

Natural connections for semi-Hamiltonian systems: The case of the ϵ -system

Paolo Lorenzoni*, Marco Pedroni**

* Dipartimento di Matematica e Applicazioni
Università di Milano-Bicocca
Via Roberto Cozzi 53, I-20125 Milano, Italy
paolo.lorenzoni@unimib.it

** Dipartimento di Ingegneria dell'Informazione e Metodi Matematici
Università di Bergamo
Viale Marconi 5, I-24044 Dalmine BG, Italy
marco.pedroni@unibg.it

Abstract

Given a semi-Hamiltonian system, we construct an F -manifold with a connection satisfying a suitable compatibility condition with the product. We exemplify this procedure in the case of the so-called ϵ -system. The corresponding connection turns out to be flat, and the flat coordinates give rise to additional chains of commuting flows.

1 Introduction

This paper is a natural continuation of [12] and deals with F -manifolds and their associated integrable hierarchies. The aim of [12] essentially was to extend the concept of F -manifold with compatible connection to the non-flat case and to show its relevance in the theory of integrable systems of hydrodynamic type. The integrability condition (i.e., the condition ensuring the existence of integrable flows) in such a framework becomes the following simple requirement on the connection ∇ and on the structure constants c_{jk}^i entering the definition of these F -manifolds:

$$R_{lmi}^k c_{pk}^n + R_{lip}^k c_{mk}^n + R_{lpm}^k c_{ik}^n = 0, \quad (1.1)$$

where R_{jkl}^i is the Riemann tensor of the connection ∇ . Thus, the starting point in [12] was an F -manifold, and the goal was the associated integrable hierarchy. The approach of this paper

is different, since here we move in the opposite direction. The starting point is an integrable hierarchy of hydrodynamic type, and the goal is the construction of an F -manifold with compatible connection. More precisely, we consider the case of *semi-Hamiltonian systems*, that is, diagonal systems of hydrodynamic type [20],

$$u_t^i = v^i(u) u_x^i, \quad u = (u^1, \dots, u^n), \quad i = 1, \dots, n, \quad (1.2)$$

whose coefficients $v^i(u)$ (usually called *characteristic velocities*) satisfy the system of equations

$$\partial_j \left(\frac{\partial_k v^i}{v^i - v^k} \right) = \partial_k \left(\frac{\partial_j v^i}{v^i - v^j} \right) \quad \forall i \neq j \neq k \neq i, \quad (1.3)$$

where $\partial_i = \frac{\partial}{\partial u^i}$. Equations (1.3) are the integrability conditions for three different systems: the first one, given by

$$\frac{\partial_j w^i}{w^i - w^j} = \frac{\partial_j v^i}{v^i - v^j}, \quad (1.4)$$

provides the characteristic velocities of the symmetries

$$u_\tau^i = w^i(u) u_x^i \quad i = 1, \dots, n,$$

of (1.2); the second one is the system

$$\partial_i \partial_j H - \Gamma_{ij}^i \partial_i H - \Gamma_{ji}^j \partial_j H = 0, \quad \Gamma_{ij}^i = \frac{\partial_j v^i}{v^j - v^i}, \quad (1.5)$$

whose solutions H are the conserved densities of (1.2); the third one is

$$\partial_j \ln \sqrt{g_{ii}} = \frac{\partial_j v^i}{v^j - v^i}, \quad (1.6)$$

and relates the characteristic velocities of the system with a class of diagonal metrics. These metrics and the associated Levi-Civita connection play a crucial role [3] in the Hamiltonian formalism of (1.2).

The first result of this paper is the existence of a connection ∇ canonically associated with a given semi-Hamiltonian system, to be called the *natural connection* of the system. By definition, it is compatible—in the sense of (1.1)—with the product $c_{jk}^i = \delta_j^i \delta_k^i$, and leads to a structure of F -manifold with compatible connection. We point out that the natural connection *does not coincide*, in general, with the Levi-Civita connections of the diagonal metrics satisfying the system (1.6)—whose Christoffel symbols are the Γ_{ij}^i appearing in (1.5).

This alternative approach to the theory of semi-Hamiltonian systems might be useful not only from a geometric viewpoint, but also for applications. Indeed, it happens that the connection ∇ mentioned above turns out to be flat even in cases where the solutions of (1.6) neither are flat nor have the Egorov property¹. In such a situation the metrics that are

¹We recall that a metric is said to have the Egorov property (or to be potential) if there exist coordinates such that the metric is diagonal and $g_{ii} = \partial_i \phi$. In other words, the *rotation coefficients*

$$\beta_{ij} = \frac{\partial_i \sqrt{g_{jj}}}{\sqrt{g_{ii}}} \quad (1.7)$$

must be symmetric

compatible with the connection ∇ are not invariant with respect to the product. Nevertheless the flatness of the connection and the condition

$$\nabla_l c_{jk}^i = \nabla_j c_{lk}^i,$$

entering the definition of F -manifold with compatible connection, are sufficient to define the so-called principal hierarchy, as it was shown in [12] extending a standard construction of Dubrovin.

As an example, we study the semi-Hamiltonian system

$$u_t^i = \left(u^i - \epsilon \sum_{k=1}^n u^k \right) u_x^i, \quad i = 1, \dots, n, \quad (1.8)$$

known in the literature as ϵ -system. It has been studied by several authors. In particular, we refer to [5, 15, 19] for the case $\epsilon = -1$, to [6] for the case $\epsilon = -\frac{1}{2}$, to [4] for the case $\epsilon = 1$, and to [18, 11, 10] for the general case. For $n > 2$ the metrics

$$g_{ii} = \frac{\varphi_i(u^i)}{\left[\prod_{l \neq i} (u^i - u^l)^2 \right]^\epsilon}, \quad i = 1, \dots, n, \quad (1.9)$$

(where $\varphi_i(u^i)$ are arbitrary non-vanishing functions of a single variable) satisfying the system (1.6) are not of Egorov type. Indeed, their rotation coefficients

$$\beta_{ij} = \left[\frac{\prod_{l \neq i} (u^i - u^l)^2}{\prod_{l \neq j} (u^j - u^l)^2} \right]^{\frac{\epsilon}{2}} \frac{\epsilon}{u^j - u^i} \quad (1.10)$$

are not symmetric². Therefore the natural connection of the ϵ -system does not coincide with the Levi-Civita connection of any of the metrics (1.9).

In the second half of the paper we study in details such natural connection, that turns out to be flat. Thus, besides the usual higher flows of (1.8), we obtain $(n - 1)$ additional chains of commuting flows associated with the non-trivial flat coordinates of the natural connection. Remarkably, the recursive identities relating these flows have a double geometrical interpretation: the first one, in terms of the usual recursive procedure involved in the construction of the principal hierarchy; the second one, in terms of certain cohomological equations appearing in [11, 10]. Finally, we explicitly construct the associated principal hierarchy in the cases $n = 2$ and $n = 3$.

