^2}=0$ and consequently the approximation term $\frac{(D+2l-1)(D+2l-3)\alpha^2}{4}\left[c_0+\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2}\right]=0$, too. Thus, the energy eigenvalues turn to become

$$\sqrt{(m_0c^2)^2-E_n^{\pm 2}}=\frac{2\left[m_0c^2\widetilde{S}_0+E_n^{\pm}V_0\right]+\widetilde{S}_0^2-V_0^2}{2\hbar c\alpha(n+\delta)}-\hbar c\alpha\left(\frac{n+\delta}{2}\right),$$

$$\delta=\frac{1}{2}\left[1+\frac{1}{(\hbar c\alpha)}\sqrt{(\hbar c\alpha)^2+4\left(\widetilde{S}_0^2-V_0^2\right)}\right],\ (n=0,1,2,3,\cdots) \tag{51}$$

which gives

$$E_n^{\pm}=\frac{V_0}{2}\left(1-\frac{4\widetilde{S}_0\left(\widetilde{S}_0+2m_0c^2\right)}{4V_0^2+\xi_n^2}\right)\pm\varsigma_n\sqrt{\frac{(m_0c^2)^2}{4V_0^2+\varsigma_n^2}-\frac{1}{4}\left(1-\frac{4\widetilde{S}_0\left(\widetilde{S}_0+2m_0c^2\right)}{4V_0^2+\varsigma_n^2}\right)^2},$$

$$\varsigma_n=\hbar c\alpha\,(2n+1)+\sqrt{(\hbar c\alpha)^2+4\left(\widetilde{S}_0^2-V_0^2\right)},$$

$$(\hbar c\alpha)^2+4\widetilde{S}_0^2\geq 4V_0^2,\ (4m_0c^2)^2\geq\left(4V_0^2+\varsigma_n^2\right)\left(1-\frac{4\widetilde{S}_0\left(\widetilde{S}_0+2m_0c^2\right)}{4V_0^2+\varsigma_n^2}\right)^2 \tag{52}$$

(iv) For equal scalar and vector usual Hulthén potential (i.e., $S_0=V_0$), Eq. (36) with the aid of Eq. (25) can be reduced to the relativistic energy equation (in the conventional atomic units $\hbar=c=1$):

$$\sqrt{m_0^2+\frac{(D+2l-1)(D+2l-3)c_0}{4r_0^2}-E_R^2}$$

$$=\frac{2r_0V_0\left[m_0+E_R-m_1\right]+r_0(m_1-2m_0)m_1}{2(n+\delta)}-\frac{n+\delta}{2r_0},$$

$$\delta=\frac{1}{2}\left[1+\sqrt{(D+2l-2)^2+\left(2r_0m_1c^2\right)^2-8r_0^2V_0m_1c^2}\right],\ (n=l=0,1,2,3,\cdots), \tag{53}$$

which is Eq. (22) of Ref. [58] if the perturbed mass $m_1=0$ and shifting parameter $c_0=0$.

(v) We discuss non-relativistic limit of the energy equation (53). When $V_0=S_0$, Eq. (23) reduces to a Schrödinger-like equation for the potential $2V(r)$. In other words, the non-relativistic limit is the Schrödinger equation for the potential $-2V_0e^{-r/r_0}/\left[1-e^{-r/r_0}\right]$, $r_0=\alpha^{-1}$. After making the parameter replacements $m_0+E_R\rightarrow 2m_0$ and $m_0-E_R\rightarrow -E_{NR}$ in Eq. (53)[58], it reduces into the non-relativistic energy equation of Refs. [10,12,13,15,57,59]:

$$E_{NR}=\frac{\alpha^2(D+2l-1)(D+2l-3)c_0}{8m_0}-\frac{1}{8m_0\alpha^2}\left[\frac{(2V_0-m_1)(2m_0-m_1)-\alpha^2\left(n+\delta\right)^2}{(n+\delta)}\right]^2,$$

$$\delta=\frac{1}{2}\left[1+\frac{1}{\alpha}\sqrt{\alpha^2(D+2l-2)^2+(2m_1c^2)^2-8V_0m_1c^2}\right],\ (n=l=0,1,2,3,\cdots) \qquad (54)$$

which is Eq. (23) of Ref. [57] when c_0 and $m_1$are set to zero. It is noted that Eq. (54) is identical to Eq. (59) of Ref. [56] for s-wave in $1D$ when the potential is $2V(r)$, when α becomes pure imaginary, i.e., $\alpha\to i\alpha$ and when we set $m_0=1$, $m_1=0$ and $c_0=0$. Equation (54) can be reduced to the constant mass ($m_1=0$) case in the three-dimensional Schrödinger equation:

