The relevance of random choice in tests of Bell inequalities with two atomic qubits

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Abstract

It is pointed out that a loophole exists in experimental tests of Bell inequality using atomic qubits, due to possible errors in the rotation angles of the atomic states. A sufficient condition is derived for closing the loophole.

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A recent experiment, by a group of Maryland, has measured the correlation between the quantum states of two Yb^{+} ions separated by a distance of about 1 meter[1]. The authors claim that the experiment is relevant because it violates a CHSH[2] (Bell) inequality, modulo the locality loophole, closing the detection loophole. In my opinion that assertion does not make full justice to the relevance of the experiment. The truth is that it is the first experiment which has tested a genuine Bell inequality. Actually the results of previous experiments, in particular those involving optical photon pairs[3], did not test any *genuine* Bell inequality, that is an inequality which is a necessary condition for the existence of local hidden variables (LHV) models. The inequalities tested in those experiments should not be qualified as Bell´s because their derivation involves additional assumptions. Consequently their violation refutes only restricted families of LHV models, namely those fulfilling the additional assumption. (For details see[4].)

The aim of the present letter is to point out the existence of a loophole in the Maryland experiment[1], or more generally in Bell tests with atomic qubits, in addition to the locality loophole. Blocking that loophole will be straightforward using random choice of the measurements, as is explained below.

In general I will consider experiments where a pair of atoms (or ions) is prepared in an entangled state. Then Alice performs a rotation of the state of her atom by an angle θ_a and, after a short time, she may detect fluorescence of the atom illuminated by an appropriate laser. Similarly Bob performs a rotation of his atom by an angle θ_b and, after that, he may detect fluorescence too. I shall label $p_{++}(\theta_a, \theta_b)$ the probability of coincidence detection and $p_{--}(\theta_a, \theta_b)$ the probability that neither Alice nor Bob detect fluorescence. Similarly $p_{-+}(\theta_a, \theta_b)$ ($p_{+-}(\theta_a, \theta_b)$) will be the probability that only Bob (Alice) detects fluorescence. In the Maryland experiment[1] (see their eq.(6)), a function $E(\theta_a, \theta_b)$ is defined by

$$E(\theta_a, \theta_b) = p_{++}(\theta_a, \theta_b) + p_{--}(\theta_a, \theta_b) - p_{+-}(\theta_a, \theta_b) - p_{-+}(\theta_a, \theta_b). \tag{1}$$

Then the authors define a parameter S by

$$S = |E(\theta_a, \theta_b) + E(\theta_a', \theta_b)| + |E(\theta_a, \theta_b') - E(\theta_a', \theta_b')|, \tag{2}$$

and claim that the CHSH[2] inequality $S \leq 2$ is violated. The notation used by the authors is, however, somewhat misleading. Instead of eq.(1) they write

$$E(\theta_a, \theta_b) = p(\theta_a, \theta_b) + p(\theta_a + \pi, \theta_b + \pi) - p(\theta_a, \theta_b + \pi) - p(\theta_a + \pi, \theta_b), \tag{3}$$

where they label $p(\theta_a, \theta_b)$ the quantity which I have labeled $p_{++}(\theta_a, \theta_b)$. Definition eq.(3), in place of eq.(1), rests upon assuming the equalities

$$\begin{aligned} p_{-+}(\theta_a, \theta_b) &= p(\theta_a + \pi, \theta_b), p_{+-}(\theta_a, \theta_b) = p(\theta_a + \pi, \theta_b), \\ p_{--}(\theta_a, \theta_b) &= p(\theta_a + \pi, \theta_b + \pi), \end{aligned}$$

which are true according to quantum mechanics, but may not be true in LHV theories. In any case the authors measured $E(\theta_a, \theta_b)$ as defined in eq.(1)[5].

In order to show that there is a loophole in the experiment, in addition to the locality loophole, I begin remembering that, according to Bell[6], a LHV model will contain a set of hidden variables, λ, a positive normalized density function, $\rho(\lambda)$, and two functions $M_a(\lambda, \theta_a)$, $M_b(\lambda, \theta_b)$, θ_a and θ_b being parameters which may be controlled by Alice and Bob respectively. The latter functions fulfil

$$M_a(\lambda, \theta_a), M_b(\lambda, \theta_b) \in \{0, 1\}. \tag{4}$$

In the Maryland experiment[1] the parameters θ_a and θ_b are angles defining the quantum states of the two ions. The probability, $p_{++}(\theta_a, \theta_b)$, that the coincidence measurement of two dichotomic variables, in two distant regions, gives a positive answer for both variables should be obtained in the LHV model by means of the integral

$$p_{++}(\theta_a, \theta_b) = \int \rho(\lambda) M_a(\lambda, \theta_a) M_b(\lambda, \theta_b) d\lambda. \tag{5}$$