²In [10] it was observed that they satisfy the Egorov-Darboux system

$$\begin{aligned} \partial_k \beta_{ij} &= \beta_{ik} \beta_{kj} & i \neq j \neq k \\ \sum_k \partial_k \beta_{ij} &= 0 & i \neq j \\ \sum_k u^k \partial_k \beta_{ij} &= -\beta_{ij} & i \neq j \end{aligned}$$

if the φ_i are constant.

Acknowledgments

We thank Andrea Raimondo for useful discussions; many ideas of the present paper have their origin in [12], written in collaboration with him. M.P. would like to thank the Department *Matematica e Applicazioni* of the Milano-Bicocca University for the hospitality. This work has been partially supported by the European Community through (ESF) Scientific Programme *MISGAM*. Some of the computations have been performed with Maple Software.

2 F -manifolds with compatible connection and related integrable systems

In this section we review the most important results of [12]. First of all let us recall the main definition of that paper.

Definition 2.1 *An F -manifold with compatible connection is a manifold endowed with an associative commutative multiplicative structure given by a $(1, 2)$ -tensor field c and a torsionless connection ∇ satisfying condition*

$$\nabla_l c_{jk}^i = \nabla_j c_{lk}^i \quad (2.1)$$

and condition

$$R_{lmi}^k c_{pk}^n + R_{lip}^k c_{mk}^n + R_{lpm}^k c_{ik}^n = 0, \quad (2.2)$$

where $R_{ijl}^k = \partial_j \Gamma_{li}^k - \partial_l \Gamma_{ji}^k + \Gamma_{jm}^k \Gamma_{li}^m - \Gamma_{lm}^k \Gamma_{ji}^m$ is the Riemann tensor of ∇ .

Definition 2.2 *An F -manifold with compatible connection is called semisimple if around any point there exist coordinates (u^1, \dots, u^n) —called canonical coordinates—such that the structures constants become*

$$c_{jk}^i = \delta_j^i \delta_k^i.$$

On the loop space of a semisimple F -manifold with compatible connection, one can define a semi-Hamiltonian hierarchy. The flows of this hierarchy are

$$u_t^i = c_{jk}^i X^k u_x^j, \quad (2.3)$$

where the vector fields X satisfy the system

$$c_{jm}^i \nabla_k X^m = c_{km}^i \nabla_j X^m. \quad (2.4)$$

Using the generalized hodograph method [20], one obtains the general solution of (2.3) in implicit form as

$$tX^i + xe^i = Y^i, \quad (2.5)$$

where

$$e = \sum_{i=1}^n \frac{\partial}{\partial u^i}$$

is the unity of the algebra and Y is an arbitrary solution of (2.4). Notice that the above representation of the solution in terms of critical points of a family of vector fields—as well as the expressions (2.3,2.4)—holds true in any coordinate system.

If the connection ∇ is flat, the definition of F -manifold with compatible connection reduces to the following definition, due to Manin [14].

Definition 2.3 *An F -manifold with compatible flat connection is a manifold endowed with an associative commutative multiplicative structure given by a $(1, 2)$ -tensor field c and a torsionless flat connection ∇ satisfying condition (2.1).*

In flat coordinates, condition (2.1) reads

$$\partial_l c_{jk}^i = \partial_j c_{lk}^i.$$

This, together with the commutativity of the algebra, implies that

$$c_{jk}^i = \partial_j C_k^i = \partial_j \partial_k C^i.$$

Therefore, condition (2.1) is equivalent to the local existence of a vector field C satisfying, for any pair (X, Y) of flat vector fields, the condition [14]:

$$X \circ Y = [X, [Y, C]],$$

where $(X \circ Y)^k = c_{ij}^k X^i Y^j$.

The hierarchy associated with an F -manifold with compatible *flat* connection is usually called *principal hierarchy*. It can be defined in the following way, which is a straightforward generalization of the original definition given by Dubrovin in the case of Frobenius manifolds. First of all, one defines the so-called *primary flows*:

$$u_{t_{(p,0)}}^i = c_{jk}^i X_{(p,0)}^k u_x^j, \quad (2.6)$$

where $(X_{(1,0)}, \dots, X_{(n,0)})$ is a basis of flat vector fields. Then, starting from these flows, one can define the “higher flows” of the hierarchy,

$$u_{t_{(p,\alpha)}}^i = c_{jk}^i X_{(p,\alpha)}^k u_x^j, \quad (2.7)$$

by means of the following recursive relations:

$$\nabla_j X_{(p,\alpha)}^i = c_{jk}^i X_{(p,\alpha-1)}^k. \quad (2.8)$$

Remark 2.4 *The vector fields obtained by means of the recursive relations (2.8) are nothing but the z -coefficients of a basis of flat vector fields of the deformed connection [2] defined, for any pair of vector fields X and Y , by*

$$\tilde{\nabla}_X Y = \nabla_X Y + zX \circ Y, \quad z \in \mathbb{C}.$$

In the following section we will show how to construct a semisimple F -manifold with compatible connection starting from a semi-Hamiltonian system.

3 From semi-Hamiltonian system to F -manifolds with compatible connection: The natural connection

Let

$$u_t^i = v^i(u) u_x^i, \quad u = (u^1, \dots, u^n), \quad i = 1, \dots, n, \quad (3.1)$$

be a semi-Hamiltonian system, that is, suppose that the characteristic velocities $v^i(u)$ satisfy (1.3). We want to define a semisimple F -manifold with compatible connection whose associated semi-Hamiltonian system contains (3.1). First of all, we define the structure constants c_{jk}^i simply assigning to the Riemann invariants u^i the role of canonical coordinates. This means that in the coordinates (u^1, \dots, u^n) we have

$$c_{jk}^i = \delta_j^i \delta_k^i. \quad (3.2)$$

Once given the structure constants, the definition of the connection ∇ is quite rigid, apart from the freedom in the choice of the Christoffel symbols Γ_{ii}^i . Indeed, such a connection must be torsion-free,

$$\Gamma_{jk}^i = \Gamma_{kj}^i,$$

and must satisfy condition (2.1), that in canonical coordinates reduces to

$$\Gamma_{jk}^i = 0 \quad \text{for } i \neq j \neq k \neq i, \quad (3.3)$$

$$\Gamma_{jj}^i = -\Gamma_{ji}^i \quad \text{for } i \neq j. \quad (3.4)$$

Moreover, the space of solutions of (2.4) must contain the characteristic velocities of the semi-Hamiltonian system we started with. Putting $X^i = v^i$ in (2.4), we obtain

$$\Gamma_{ji}^i = \frac{\partial_j v^i}{v^j - v^i} \quad \text{for } i \neq j, \quad (3.5)$$

where the v^i satisfy (1.3). The compatibility condition (2.2) between c and ∇ is now automatically satisfied, thanks to the following lemmas.