$$E_{NR}=\frac{\alpha^2}{2m_0}\left\{l(l+1)c_0-\left[\frac{2V_0m_0}{\alpha^2(n+l+1)}-\frac{n+l+1}{2}\right]^2\right\},$$

which is identical to the expressions given in Refs. [50,52] when the vector potential is taken as $2V(r)$, $c_0=0$ and $\omega=1$ in Ref. [50]. The numerical approximation to the energy eigenvalues in Ref. [50] for the last energy equation was found to be more efficient than the approximation given in Eq. (19) of Ref. [50]. Taking $V_0=Z\alpha e^2$ as in [50], we obtain

$$E_{NR}=\frac{\alpha^2}{2m_0}\left\{l(l+1)c_0-\left[\frac{2m_0Ze^2}{\alpha(n+l+1)}-\frac{(n+l+1)}{2}\right]^2\right\}.$$

For the s-wave ($l=0$), the above energy spectrum is identical to the factorization method [23], SUSYQM [12,13] and NU [46] methods. Expanding the energy equation (53) under the weak coupling conditions $[(n+\delta)/m_0r_0]^2\ll 1$ and $[V_0r_0/(n+\delta)]^2\ll 1$, retaining only the terms containing the power of $(1/m_0r_0)^2$ and $(r_0V_0)^4$, we obtain the relativistic energy equation

$$E_R\approx E_{NR}+m_0+2(2m_0-m_1)\left(\frac{(2V_0-m_1)}{2\alpha(n+\delta)}\right)^4, \qquad (55)$$

which is simply Eq. (24) of Ref. [57], where δ is given in Eq. (54). The first term is the non-relativistic energy and third term is the relativistic approximation to energy.

Now, let us find the wave function $y_n(s)$, which is the polynomial solution of hypergeometric-type equation. We multiply Eq. (4) by the weight function $\rho(s)$ so that it can be rewritten in self-adjoint form [45,46]

$$[\omega(s)y_n'(s)]'+\lambda\rho(s)y_n(s)=0. \qquad (56)$$

The weight function $\rho(s)$ that satisfies Eqs. (9) takes the following form

$$\rho(z)=z^{2\varepsilon_{nl}}(1-z)^{\beta},\ \beta=2\delta-1 \qquad (57)$$

which gives the Rodrigues relation:

$$y_{nl}(z) = B_{nl} z^{-2\varepsilon_{nl}} (1-z)^{-\beta} \frac{d^n}{dz^n} \left[z^{n+2\varepsilon_{nl}} (1-z)^{n+\beta} \right]$$
$$= B_{nl} P_n^{(2\varepsilon_{nl},\beta)} (1-2z). \tag{58}$$

On the other hand, inserting the values of $\sigma(s), \pi(s)$ and $\tau(s)$ given in Eqs. (26), (31) and (32) into Eq. (10), we get the other part of the wave function

$$\phi(s) = z^{\varepsilon_{nl}} (1-z)^{\delta}. \tag{59}$$

Hence, the wave function $g_n(z) = \phi_n(z) y_n(z)$ becomes

$$g(z) = C_{nl} z^{\varepsilon_{nl}} (1-z)^{\delta} P_n^{(2\varepsilon_{nl},\beta)} (1-2z)$$
$$= C_{nl} z^{\varepsilon_{nl}^{(D)}} (1-z)^{\delta} P_n^{(2\varepsilon_{nl}^{(D)},\beta)} (1-2z), \ z \in [1,0). \tag{60}$$

Finally, the radial wave functions of the Klein-Gordon equation are obtained as

$$R_l(r) = N_{nl} r^{-(D-1)/2} e^{-\varepsilon_{nl}^{(D)} \alpha r} (1-e^{-\alpha r})^{\delta} P_n^{(2\varepsilon_{nl}^{(D)},\beta)} (1-2e^{-\alpha r}), \tag{61}$$

with

$$\varepsilon_{nl}^{(D)} = \frac{1}{\hbar c \alpha} \sqrt{(m_0 c^2)^2 + \frac{\hbar^2 c^2 \alpha^2 (D+2l-1)(D+2l-3) c_0}{4} - E_{nl}^2},$$
$$\beta = \frac{1}{\hbar c \alpha} \sqrt{4\left(\widetilde{S}_0^2 - V_0^2\right) + (\hbar c \alpha)^2 (D+2l-2)^2}, \ \delta = \frac{1}{2}(1+\beta), \tag{62}$$

where E_{nl} is given in Eq. (37) and N_{nl} is the radial normalization factor. The Jacobi polynomials $P_n^{(2\varepsilon_{nl}^{(D)},\beta)}(1-2e^{-\alpha r})$ [60] in the last result can be written in terms of the hypergeometric function ${}_2F_1(-n, n+2\varepsilon_{nl}^{(D)}+\beta+1, 2\varepsilon_{nl}^{(D)}; e^{-\alpha r})$ which gives the same result obtained in Ref. [57].