Similarly the probability, $p_{+-}(\theta_a, \theta_b)$, that Alice gets the answer "yes" and Bob the answer "no" is given by

$$p_{+-}(\theta_a, \theta_b) = \int \rho(\lambda) M_a(\lambda, \theta_a) [1 - M_b(\lambda, \theta_b)] d\lambda, \tag{6}$$

and analogous expressions for p_{-+} and p_{--}.

A LHV model for an atomic experiment may be obtained by choosing

$$\begin{aligned} \rho(\lambda) &= \frac{1}{2\pi}, \lambda \in [0, 2\pi], \ M_a(\lambda, \theta_a) = \Theta\left(\frac{\pi}{2} - |\lambda - \theta_a|\right), \\ M_b(\lambda, \theta_b) &= \Theta\left(\frac{\pi}{2} - |\lambda - \theta_b - \pi|\right), \mathrm{mod}(2\pi), \end{aligned} \tag{7}$$

where $\Theta(x) = 1$ if $x > 0$, $\Theta(x) = 0$ if $x < 0$. It is easy to see, taking eqs.(5) and (6) into account, that model predictions are (assuming $\theta_a, \theta_b \in [0, \pi]$)

$$\begin{aligned} p_{++}(\theta_a, \theta_b) &= p_{--}(\theta_a, \theta_b) = \frac{|\theta_a - \theta_b|}{2\pi}, \\ p_{+-}(\theta_a, \theta_b) &= p_{-+}(\theta_a, \theta_b) = \frac{1}{2} - \frac{|\theta_a - \theta_b|}{2\pi}. \end{aligned} \tag{8}$$

Hence I get

$$E(\theta_a, \theta_b) = \frac{2}{\pi} |\theta_a - \theta_b| - 1, \tag{9}$$

and it is not difficult to show that, for any choice of the angles $\theta_a, \theta_b, \theta_a', \theta_b'$, the model predicts $S \leq 2$ with S given by eq.(2).

Now let us assume that the experiment is performed so that Alice and Bob start measuring the quantity $E(\theta_a, \theta_b)$ in a sequence of runs of the experiment. After that they measure $E(\theta_a, \theta_b')$ in another sequence, then they measure $E(\theta_a', \theta_b)$ and, finally, they measure $E(\theta_a', \theta_b')$. Let α be the error in the rotation performed by Bob on his atom in the first sequence of runs, so that the rotation angle is $\theta_b+\alpha$ rather than θ_b in the measurement of $E(\theta_a,\theta_b)$. Similarly I shall assume that the rotation angles are $\theta_b'+\beta,\theta_b+\gamma$ and $\theta_b'+\delta$ in the measurements of $E(\theta_a,\theta_b')$, $E(\theta_a',\theta_b)$ and $E(\theta_a',\theta_b')$, respectively. For simplicity I will assume that no error appears in Alice rotations. The errors are considered small, specifically $|\alpha|,|\beta|,|\gamma|,|\delta|<\pi/4$. I shall prove that, taking into account the errors in the measurement of the angles, the LHV model prediction for the parameter S, eq.(2) may apparently violate the CHSH[2] inequality $S\leq 2$. To do that let us choose, as in the Maryland experiment[1],

$$\theta_a=\frac{\pi}{2},\theta_b=\frac{\pi}{4},\theta_a'=0,\theta_b'=\frac{3\pi}{4}. \tag{10}$$

The values predicted by the LHV model for the relevant quantities are

$$\begin{aligned} E(\theta_a,\theta_b+\alpha) &= -0.5-\frac{2\alpha}{\pi},\ E(\theta_a,\theta_b'+\beta)=-0.5+\frac{2\beta}{\pi}, \\ E(\theta_a',\theta_b+\gamma) &= -0.5+\frac{2\gamma}{\pi},\ E(\theta_a',\theta_b'+\delta)=0.5+\frac{2\delta}{\pi}. \end{aligned} \tag{11}$$