Lemma 3.1 *The only non-vanishing components of the Riemann tensor of a connection satisfying conditions (3.3,3.4,3.5) are*

$$R_{iki}^i = -R_{iik}^i = \partial_k \Gamma_{ii}^i - \partial_i \Gamma_{ik}^i \quad (3.6)$$

and

$$R_{qqi}^i = -R_{qiq}^i = \partial_q \Gamma_{qi}^i - \partial_i \Gamma_{qq}^i + (\Gamma_{iq}^i)^2 + \Gamma_{qq}^i \Gamma_{qi}^q - \sum_{p=1}^n \Gamma_{pi}^i \Gamma_{qq}^p. \quad (3.7)$$

Proof. First of all we have

$$R_{qkl}^i = 0 \quad \text{for distinct indices} \quad (3.8)$$

$$R_{ikl}^i = -R_{ilk}^i = \partial_k \Gamma_{il}^i - \partial_l \Gamma_{ik}^i = 0 \quad \text{if } i \neq k \neq l \neq i \quad (3.9)$$

$$R_{qki}^i = -R_{qik}^i = \partial_k \Gamma_{iq}^i + \Gamma_{ik}^i \Gamma_{iq}^i - \Gamma_{iq}^i \Gamma_{qk}^q - \Gamma_{ik}^i \Gamma_{kq}^k = 0 \quad \text{if } i \neq q \neq k \neq i. \quad (3.10)$$

The first identity is a consequence of the vanishing of the Christoffel symbols Γ_{jk}^i when the three indices are distinct, the second one is a consequence of the semi-Hamiltonian property (1.3), and the third one is a consequence of the identity (see [20])

$$\partial_k \Gamma_{iq}^i + \Gamma_{ik}^i \Gamma_{iq}^i - \Gamma_{iq}^i \Gamma_{qk}^q - \Gamma_{ik}^i \Gamma_{kq}^k = \frac{v^k - v^i}{v^q - v^k} \left[\partial_q \left(\frac{\partial_k v^i}{v^k - v^i} \right) - \partial_k \left(\frac{\partial_q v^i}{v^q - v^i} \right) \right]$$

and of the semi-Hamiltonian property. Finally, using (3.4) and (3.10) one can easily prove that

$$R_{qq^l}^i = -R_{q^l q}^i = -\partial_l \Gamma_{qq}^i + \Gamma_{qq}^i \Gamma_{ql}^q - \Gamma_{il}^i \Gamma_{qq}^i - \Gamma_{ll}^i \Gamma_{qq}^l = 0 \text{ if } i \neq q \neq l \neq i. \quad (3.11)$$

All the other components vanish apart from (3.6,3.7). \square

Lemma 3.2 *Condition (2.2) follows from (3.3), (3.4), and (3.5).*

Proof. In canonical coordinates, condition (2.2) takes the form

$$R_{lmi}^n \delta_p^n + R_{lip}^n \delta_m^n + R_{lpm}^n \delta_i^n = 0. \quad (3.12)$$

It is clearly satisfied if $n \neq p, m, i$. Since p, m, i appear cyclicly in (3.12), it is sufficient to prove it for $p = n$, that is,

$$R_{lmi}^n + R_{lin}^n \delta_m^n + R_{lnm}^n \delta_i^n = 0. \quad (3.13)$$

In turn, this condition needs a check only for $i \neq n$, leading to $R_{lmi}^n + R_{lin}^n \delta_m^n = 0$. We end up with $R_{lmi}^n = 0$, where $i \neq n$ and $m \neq n$, which is the content of Lemma 3.1. \square

Remark 3.3 *Let us consider a diagonal metric solving the system (1.6). Its Levi-Civita connection clearly satisfies (3.3) and (3.5). It fulfills also (3.4) if and only if the metric is potential in the coordinates (u^1, \dots, u^n) . Therefore, only in this case one can choose the Γ_{ii}^i in such a way that ∇ is such a Levi-Civita connection. In other words, a connection ∇ satisfying conditions (3.3,3.4,3.5) does not necessarily coincide with the Levi-Civita connection of a metric solving (1.6). A (counter)example is given by the ϵ -system discussed in Section 5.*

A natural way to eliminate the residual freedom in the choice of the Christoffel coefficients Γ_{ii}^i ($i = 1, \dots, n$) is to impose the additional requirement

$$\nabla e = 0, \quad (3.14)$$

where $e = \sum_{i=1}^n \frac{\partial}{\partial u^i}$ is the unity of the algebra. Indeed, condition (3.14) means that

$$\sum_{k=1}^n \Gamma_{jk}^i = 0. \quad (3.15)$$

If $i \neq j$, it coincides with (3.4). If $i = j$, it gives

$$\Gamma_{ii}^i = - \sum_{k \neq i} \Gamma_{ik}^i, \quad i = 1, \dots, n. \quad (3.16)$$

Definition 3.4 We call the connection ∇ defined by conditions (3.3,3.4,3.5,3.16) the natural connection associated with the semi-Hamiltonian system (3.1).

By construction, the natural connection and the product (3.2) satisfy Definition 2.1 of F -manifold with compatible connection, and the semi-Hamiltonian system (3.1) is one of the integrable flows associated with this F -manifold.

Remark 3.5 We added condition (3.16) in order to associate a unique connection to a given semi-Hamiltonian system. We will see in Section 5 that this is a very convenient choice for the ϵ -system. Nevertheless, there might be situations where a different condition has to be chosen.

We close this section with a remark on the Euler vector field

$$E = \sum_{k=1}^n u^k \frac{\partial}{\partial u^k}. \quad (3.17)$$

Proposition 3.6 The Euler vector field E and the unity of the algebra e satisfy the identity

$$\nabla_e E = e,$$

where ∇ is the natural connection.

Proof. We have that

$$\begin{aligned} (\nabla_e E)^i &= e^k \nabla_k E^i = e^k (\partial_k E^i + \Gamma_{kl}^i E^l) = \sum_{k=1}^n (\delta_k^i + \Gamma_{kl}^i E^l) \\ &= 1 + \left(\sum_{k=1}^n \Gamma_{kl}^i \right) E^l = 1 = e^i, \end{aligned}$$

where we have used (3.15). □

4 Special recurrence relations for semi-Hamiltonian systems

In view of the example of the ϵ -system (to be discussed in the next section), we recall some results obtained in [10, 11].