(i) The exact radial wave functions for the s-wave Klein-Gordon equation in $1D$ reduces to the following form (in $\hbar = c = 1$) :

$$R_n(x) = C_n e^{-\sqrt{m_0^2 - E_n^2} x} (1-e^{-x/r_0})^{(1+a)/2} P_n^{(2r_0\sqrt{m_0^2-E_n^2},a)} (1-2e^{-x/r_0}),$$
$$a = \sqrt{1+4r_0^2\left(\widetilde{S}_0^2 - V_0^2\right)}, \tag{63}$$

where E_n is given in Eq. (41). The last formula is identical to Eq. (35) of Ref. [56] when the modified scalar potential, $\widetilde{S}_0$, is set to zero.

(ii) Choosing the atomic units $h/2\pi = \hbar = c = 1$, the exact radial wave functions for the s-wave Klein-Gordon equation in $3D$ reduces to the following form:

$$R_n(r) = N_n e^{-\sqrt{m_0^2 + -E_n^2} r} \left(1 - e^{-r/r_0}\right)^{(1+a)/2} P_n^{(2r_0\sqrt{m_0^2 - E_n^2}, a)}(1 - 2e^{-r/r_0}),$$

$$P_n^{(2r_0\sqrt{m_0^2 - E_n^2}, a)}(1 - 2e^{-r/r_0}) =_2 F_1(-n, n + 2r_0\sqrt{m_0^2 - E_n^2} + a + 1, 2r_0\sqrt{m_0^2 - E_n^2}; e^{-\alpha r}), \quad (64)$$

where E_n and a are given in Eq. (52) and Eq. (63), respectively. The last formula is identical to Eq. (22) of Ref. [57] when the perturbed mass m_1 is set to zero.

(iii) The quasi-exact radial wave functions for the l-wave Klein-Gordon equation in $3D$ reduces to the following form (in $\hbar = c = 1$) :

$$R_{nl}(r) = N_{nl} e^{-\sqrt{m_0^2 + \frac{l(l+1)c_0}{r_0^2} - E_{nl}^2} r} \left(1 - e^{-r/r_0}\right)^{(1+a_l)/2} P_n^{(2r_0\sqrt{m_0^2 + \frac{l(l+1)c_0}{r_0^2} - E_{nl}^2}, a_l)}(1 - 2e^{-r/r_0}),$$

$$P_n^{(2r_0\sqrt{m_0^2 + \frac{l(l+1)c_0}{r_0^2} - E_{nl}^2}, a_l)}(1 - 2e^{-r/r_0})$$

$$=_2 F_1(-n, n + 2r_0\sqrt{m_0^2 + \frac{l(l+1)c_0}{r_0^2} - E_{nl}^2} + a_l + 1, 2r_0\sqrt{m_0^2 + \frac{l(l+1)c_0}{r_0^2} - E_{nl}^2}; e^{-\alpha r}),$$

$$a_l = \sqrt{(2l+1)^2 + 4r_0^2\left(\widetilde{S}_0^2 - V_0^2\right)}, \quad (65)$$

where E_{nl} is given in Eq. (43) and $\alpha = r_0^{-1}$. It is identical to Ref. [57] when $m_1 = 0$. The eigenfunctions in the constant mass case are written as

$$R_{nl}(r) = N_{nl} e^{-\sqrt{m_0^2 + \frac{l(l+1)c_0}{r_0^2} - E_{nl}^2} r} \left(1 - e^{-r/r_0}\right)^{(1+a_l)/2} P_n^{(2r_0\sqrt{m_0^2 + \frac{l(l+1)c_0}{r_0^2} - E_{nl}^2}, b_l)}(1 - 2e^{-r/r_0}),$$

$$b_l = \sqrt{(2l+1)^2 + 4r_0^2\left(S_0^2 - V_0^2\right)}. \quad (66)$$

At the end of these calculations, the total wave functions of the Klein-Gordon equation with position-dependent mass for the scalar and vector Hulthén potentials are