Then the parameter actually measured in the experiment is

$$S'=|E(\theta_a,\theta_b+\alpha)+E(\theta_a',\theta_b+\gamma)|+|E(\theta_a,\theta_b'+\beta)-E(\theta_a',\theta_b'+\delta)|, \tag{12}$$

and the LHV prediction for that parameter is

$$S'==2+\frac{2}{\pi}(\alpha-\beta-\gamma+\delta),$$

which may violate the inequality $S'\leq 2$ for some values of the parameters α,β,γ and δ. In particular the results of the Maryland experiment[1] are reproduced by choosing

$$2\alpha/\pi=0.018, 2\beta/\pi=-0.046, 2\gamma/\pi=-0.081, 2\delta/\pi=-0.073.$$

The errors in the angles are of order 7^o or less. It is plausible that errors as high as these may appear in experiments with atomic qubits but not in optical photon experiments. I stress that no violation of a Bell inequality by a LHV model is produced. Actually the parameter S' of eq.(12) is not a CHSH parameter as defined in eq.(2).

In the following I shall prove that the loophole may be closed by random choice of the angles to be measured. To begin with, it is easy to see that the

LHV model predictions do not violate the inequality $S' \leq 2$ if the error in the measurement, by Bob, of the angle θ_b is the same in all measurements of that angle, and similarly for θ_b'. In fact the inequality is fulfilled if $\alpha = \beta$ and $\gamma = \delta$, as may be seen by looking at eq.(12). In the following I derive a sufficient condition for the fulfillement of the inequality, $S' \leq 2$, for the actually measurable quantity S', by the predictions of any LHV model.

Let us assume that there is a (normalized) probability distribution, $f_a(x)$, for the errors when Alice rotates her atom by an angle θ_a and another distribution, $f_a'(y)$, when she rotates her atom by an angle θ_a'. Similarly I shall assume that there are similar disitribuions $f_b(u)$ and $f_b'(v)$ for the errors in the rotations, by Bob, of the angles θ_b and θ_b'. I shall show that a sufficient condition for the inequality $S' \leq 2$ is that the distributions of errors, in the rotations made by Alice, are the same independently of what rotation performs Bob on the partner atom. And similarly for the rotations made by Bob. If this is the case the predictions of any LHV model for the quantity S' will be obtained from probabilities defined as follows (compare with eqs.(5) and (6))

$$
\begin{aligned}
p_{++}\left(\theta_a, \theta_b\right) &= \int \rho\left(\lambda\right) M_a\left(\lambda, \theta_a + x\right) M_b\left(\lambda, \theta_b + u\right) d\lambda f_a(x) dx f_b(u) du, \quad (13)\\
p_{+-}\left(\theta_a, \theta_b\right) &= \int \rho\left(\lambda\right) M_a\left(\lambda, \theta_a + x\right) \left[1 - M_b\left(\lambda, \theta_b + u\right)\right] d\lambda f_a(x) dx f_b(u) du,
\end{aligned}
$$

and similarly for the other quantities p_{ij} with $i, j = +, -$. Now we may define new quantities

$$
\begin{aligned}
Q_a\left(\lambda, a\right) &= \int M_a\left(\lambda, \theta_a + x\right) f_a(x) dx, \qquad (14)\\
Q_b\left(\lambda, b\right) &= \int M_b\left(\lambda, \theta_b + u\right) f_b(u) du,\\
Q_a\left(\lambda, a'\right) &= \int M_a\left(\lambda, \theta_a' + y\right) f_a'(y) dy,\\
Q_b\left(\lambda, b'\right) &= \int M_b\left(\lambda, \theta_b' + v\right) f_b'(v) dv,
\end{aligned}
$$

which fulfil the conditions (compare with eqs.(4))

$$
0 \leq Q_a\left(\lambda, a\right), Q_a\left(\lambda, a'\right), Q_b\left(\lambda, b\right), Q_b\left(\lambda, b'\right) \leq 1. \qquad (15)
$$

The consequence is that we may obtain a new LHV model for the experiment with the quantities Q, eqs.(15) , in place of the quantities M, eqs.(4) . The existence of the model implies the fulfillement of the inequality $S' \leq 2$.

From our proof it is rather obvious that the essential condition required to block the loophole is that the probability distribution of errors made by Bob are independent of what rotation is performed by Alice in the partner atom, and similarly the errors made by Alice should be independent of the rotation performed by Bob. A simple method to insure that independence is that Alice makes at random the choice whether to rotate her atom by the angle θ_a or by the angle θ'_a, and similarly Bob. That is, after every preparation of the entangled state of the atom, Alice should make a random choice (with equal probabilities) between the rotation angles θ_a and θ'_a and similarly Bob should make a random choice, *independently of Alice*, between θ_b and θ'_b.

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