Let M be an n -dimensional manifold. A tensor field $L : TM \rightarrow TM$ of type $(1, 1)$ is said to be *torsionless* if the identity

$$[LX, LY] - L[LX, Y] - L[X, LY] + L^2[X, Y] = 0$$

is verified for any pair of vector fields X and Y on M . According to the theory of graded derivations of Frölicher-Nijenhuis [7], a torsionless tensor field L of type $(1, 1)$ defines a differential operator d_L on the Grassmann algebra of differential forms on M , satisfying the fundamental conditions

$$d \cdot d_L + d_L \cdot d = 0, \quad d_L^2 = 0.$$

On functions and 1-forms this derivation is defined by the following equations:

$$\begin{aligned} d_L f(X) &= df(LX) && \text{(that is, } d_L f = L^* df) \\ d_L \alpha(X, Y) &= Lie_{LX}(\alpha(Y)) - Lie_{LY}(\alpha(X)) - \alpha([X, Y]_L), \end{aligned}$$

where

$$[X, Y]_L = [LX, Y] + [X, LY] - L[X, Y].$$

For instance, if $L = \text{diag}(f^1(u^1), \dots, f^n(u^n))$, the action of d_L on functions is given by the formula

$$d_L g = \sum_{i=1}^n f^i \frac{\partial g}{\partial u^i} du^i.$$

We assume $a : M \rightarrow \mathbb{R}$ to be a function which satisfies the cohomological condition

$$dd_L a = 0, \tag{4.1}$$

and, from now on, that $L = \text{diag}(u^1, \dots, u^n)$. Then, according to the results of [10], to any solution $h = H(u)$ of the equation

$$dd_L h = dh \wedge da \tag{4.2}$$

we can associate a semi-Hamiltonian hierarchy. Indeed, it is easy to prove that the system of quasilinear PDEs

$$u_t^i = \left[-\frac{\partial_i K}{\partial_i H} \right] u_x^i, \quad i = 1, \dots, n, \tag{4.3}$$

is semi-Hamiltonian for any solution $h = K(u)$ of the equation (4.2), and that, once fixed H , the flows associated to any pair (K_1, K_2) of solutions of (4.2) commute. Moreover, there is a recursive procedure to obtain solutions of (4.2).

Lemma 4.1 ([1, 13]) *Let K_0 be a solution of (4.2). Then the functions K_α recursively defined by*

$$dK_{\alpha+1} = d_L K_\alpha - K_\alpha da, \quad \alpha \geq 0, \tag{4.4}$$

satisfy equation (4.2).

Let us illustrate how to apply the previous procedure in the case of the (trivial) solutions $H = a$ and $K_0 = -a$ of equation (4.2). Using the recursive relations (4.4), we get

$$\begin{aligned}
u_{t_0}^i &= -\frac{\partial_i K_0}{\partial_i a} u_x^i = u_x^i \\
u_{t_1}^i &= -\frac{\partial_i K_1}{\partial_i a} u_x^i = [u^i - a] u_x^i = [u^i + K_0] u_x^i \\
u_{t_2}^i &= -\frac{\partial_i K_2}{\partial_i a} u_x^i = [(u^i)^2 + K_0 u^i + K_1] u_x^i \\
&\vdots \\
u_{t_n}^i &= -\frac{\partial_i K_n}{\partial_i a} u_x^i = [(u^i)^n + K_0 (u^i)^{n-1} + K_1 (u^i)^{n-2} + \dots + K_{n-1}] u_x^i \\
&\vdots
\end{aligned}$$

The choice $H = -K_0 = a = \epsilon \operatorname{Tr}(L) = \epsilon \sum_{i=1}^n u^i$ gives rise to the ϵ -system discussed in the Introduction and in the following section.

5 The ϵ -system

In this section we exemplify our construction in the case of the ϵ -system (1.8). In particular, we show that its natural connection is flat, so that we obtain an F -manifold with flat compatible connection and its principal hierarchy.

5.1 The natural connection of the ϵ -system

According to Definition 3.4, the natural connection of the ϵ -system is given by

$$\begin{aligned}
\Gamma_{jk}^i &= 0 \quad \text{for } i \neq j \neq k \neq i \\
\Gamma_{ji}^i &= \frac{\epsilon}{u^i - u^j} \quad \text{for } i \neq j \\
\Gamma_{jj}^i &= -\Gamma_{ji}^i = \frac{\epsilon}{u^j - u^i} \quad \text{for } i \neq j \\
\Gamma_{ii}^i &= -\sum_{k \neq i} \Gamma_{ik}^i = -\sum_{k \neq i} \frac{\epsilon}{u^i - u^k}.
\end{aligned} \tag{5.1}$$

Proposition 5.1 *The natural connection of the ϵ -system is flat.*

Proof. We have that

$$R_{iki}^i = \partial_k \Gamma_{ii}^i - \partial_i \Gamma_{ik}^i = \partial_k \left(-\sum_{j \neq i} \frac{\epsilon}{u^i - u^j} \right) - \partial_i \left(\frac{\epsilon}{u^i - u^k} \right) = -\frac{\epsilon}{(u^i - u^k)^2} + \frac{\epsilon}{(u^i - u^k)^2} = 0$$

and

$$\begin{aligned}
R_{qqi}^i &= \partial_q \Gamma_{qi}^i - \partial_i \Gamma_{qq}^i + (\Gamma_{iq}^i)^2 + \Gamma_{qq}^i \Gamma_{qi}^q - \sum_{p=1}^n \Gamma_{pi}^i \Gamma_{qq}^p \\
&= \partial_q \Gamma_{qi}^i - \partial_i \Gamma_{qq}^i + \Gamma_{iq}^i (\Gamma_{iq}^i - \Gamma_{iq}^q) - \sum_{p \neq i, q} \Gamma_{pi}^i \Gamma_{qq}^p - \Gamma_{ii}^i \Gamma_{qq}^i - \Gamma_{qi}^i \Gamma_{qq}^q \\
&= (\partial_q + \partial_i) \left[\frac{\epsilon}{u^i - u^q} \right] + 2 \frac{\epsilon^2}{(u^i - u^q)^2} + \sum_{p \neq i, q} \frac{\epsilon^2}{(u^i - u^p)(u^p - u^q)} \\
&\quad + \frac{\epsilon}{u^i - u^q} \sum_{p \neq i, q} \left[-\frac{\epsilon}{u^i - u^p} + \frac{\epsilon}{u^q - u^p} \right] + \frac{\epsilon}{u^i - u^q} \left[-\frac{\epsilon}{u^i - u^q} + \frac{\epsilon}{u^q - u^i} \right] \\
&= \sum_{p \neq i, q} \frac{\epsilon^2}{(u^i - u^p)(u^p - u^q)} + \frac{\epsilon}{u^i - u^q} \sum_{p \neq i, q} \frac{\epsilon(u^i - u^p - u^q + u^p)}{(u^i - u^p)(u^q - u^p)} = 0.
\end{aligned}$$

Due to Lemma 3.1, there are no more components of R to be checked, so that ∇ is flat. \square

The next proposition is devoted to the relations between the natural connection (5.1) and the Euler vector field (3.17).