$$\psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(\mathbf{x}) = N_{nl} r^{-(D-1)/2} e^{-\varepsilon_{nl}^{(D)} \alpha r} \left(1 - e^{-\alpha r}\right)^{\delta} P_n^{(2\varepsilon_{nl}^{(D)}, \beta)}(1 - 2e^{-\alpha r})$$

$$\frac{1}{\sqrt{2\pi}} \exp(\pm i l_1 \theta_1) \prod_{j=2}^{D-2} \sqrt{\frac{(2l_j + j - 1)\, n_j!}{2\Gamma\left(l_j + l_{j-1} + j - 2\right)}} \left(\sin\theta_j\right)^{l_j - n_j} P_{n_j}^{(l_j - n_j + (j-2)/2, l_j - n_j + (j-2)/2)}(\cos\theta_j)$$

$$\sqrt{\frac{(2n_{D-1} + 2m' + 1)\, n_{D-1}!}{2\Gamma\left(n_{D-1} + 2m'\right)}} \left(\sin\theta_{D-1}\right)^{l_{D-2}} P_{n_{D-1}}^{(m', m')}(\cos\theta_{D-1}), \quad (67)$$

where $\varepsilon_{nl}^{(D)}$ and β are given in Eq. (62) and E_{nl} is given in Eq. (37) [46].

To check the accuracy of the resulting analytical expressions. We give a few numerical real eigenvalues for some selected values of the mass m_0 and $m_1$and potential parameters S_0 and V_0. In Tables 1 and 2, taking $\alpha = 1$ and $m_0 = 1$, we present some numerical values for the energy spectrum of the constant mass Klein-Gordon equations with the condition $S_0 = V_0$ for all possible real eigenvalues. To get more real energy eigenvalues in the constant mass case (e.g., $m_0 = 1$, $m_1 = 0$), the vector parameter V_0 of the Hulthén potential should be increased. As shown in Tables 1 and 2, when the parameter $V_0 = S_0 = 1, 2, 3, 6, 8, 20$, we obtain $N =$ 1,3,6,10,15,36 real energy eigenvalues, respectively. The numerical solution of the position-dependent mass case with vector and scalar Hulthén potential parameters satisfying the conditions $S_0 = \pm V_0$ and $S_0 > V_0$ are presented in Table 3. For example, in Table 3, when the Hulthén potential parameter $V_0 = S_0 = 1$, $m_0 = 5$ and $m_1 \neq 0$, we obtain $N = 46$ real energy eigenvalues. Obviously, the number of real eigenvalues increases in the solution of the position-dependent case than in the constant mass case where the condition $S_0 \geq V_0$ must be fulfilled.

IV. COCLUSIONS

In summary, we have proposed an alternative approximation scheme for the centrifugal potential similar to the non-relativistic case. This is because the usual approximation [10,13,15] for the centrifugal term is only valid for low-screening region, however, for the high screening region where α increases, the agreement between the old approximation and centrifugal term decreases. Using this approximation scheme, the analytical solutions of the radial Klein-Gordon equation with position-dependent mass for scalar and vector Hulthén potentials can be approximately obtained for any dimension D and orbital angular momentum quantum number l. It is found that the expressions for the eigenvalues and the corresponding eigenfunctions become complicated and tedious since the eigenvalues are related to the parameters m_o, m_1, S_0, V_0, c_0 and α. We have investigated the possibility to obtain the bound-state (real) energy spectra with some constraints to be imposed on the parameters and, further, the relationship between the strengths of vector V_0 and scalar S_0 coupling parameters. In one- and three-dimensions, the special cases for the angular momentum $l = 0, 1$ are carried out in detail. We find that the analytical expressions of the energy eigenvalues and eigenfunctions are identical with the results obtained by other methods. The analytical energy equation and the unnormalized radial wavefunctions are expressed in terms of hypergeometric polynomials. For constant mass case ($m_1 = 0$) and s-wave ($l = 0$), the results are reduced to exact solution of bound states of s-wave Klein-Gordon equation with scalar and vector Hulthén potentials. To test our results, we have also calculated the energy eigenvalues of a particle and antiparticle for the constant mass limit as well as the position-dependent mass case. The case of spatial-dependent mass with scalar potential $S_0 = m_1 c^2$ is found to be equivalent to the constant mass with scalar potential $S_0 = 0$ in a pure vector case. Hence, the spectrum is found to be same.

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[1] F. Dominguez-Adame, Phys. Lett. A 136 (1989) 175.

[2] B. Talukdar, A. Yunus and M.R. Amin, Phys. Lett. A 141 (1989) 326; N. Saad, Phys. Scr. 76 (2007) 623; W.C. Qiang, R.S. Zhou and Y. Gao, Phys. Lett. A 371 (2007) 201.