Proposition 5.2 *The covariant derivative of E is given by $\nabla_j E^k = (1 - n\epsilon)\delta_j^k + \epsilon$, that is,*

$$\nabla E = (1 - n\epsilon)I + \epsilon e \otimes d(\text{Tr}L), \quad (5.2)$$

where I is the identity on the tangent bundle and e is the unity. Therefore, for any vector field X ,

$$\nabla_X (E - \epsilon(\text{Tr}L)e) = (1 - n\epsilon)X. \quad (5.3)$$

Moreover, the Euler vector field E is linear in flat coordinates, i.e.,

$$\nabla \nabla E = 0. \quad (5.4)$$

Proof. Formula (5.2) follows from the very definitions of ∇ and E . Then, using the flatness of e , we obtain (5.3). Finally,

$$\nabla \nabla E = \nabla ((1 - n\epsilon)I + \epsilon e \otimes d(\text{Tr}L)) = 0$$

since also $\nabla I = 0$ (due to the fact that ∇ is torsionless) and $\nabla (d(\text{Tr}L)) = 0$ (as noticed in the next subsection, before Proposition 5.4). \square

We remark that (5.4) is one of the property entering the definition of Frobenius manifold.

5.2 Flat coordinates

In this subsection we discuss some properties of the flat coordinates of the natural connection (5.1) of the ϵ -system.

We have to find a basis of flat exact 1-forms $\theta = \theta_i du^i$, that is, n independent solutions of the linear system of PDEs

$$\begin{aligned} \partial_j \theta_i - \epsilon \frac{\theta_i - \theta_j}{u^i - u^j} &= 0, & i = 1, \dots, n, j \neq i \\ \partial_i \theta_i + \epsilon \sum_{k \neq i} \frac{\theta_k - \theta_i}{u^k - u^i} &= 0, & i = 1, \dots, n, \end{aligned} \quad (5.5)$$

which is equivalent to

$$\begin{aligned} \partial_j \theta_i - \epsilon \frac{\theta_i - \theta_j}{u^i - u^j} &= 0, & i = 1, \dots, n, j \neq i \\ \sum_{k=1}^n \partial_k \theta_i &= 0, & i = 1, \dots, n. \end{aligned} \quad (5.6)$$

In particular, we have that

$$0 = \sum_{k=1}^n \partial_k \theta_i = \sum_{k=1}^n \partial_i \theta_k = \partial_i \left(\sum_{k=1}^n \theta_k \right),$$

showing that $\sum_{k=1}^n \theta_k$ is constant if $\theta = \theta_k du^k$ is flat.

Remark 5.3 *It trivially follows from (5.6) that f is a flat coordinate if and only if*

$$\begin{aligned} (u^i - u^j) \partial_j \partial_i f - \epsilon (\partial_i f - \partial_j f) &= 0, & i = 1, \dots, n, j \neq i \\ \sum_{k=1}^n \partial_k \partial_i f &= 0, & i = 1, \dots, n. \end{aligned} \quad (5.7)$$

Since

$$(dd_L f - d(\epsilon \operatorname{Tr} L) \wedge df)_{ij} = (u^i - u^j) \partial_j \partial_i f - \epsilon (\partial_i f - \partial_j f),$$

any flat coordinate of the natural connection of the ϵ -system solves equation (4.2) with $a = -\epsilon \operatorname{Tr} L = -\epsilon \sum_{i=1}^n u^i$.

A trivial solution of the system (5.6) is given by $\theta_j = 1$ for all j , corresponding to the flat 1-form $\theta^{(1)} = \sum_{j=1}^n du^j = df_\epsilon^1$, where $f_\epsilon^1 = \sum_{j=1}^n u^j$. The other flat coordinates can be chosen according to

Proposition 5.4 *There exist flat coordinates $(f_\epsilon^1, f_\epsilon^2, \dots, f_\epsilon^n)$ whose partial derivatives $\partial_i f_\epsilon^p(u)$ are homogeneous functions of degree $-n\epsilon$ for all $p = 2, \dots, n$ and $i = 1, \dots, n$. In particular, if $\epsilon \neq \frac{1}{n}$ there exist flat coordinates $(f_\epsilon^1, f_\epsilon^2, \dots, f_\epsilon^n)$ such that $f_\epsilon^p(u)$ is a homogeneous function of degree $(1 - n\epsilon)$ for all $p = 2, \dots, n$.*

Proof. Suppose that $\phi = \phi_j dw^j$ is a flat 1-form. Then $\sum_{j=1}^n \phi_j = c$ constant, and $\theta := \phi - \frac{c}{n}\theta^{(1)}$ is still a flat form and satisfies $\sum_{j=1}^n \theta_j = 0$. Then equations (5.5) entail that

$$\sum_{j=1}^n u^j \partial_j \theta_i = -n\epsilon \theta_i,$$

so that $\theta_i(u)$ is homogeneous of degree $-n\epsilon$ for all i . This shows that we can always find a basis $(\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(n)})$ of flat forms such that the components of $\theta^{(p)}$, for all $p \geq 2$, are homogeneous of degree $-n\epsilon$. Since flat forms are exact, the first assertion is proved.