[3] L. Chetouania, L. Guechia, A. Lecheheba, T.F. Hammannb and A. Messouberc, Physica A 234 (1996) 529; Y.C. Chen, F.L. Lu and D.S. Sun, Cent. Eur. J. Phys. 6(4) (2008) 884.

[4] Y.P. Varshni, Phys. Rev. A 41 (1990) 4682.

[5] M.A. Nuñez, Phys. Rev. A 47 (1993) 3620.

[6] S.H. Patil, J. Phys. A: Math. Gen. 34 (2001) 3153.

[7] P. Matthys and H. De. Meyer, Phys. Rev. 38 (1988) 1168.

[8] A.Z. Tang and F.T. Chan, Phys. Rev. A 35 (1987) 911.

[9] B. Roy and R. Roychoudhury, J. Phys. A: Math. Gen. 20 (1987) 3051.

[10] S.M. Ikhdair and R. Sever, J. Math. Chem. 42 (2007) 461; S.M. Ikhdair and R. Sever, Ann. Phys. (Leibzig) 16 (2007) 218.

[11] S.M. Ikhdair and R. Sever, Int. J. Theor. Phys. 46 (2007) 1643; S.M. Ikhdair and R. Sever, Int. J. Theor. Phys. 46 (2007) 2384.

[12] E.D. Filho and R.M. Ricotta, Mod. Phys. Lett. A 10 (1995) 1613.

[13] B. Gönül, O. Özer, Y. Gancelik and M. Kocak, Phys. Lett. A 275 (2000) 238.

[14] S.W. Qian, B.W. Huang and Z.Y. Gu, New J. Phys. 4 (2002) 13.

[15] O. Bayrak, G. Kocak and I. Boztosun, J. Phys. A: Math. Gen. 39 (2006) 11521; C.Y. Chen, D.S. Sun and F.L. Lu, Phys. Lett. A 370 (2007) 219; W.C. Qiang and S.H. Dong, Phys. Lett. A 368 (2007) 13.

[16] L. Hulthén, Ark. Mat. Astron. Fys. A 28 (1942) 5.

[17] B. Durand and L. Durand, Phys. Rev. D 23 (1981) 1092.

[18] R. Dutt, K. Chowdhury and Y.P. Varshni, J. Phys. A: Math. Gen. 18 (1985) 1379.

[19] T. Xu, Z.Q. Cao, Y.C. Ou, Q.S. Shen and G.L. Zhu, Chin. Phys. 15 (2006) 1172.

[20] I.S. Bitensky, V.K. Ferleger and I.A. Wojciechowski, Nucl. Instrum. Methods Phys. Res., Sect. B 125 (1997) 201.

[21] C.S. Jia, J.Y. Wang, S. He and L.T. Sun, J. Phys. A: Math. Gen. 33 (2000) 6993.

[22] J.A. Olson, D.A. Micha, J. Chem. Phys. 68 (1978) 4252.

[23] S. Flügge, Practical Quantum Mechanics, Vol. 1, Springer, Berlin, 1994.

[24] J.V. Guo, J. Meng and F.X. Xu, Chin. Phys. Lett. 20 (2003) 602.

[25] A.D. Alhaidari, J. Phys. A 37 (2004) 5805.

[26] R.K. Su and Z.Q. Ma, J. Phys. A 19 (1986) 1739.

[27] W.C. Qiang, Chin. Phys. 11 (2002) 757.

[28] R.S. Tutik, J. Phys. A 25 (1992) L413.

[29] X.Y. Gu, Z.Q. Ma and S.H. Dong, Int. J. Mod. Phys. E 11 (2002) 335.

[30] S.H. Dong, J. Phys. A 36 (2003) 4977.

[31] S.H. Dong, X.Y. Gu and Z.Q. Ma, Int. J. Mod. Phys. E 12 (2003) 555.

[32] Z.Q. Ma, S.H. Dong, X.Y. Gu, J. Yu and M. Lozada-Cassou, Int. J. Mod. Phys. E 13 (2004) 597.

[33] C.Y. Chen, D.S. Sun and F.L. Lu, Phys. Lett. A 330 (2004) 424.

[34] A. de Souza Dutra and G. Chen, Phys. Lett. A 349 (2006) 297; G. Chen, Z.D. Chen and P.E. Xuan, Phys. Scr. 74 (2006) 367.

[35] G. Chen, Phys. Lett. A 339 (2005) 300; A. de Souza Dutra and C.A.S. Almeida, Phys. Lett. A 275 (2000) 25.