The second assertion simply follows from the general fact that a function $f(u)$, whose partial derivatives are homogeneous of degree $r \neq -1$, is (up to an additive constant) homogeneous of degree $(r + 1)$. Even though this is well known, we give a proof for the reader's sake. We know that

$$\sum_{k=1}^n u^k \partial_k (\partial_j f) = r \partial_j f,$$

therefore we have

$$\partial_j \left(\sum_{k=1}^n u^k \partial_k f - (r + 1)f \right) = 0,$$

meaning that

$$\sum_{k=1}^n u^k \partial_k f = (r + 1)f + c$$

for some constant c that can be eliminated if $r \neq -1$. □

In the case $n = 2$, we have that

$$\theta^{(2)} = (u^1 - u^2)^{-2\epsilon} (du^1 - du^2),$$

so that the flat coordinates are

$$f_\epsilon^1 = u^1 + u^2, \quad f_\epsilon^2 = (u^1 - u^2)^{1-2\epsilon} \quad \text{if } \epsilon \neq \frac{1}{2} \quad (5.8)$$

$$f_\epsilon^1 = u^1 + u^2, \quad f_\epsilon^2 = \ln(u^1 - u^2) \quad \text{if } \epsilon = \frac{1}{2}. \quad (5.9)$$

In the case $n = 3$, assuming $\epsilon \neq \frac{1}{3}$ and taking into account the homogeneity of the flat coordinates, it is possible to reduce the system (5.7) to a third order ODE whose solutions can be explicitly written in terms of hypergeometric functions ${}_2F_1(\alpha; \beta; \gamma; z)$ (see the Appendix

for more details). If $\epsilon \neq \frac{1}{3}$ we obtain the flat coordinates (in the domain where $u^3 > u^1$)

$$\begin{aligned} f_\epsilon^1 &= u^1 + u^2 + u^3 \\ f_\epsilon^2 &= (1 - 3\epsilon)(2u^2 - u^3 - u^1)[(u^3 - u^1)(u^1 - u^2)^2]^{-\epsilon} {}_2F_1\left(\epsilon; 1 - \epsilon; 1 + 2\epsilon; \frac{u^2 - u^3}{u^1 - u^3}\right) + \\ &\quad (1 + \epsilon)[(u^3 - u^1)(u^1 - u^2)^2]^{-\epsilon}(u^1 - u^3) {}_2F_1\left(2 - \epsilon; \epsilon - 1; 1 + 2\epsilon; \frac{u^2 - u^3}{u^1 - u^3}\right) \\ f_\epsilon^3 &= (2u^2 - u^1 - u^3)[(u^3 - u^1)(u^3 - u^2)^2]^{-\epsilon} {}_2F_1\left(\epsilon; 1 - \epsilon; 1 - 2\epsilon; \frac{u^3 - u^2}{u^3 - u^1}\right) + \\ &\quad -[(u^3 - u^1)(u^3 - u^2)^2]^{-\epsilon}(u^3 - u^1) {}_2F_1\left(\epsilon - 1; 2 - \epsilon; 1 - 2\epsilon; \frac{u^3 - u^2}{u^3 - u^1}\right). \end{aligned}$$

It turns out that in the case $\epsilon = \frac{1}{3}$ the functions f_ϵ^2 and f_ϵ^3 reduce to a constant. For integer values of the parameter ϵ one obtains simpler expressions. For instance, in the case $\epsilon = -1$ (up to inessential constant factors) we have

$$\begin{aligned} f_\epsilon^1 &= u^1 + u^2 + u^3 \\ f_\epsilon^2 &= 4(u^3 - u^1)^3(u^1 + u^3 - 2u^2) \\ f_\epsilon^3 &= 4(u^3 - u^2)^3(u^2 + u^3 - 2u^1), \end{aligned}$$

and in the case $\epsilon = 2$ we have

$$\begin{aligned} f_\epsilon^1 &= u^1 + u^2 + u^3 \\ f_\epsilon^2 &= \frac{4(u^2 + u^3 - 2u^1)}{(u^2 - u^1)^3(u^3 - u^1)^3} \\ f_\epsilon^3 &= \frac{4(u^1 + u^2 - 2u^3)}{(u^3 - u^2)^3(u^3 - u^1)^3}. \end{aligned}$$

This concludes the discussion of the case $\epsilon \neq \frac{1}{3}$. We will make later some consideration for the case $\epsilon = \frac{1}{3}$.

Remark 5.5 Given any set $(f_\epsilon^1, \dots, f_\epsilon^n)$ of flat coordinates, the natural connection ∇ is the Levi-Civita connection of the metric $\eta = \eta_{ij} df_\epsilon^i \otimes df_\epsilon^j$ for any choice of the invertible symmetric matrix (η_{ij}) . For $n = 2$, choosing f_ϵ^1 and f_ϵ^2 as above, and

$$(\eta)_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we obtain

$$g_{ii} = \frac{(-1)^i 2(2\epsilon - 1)}{|u^1 - u^2|^{2\epsilon}}, \quad i = 1, 2, \quad (5.10)$$

which is one of the metrics (1.9). On the contrary—as we have said in the Introduction—the metrics (1.9) are not of Egorov type if $n > 2$. This means that for $n > 2$ the natural connection for the ϵ -system does not coincide with the Levi-Civita connection of the metrics (1.9).

5.3 The structure constants

Let us discuss in details the case $n = 2$. In the coordinates (5.8) and (5.9) the structure constants are given by

$$c_{11}^1 = c_{12}^2 = c_{21}^2 = \frac{1}{2}, \quad c_{11}^2 = c_{12}^1 = c_{12}^2 = c_{22}^2 = 0, \quad c_{22}^1 = \frac{(f_\epsilon^2)^{\frac{4\epsilon}{1-2\epsilon}}}{2(2\epsilon-1)^2}$$

for $\epsilon \neq \frac{1}{2}$ and by

$$c_{11}^1 = c_{12}^2 = c_{21}^2 = \frac{1}{2}, \quad c_{11}^2 = c_{12}^1 = c_{12}^2 = c_{22}^2 = 0, \quad c_{22}^1 = \frac{1}{8f_\epsilon^2}$$

for $\epsilon = \frac{1}{2}$. Hence the vector potential C has components

$$\begin{aligned} C^1 &= \frac{(f_\epsilon^2)^{-\frac{2}{2\epsilon-1}}}{4(2\epsilon+1)} + \frac{1}{4}(f_\epsilon^1)^2 \\ C^2 &= \frac{1}{2}f_\epsilon^1 f_\epsilon^2 \end{aligned}$$

for $\epsilon \neq \pm\frac{1}{2}$ and

$$\begin{aligned} C^1 &= \frac{1}{16}f_\epsilon^2 (\ln f_\epsilon^2 - 1) + \frac{1}{4}(f_\epsilon^1)^2 \\ C^2 &= \frac{1}{2}f_\epsilon^1 f_\epsilon^2 \end{aligned}$$

for $\epsilon = \pm\frac{1}{2}$.

It is easy to check that, lowering the index of the vector potential with the metric (5.10), that in flat coordinates is antidiagonal with components $g_{12} = g_{21} = 1$, we obtain an exact 1-form. In other words, for $n = 2$ we obtain a scalar potential F satisfying WDVV equations:

$$F = \frac{2\epsilon-1}{4(2\epsilon+1)(2\epsilon-3)}(f_\epsilon^2)^{-\frac{2\epsilon-3}{2\epsilon-1}} + \frac{1}{4}(f_\epsilon^1)^2 f_\epsilon^2, \quad \text{if } \epsilon \neq \pm\frac{1}{2}, \frac{3}{2} \quad (5.11)$$

$$F = \frac{1}{16}(f_\epsilon^2)^2 \ln f_\epsilon^2 - \frac{3}{32}(f_\epsilon^2)^2 + \frac{1}{4}(f_\epsilon^1)^2 f_\epsilon^2, \quad \text{if } \epsilon = \pm\frac{1}{2} \quad (5.12)$$

$$F = \frac{1}{16} \ln f_\epsilon^2 + \frac{1}{4}(f_\epsilon^1)^2 f_\epsilon^2, \quad \text{if } \epsilon = \frac{3}{2} \quad (5.13)$$

Let us finally consider the case $n = 3$, $\epsilon = 1$. As flat coordinates we can choose

$$\begin{aligned} f^1 &= u^1 + u^2 + u^3 \\ f^2 &= \frac{1}{2(u^1 - u^2)(u^3 - u^1)} \\ f^3 &= \frac{1}{2(u^2 - u^3)(u^1 - u^2)}. \end{aligned}$$

In such coordinates the structure constants read:

$$\begin{aligned}
c_{11}^1 &= \frac{1}{3}, c_{11}^2 = 0, c_{11}^3 = 0, c_{12}^1 = 0, c_{12}^2 = \frac{1}{3}, c_{12}^3 = 0, c_{13}^1 = 0, c_{13}^2 = 0, c_{13}^3 = \frac{1}{3} \\
c_{22}^1 &= -\frac{1}{12} \frac{f^3 (3(f^2)^2 + 3f^2 f^3 + (f^3)^2)}{(f^2)^3 (f^2 + f^3)^3} \\
c_{22}^2 &= -\frac{1}{24} \frac{(10(f^2)^2 + 9f^2 f^3 + 2(f^3)^2)(f^3)^2 \sqrt{2}}{(f^2 + f^3)^2 f^2 \sqrt{-f^3 f^2 (f^2 + f^3) (f^3 + 2f^2)^2}} \\
c_{22}^3 &= -\frac{1}{8} \frac{(f^3 + 2f^2)^2 f^3 \sqrt{2}}{(f^2)^2 (f^2 + f^3)^2 \sqrt{-f^3 f^2 (f^2 + f^3) (f^3 + 2f^2)^2}} \\
c_{23}^1 &= \frac{1}{12} (f^2 + f^3)^{-3} \\
c_{23}^2 &= -\frac{1}{24} \frac{(f^2)^2 (4(f^2)^2 + 12f^2 f^3 + 5(f^3)^2) \sqrt{2}}{(f^2 + f^3)^2 f^3 \sqrt{-f^3 f^2 (f^2 + f^3) (f^3 + 2f^2)^2}} \\
c_{23}^3 &= \frac{1}{24} \frac{(10(f^2)^2 + 9f^2 f^3 + 2(f^3)^2)(f^3)^2 \sqrt{2}}{(f^2 + f^3)^2 f^2 \sqrt{-f^3 f^2 (f^2 + f^3) (f^3 + 2f^2)^2}} \\
c_{33}^1 &= -\frac{1}{12} \frac{f^2 ((f^2)^2 + 3f^2 f^3 + 3(f^3)^2)}{(f^3)^3 (f^2 + f^3)^3} \\
c_{33}^2 &= \frac{1}{8} \frac{(f^3 + 2f^2)(f^2 + 2f^3)(f^2)^3 \sqrt{2}}{(f^3)^2 (f^2 + f^3)^2 \sqrt{-f^3 f^2 (f^2 + f^3) (f^3 + 2f^2)^2}} \\
c_{33}^3 &= \frac{1}{24} \frac{(f^2)^2 (4(f^2)^2 + 12f^2 f^3 + 5(f^3)^2) \sqrt{2}}{(f^2 + f^3)^2 f^3 \sqrt{-f^3 f^2 (f^2 + f^3) (f^3 + 2f^2)^2}}
\end{aligned}$$

We do not display the components of the vector potential C , since the corresponding expressions are quite cumbersome.

5.4 The primary flows

In order to define the primary flows we need a basis of flat vector fields $X = X^i \frac{\partial}{\partial u^i}$, that is, n independent solutions of the linear system of PDEs

$$\begin{aligned}
\partial_j X^i + \epsilon \frac{X^i - X^j}{u^i - u^j} &= 0, \quad i = 1, \dots, n, j \neq i \\
\partial_i X^i - \epsilon \sum_{k \neq i} \frac{X^k - X^i}{u^k - u^i} &= 0, \quad i = 1, \dots, n
\end{aligned} \tag{5.14}$$

which is equivalent to

$$\partial_j X^i + \epsilon \frac{X^i - X^j}{u^i - u^j} = 0, \quad i = 1, \dots, n, j \neq i \quad (5.15)$$

$$[e, X] = 0. \quad (5.16)$$

Comparing (5.14) with (5.5), one notices that the components X^i of a flat vector fields for ϵ are given by the components of a flat 1-form for $-\epsilon$. Therefore, from Proposition 5.4 we have that there always exists a basis of flat vector fields $(X_{(1)} = e, X_{(2)}, \dots, X_{(n)})$ such that the components $X_{(p)}^i(u)$, for $p = 2, \dots, n$, are homogeneous functions of degree $n\epsilon$.

In the case $n = 2$ we have, for $\epsilon \neq -\frac{1}{2}$,

$$\begin{aligned} df_{-\epsilon}^1 &= du^1 + du^2 \\ df_{-\epsilon}^2 &= (1 + 2\epsilon)(u^1 - u^2)^{2\epsilon}(du^1 - du^2) \end{aligned}$$

and therefore

$$\begin{aligned} X_{(1,0)} &= \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^2} = e \\ X_{(2,0)} &= (1 + 2\epsilon)(u^1 - u^2)^{2\epsilon} \left(\frac{\partial}{\partial u^1} - \frac{\partial}{\partial u^2} \right). \end{aligned}$$

In canonical coordinates the primary flows are thus given by

$$\begin{aligned} u_{t(1,0)}^1 &= u_x^1 \\ u_{t(1,0)}^2 &= u_x^2 \end{aligned}$$

and

$$\begin{aligned} u_{t(2,0)}^1 &= (1 + 2\epsilon)(u^1 - u^2)^{2\epsilon} u_x^1 \\ u_{t(2,0)}^2 &= -(1 + 2\epsilon)(u^1 - u^2)^{2\epsilon} u_x^2. \end{aligned}$$

The case $n = 3$, $\epsilon \neq -\frac{1}{3}$ can be treated similarly since we know the flat coordinates.