[36] A.D. Alhaideri, Phys. Lett. A 322 (2004) 72; R. Koç and M. Koca, J. Phys. A: Math. Gen. 36 (2003) 8105.

[37] O. Mustafa and S.H. Mazharimousavi, Int. J. Theor. Phys. 47 (2008) 1112.

[38] C.S. Jia and A. de Souza Dutra, Annals of Phys. 323 (2008) 566; C.S. Jia, P.Q. Wang, J.Y. Liu and S. He, Int. J. Theor. Phys. 47 (2008) 2513.

[39] J. Yu, S.H. Dong and G.H. Sun, Phys. Lett. A322 (2004) 290; C. Gang, Phys. Lett. A 329 (2004) 22; C. Berkdemir, Nucl. Phys. A 770 (2006) 32.

[40] Y. Xu, S. He and C.S. Jia, J. Phys. A: Math. Theor. 41 (2008) 255302.

[41] G. Chen and Z.D. Chen, Phys. Lett. A 331 (2004) 312.

[42] R. Sever and C. Tezcan, Int. J. Mod. Phys. E 17(7) (2008) 1327.

[43] S.M. Ikhdair and R. Sever, arXiv:0810.1159, to appear in the Int. J. Mod. Phys. C 20(3) (2009).

[44] A.F. Nikiforov and V.B. Uvarov, Special Functions of Mathematical Physics (Birkhauser, Bassel, 1988).

[45] S.M. Ikhdair, Chin. J. Phys. 46 (2008) 291; S.M. Ikhdair and R. Sever, Ann. Phys. (Berlin)

17(11) (2008) 897; S.M. Ikhdair and R. Sever, Cent. Eur. J. Phys. 6 (2008) 685; S.M. Ikhdair and R. Sever, Int. J. Mod. Phys. C 19 (2008) 221; S.M. Ikhdair and R. Sever, J. Math. Chem. (DOI:10.1007/s10910-008-9438-8).

[46] S.M. Ikhdair, arXiv:0810.1590, to appear in the Int. J. Mod. Phys. C 20(1) (2009); S.M. Ikhdair and R. Sever, Cent. Eur. J. Phys. 6 (2008) 141.

[47] S.M. Ikhdair and R. Sever, arXiv:0808.1002, submitted to Int. J. Mod. Phys. C.

[48] S.M. Ikhdair and R. Sever, Int. J. Mod. Phys. E 17 (2008) 1107.

[49] S. Haouat and L. Chetouani, Phys. Scr. 77 (2008) 025005.

[50] C.S. Jia, J.Y. Liu and P.Q. Wang, Phys. Lett. A 372 (2008) 4779.

[51] S.M. Ikhdair and R. Sever, arXiv:0807.2085; S.M. Ikhdair and R. Sever, arXiv:0809.2485, submitted to Ann. Phys. (Berlin).

[52] S.M. Ikhdair, "An improved approximation scheme for the centrifugal term and the Hulthén potential", submitted to Eur. Phys. J. A.

[53] S. Dong, J. Garcia-Ravelo and S.H. Dong, Phys. Scr. 76 (2007) 393.

[54] W. Lucha and F.F. Schöberl, Int. J. Mod. Phys. C 10 (1999) 607.

[55] S.M. Ikhdair and R. Sever, Z. Phys. C 56 (1992) 155; Z. Phys. C 58 (1993) 153; Z. Phys. D 28 (1993) 1; Hadronic J. 15 (1992) 389; Int. J. Mod. Phys. A 18 (2003) 4215; Int. J. Mod. Phys. A 19 (2004) 1771; Int. J. Mod. Phys. A 20 (2005) 4035; Int. J. Mod. Phys. A 20 (2005) 6509; Int. J. Mod. Phys. A 21 (2006) 2191; Int. J. Mod. Phys. A 21 (2006) 3989; Int. J. Mod. Phys. A 21 (2006) 6699; Int. J. Mod. Phys. E 17 (2008) 669.

[56] M. Simsek and H. Egrifes, J. Phys. A: Math. Gen. 37 (2004) 4379.

[57] C.Y. Chen, D.S. Sun and F.L. Lu, Phys. Lett. A 370 (2007) 219.

[58] S.M. Ikhdair and R. Sever, Int. J. Mod. Phys. C 19(9) (2008) 1425.

[59] M. Aktaş and R. Sever, J. Mol. Struct. 710 (2004) 219.

[60] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1964).

[61] E. Olğar, R. Koç and H. Tütüncüler, Phys. Scr. 78 (2008) 015011.

[62] G. Chen, Phys. Scr. 69 (2004) 257; G. Chen, Z.D. Chen and Z.M. Lou, Phys. Lett. A 331 (2004) 374.

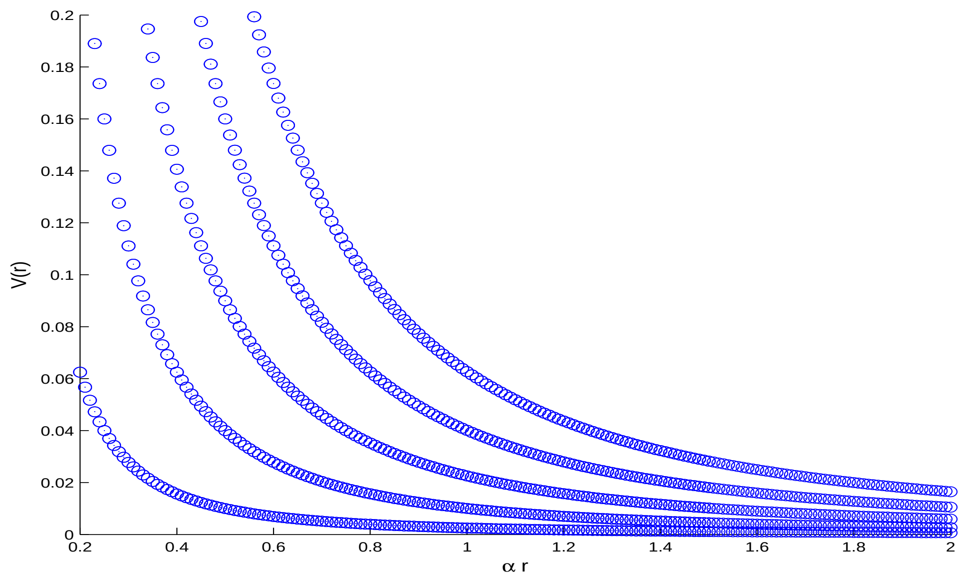

FIG. 1: A plot of the variation of the centrifugal potential, $1/r^2$ and its corresponding propose approximation form $\alpha^2\left[c_0+\frac{e^{\alpha r}}{(e^{\alpha r}-1)^2}\right]$ versus αr, where the screening parameter α changes from $\alpha=0.050$ to $\alpha=0.250$ in steps of 0.050.

TABLE I: The energy spectrum of the scalar and vector Hulthén potential for $m_0 = 1$ and $m_1 = 0$.

$V_0 = S_0$	n	l	$E_{nl}^{+\,a}$	$E_{nl}^{-\,a}$	E_{nl}^{+} [61,62]^b	E_{nl}^{-} [61,62]^b
1	1	0	1.000000	−0.600000	1.000000	−0.600000
	1	1	–	–	–	–
2	1	0	0.707107	−0.707107	0.707107	−0.707107
	1	1	0.984171	−0.214941	–	–
	1	2	–	–	–	–
	2	0	0.984171	−0.214941	0.984171	−0.214941
	2	1	–	–	–	–
3	1	0	0.302169	−0.763708	0.302169	−0.763708
	1	1	0.911438	−0.411438	–	–
	1	2	0.600000	0.600000	–	–
	1	3	–	–	–	–
	2	0	0.911438	−0.411438	0.911438	−0.411438
	2	1	0.600000	0.600000	–	–
	2	2	–	–	–	–
	3	0	0.600000	0.600000	0.600000	0.600000
	3	1	–	–	–	–
6	1	0	−0.355051	−0.844949	−0.355051	−0.844949
	1	1	0.235890	−0.635890	–	–
	1	2	0.763708	−0.302169	–	–
	1	3	0.994273	0.284416	–	–
	2	0	0.235890	−0.635890	0.235890	−0.635890
	2	1	0.763708	−0.302169	–	–
	2	2	0.994273	−0.284416	–	–
	2	3	–	–	–	–
	3	0	0.763708	−0.302169	0.763708	−0.302169
	3	1	0.994273	0.284416	–	–
	3	2	–	–	–	–
	4	0	0.994273	0.284416	0.994273	0.284416

[a]The present NU method.

[b]The results from AIM and SUSY.

TABLE II: The energy spectrum of the scalar and vector Hulthén potential for $m_0 = 1$ and $m_1 = 0$.

$V_0 = S_0$	n	l	E^+_{nl}	E^-_{nl}	$V_0 = S_0$	n	l	E^+_{nl}	E^-_{nl}
8	1	0	−0.539504	−0.872260	20	2	0	−0.662662	−0.853230
	1	1	−0.063251	−0.703872		2	1	−0.418342	−0.735504
	1	2	0.447214	−0.447214		2	2	−0.127025	−0.578857
	1	3	0.870312	−0.061324		2	3	0.194284	−0.377770
	1	4	0.800000	0.8000000		2	4	0.523260	−0.122370
	1	5	−	−		2	5	0.825665	0.208818
	2	0	−0.063251	−0.703872		2	6	0.998229	0.706553
	2	1	0.447214	−0.447214		3	0	−0.418342	−0.735504
	2	2	0.870312	−0.061324		3	1	−0.127025	−0.578857
	2	3	0.800000	0.800000		3	2	0.194284	−0.377770
	2	4	−	−		3	3	0.523260	−0.122370
	3	0	0.447214	−0.447214		3	4	0.825665	0.208818
	3	1	0.870312	−0.061324		3	5	0.998229	0.706553
	3	2	0.800000	0.800000		4	0	−0.127025	−0.578857
	3	3	−	−		4	1	0.194284	−0.377770
	4	0	0.870312	−0.061324		4	2	0.523260	−0.122370
	4	1	0.800000	0.800000		4	3	0.825665	0.208818
	4	2	−	−		4	4	0.998229	0.706553
	5	0	0.800000	0.800000		5	0	0.194284	−0.377770
	5	1	−	−		5	1	0.523260	−0.122370
	6	0	−	−		5	2	0.825665	0.208818
20	1	0	−0.846811	−0.935368		5	3	0.998229	0.706553
	1	1	−0.662662	−0.853230		6	0	0.523260	−0.122370
	1	2	−0.418342	−0.735504		6	1	0.825665	0.208818
	1	3	−0.127025	−0.578857		6	2	0.998229	0.706553
	1	4	0.194284	−0.377770		7	0	0.825665	0.208818
	1	5	0.523260	−0.122370		7	1	0.998229	0.706553
	1	6	0.825665	0.208818		8	0	0.998229	0.706553
	1	7	0.998229	0.706553		9	0	−	−

TABLE III: The energy spectrum of the scalar and vector Hulthén potential for $m_1 \neq 0$.

m_0	m_1	V_0	S_0	n	l	E^+	E^-	m_0	m_1	V_0	S_0	n	l	E^+	E^-
5	0.01	2	2	1	0	0.822925	−4.913410	5	1	−10	20	1	0	4.857570	−1.483450
				1	1	3.110670	−4.804170					1	1	4.875450	−1.571890
				2	0	3.065630	−4.807820					2	0	4.999480	−2.709050
				2	1	4.252020	−4.650830					2	1	4.999990	−2.772530
				2	2	4.795730	−4.445800					2	2	4.998750	−2.895220
				3	0	4.229630	−4.655840					3	0	4.924130	−3.601650
				3	1	4.793910	−4.447040					3	1	4.914310	−3.648140
				3	2	4.989330	−4.185200					3	2	4.893220	−3.737900
				3	3	4.956220	−3.857960					3	3	4.858140	−3.864780
5	0.01	−2	2	1	0	4.913410	−0.822930	5	0.1	1	1	1	0	3.443410	-4.868720
				1	1	4.804170	−3.110670					1	1	4.722690	−4.742880
				2	0	4.807820	−3.065630					2	0	4.618770	−4.768190
				2	1	4.650830	−4.252020					2	1	4.982510	−4.577550
				2	2	4.445800	−4.795730					2	2	4.964780	−4.347700
				3	0	4.655840	−4.229630					3	0	4.960360	−4.613290
				3	1	4.447040	−4.793910					3	1	4.967570	−4.354450
				3	2	4.185200	−4.989330					3	2	4.788530	−4.056980
				3	3	3.857960	−4.956220					3	3	4.484330	−3.682040
5	0.1	−2	5	1	0	4.871650	−3.222360					4	0	4.984480	−4.401670
				1	1	4.926240	−3.503700					4	1	4.794830	−4.065620
				2	0	5.000000	−4.245710					4	2	4.488330	−3.686650
				2	1	4.995470	−4.392630					4	3	4.054980	−3.206920
				2	2	4.965180	−4.615030					4	4	3.455290	−2.575480
				3	0	4.915250	−4.768460					5	0	4.837690	−4.126180
				3	1	4.878060	−4.836860					5	1	4.497830	−3.697630
				3	2	4.793250	−4.930300					5	2	4.060510	−3.212870
				3	3	4.647670	−4.993400					5	3	3.459590	−2.579950
												5	4	2.567010	−1.664550
												5	5	−	−