Let us consider the case $n = 3$, $\epsilon = -\frac{1}{3}$. One of the flat vector fields is the unity e of the algebra. We know that there exist two other flat vector fields $X_{(2)}$ and $X_{(3)}$, whose components are homogeneous functions of degree -1 ,

$$u^1 \partial_1 X_{(i)}^k + u^2 \partial_2 X_{(i)}^k + u^3 \partial_3 X_{(i)}^k = -X_{(i)}^k, \quad i = 2, 3, \quad k = 1, 2, 3, \quad (5.17)$$

satisfying the additional property

$$X_{(i)}^1 + X_{(i)}^2 + X_{(i)}^3 = 0, \quad i = 2, 3. \quad (5.18)$$

Since $\partial_j X_{(i)}^k = \partial_k X_{(i)}^j$, from (5.17) we obtain

$$u^1 X_{(i)}^1 + u^2 X_{(i)}^2 + u^3 X_{(i)}^3 = c, \quad (5.19)$$

where c is a constant. Two cases are possible: $c = 0$ and $c \neq 0$. In both cases, taking into account condition (5.18), we can write one of the components of the vector field $X_{(i)}$ in terms of the remaining two. Substituting the result in (5.14), we obtain a system of 3 equations whose solution is

$$\begin{aligned} X_{(2)}^1 &= \frac{(u^2 - u^3)^{1/3}}{(u^3 - u^1)^{2/3}(u^1 - u^2)^{2/3}} \\ X_{(2)}^2 &= \frac{(u^3 - u^1)^{1/3}}{(u^2 - u^3)^{2/3}(u^1 - u^2)^{2/3}} \\ X_{(2)}^3 &= \frac{(u^1 - u^2)^{1/3}}{(u^3 - u^1)^{2/3}(u^2 - u^3)^{2/3}} \end{aligned}$$

for $c = 0$ and

$$\begin{aligned} X_{(3)}^1 &= \frac{c}{u^2 - u^1} + \frac{c(u^3 - u^2)^{1/3}}{3(u^3 - u^1)^{2/3}(u^1 - u^2)} \int \frac{du^3}{(u^3 - u^2)^{1/3}(u^3 - u^1)^{1/3}} \\ X_{(3)}^2 &= \frac{c}{u^2 - u^1} + \frac{c(u^3 - u^1)^{2/3}}{3(u^3 - u^2)^{2/3}(u^1 - u^2)} \int \frac{du^3}{(u^3 - u^1)(u^3 - u^2)^{1/3}} \\ X_{(3)}^3 &= \frac{c}{3(u^3 - u^2)^{2/3}(u^3 - u^1)^{2/3}} \int \frac{du^3}{(u^3 - u^2)^{1/3}(u^3 - u^1)^{1/3}} \end{aligned}$$

for $c \neq 0$. Notice that we can choose the constants of integration in the above integrals in such a way that the $X_{(3)}^i$ be homogeneous of degree -1.

Hence we can explicitly construct the principal hierarchy (2.7) also in the case $\epsilon = -\frac{1}{3}$.

5.5 The higher flows

The higher flows are defined by vector fields $X_{(p,\alpha)}$ satisfying

$$\nabla_j X_{(p,\alpha)}^i = c_{ik}^j X_{(p,\alpha-1)}^k \quad (5.20)$$

or, more explicitly,

$$\partial_j X_{(p,\alpha)}^i + \epsilon \frac{X_{(p,\alpha)}^i - X_{(p,\alpha)}^j}{u^i - u^j} = 0, \quad i = 1, \dots, n, j \neq i \quad (5.21)$$

$$\partial_i X_{(p,\alpha)}^i - \epsilon \sum_{k \neq i} \frac{X_{(p,\alpha)}^k - X_{(p,\alpha)}^i}{u^k - u^i} = X_{(p,\alpha-1)}^i, \quad i = 1, \dots, n. \quad (5.22)$$

Taking into account (5.21), condition (5.22) can be written as

$$\sum_{k=1}^n \partial_k X_{(p,\alpha)}^i = X_{(p,\alpha-1)}^i, \quad i = 1, \dots, n \quad (5.23)$$

or, in compact form, as

$$[e, X_{(p,\alpha)}] = X_{(p,\alpha-1)}. \quad (5.24)$$

We show now that—apart from some critical values of ϵ —the higher flows of the principal hierarchy can be obtained by applying the recursive procedure described in Section 4. First of all, we recall from Remark 5.3 that the flat coordinates of the natural connection of the $(-\epsilon)$ -system satisfy equation (4.2), with $L = \text{diag}(u^1, \dots, u^n)$ and $a = \epsilon \text{Tr} L$. Therefore, they can be used as starting point for the recursive procedure (4.4), giving rise to the flows (4.3), with $H = a$.

Proposition 5.6 *Suppose that $(f_{-\epsilon}^1 = \text{Tr} L, f_{-\epsilon}^2, \dots, f_{-\epsilon}^n)$ be the flat coordinates described in Proposition 5.4 of the natural connection of the $(-\epsilon)$ -system. If $K_{(p,\alpha)}$ are the functions defined recursively by*

$$K_{(p,0)} = -\epsilon f_{-\epsilon}^p, \quad dK_{(p,\alpha+1)} = d_L K_{(p,\alpha)} - \epsilon K_{(p,\alpha)} d(\text{Tr} L), \quad \alpha \geq 0, \quad (5.25)$$

and

$$Y_{(p,\alpha)}^i = -\frac{\partial_i K_{(p,\alpha)}}{\partial_i a} = -\frac{1}{\epsilon} \partial_i K_{(p,\alpha)}, \quad \alpha \geq 0, \quad (5.26)$$

are the components of the vector fields of the corresponding hierarchy, then the vector fields $X_{(1,\alpha)} = \frac{1}{\prod_{j=1}^{\alpha} (j-n\epsilon)} Y_{(1,\alpha)}$ (for $\epsilon \neq \frac{j}{n}$ with $j = 1, \dots, \alpha$) and $X_{(p,\alpha)} = \frac{1}{\alpha!} Y_{(p,\alpha)}$, for $p = 2, \dots, n$, satisfy the recursion relations (5.20).

Proof. We know that from Lemma 4.1 that the function $K_{(p,\alpha)}$ satisfies equation (4.2). Then it is easily checked that the vector fields $Y_{(p,\alpha)}$ and $X_{(p,\alpha)}$ satisfy equation (5.21), so that there are only relations (5.23) to be proved.

Let us consider the case $p = 1$. After writing (5.25) in canonical coordinates,

$$\partial_j K_{(1,\alpha+1)} = u^j \partial_j K_{(1,\alpha)} - \epsilon K_{(1,\alpha)}, \quad \alpha \geq 0, \quad (5.27)$$

and recalling that $K_{(1,0)}(u) = -\epsilon \sum_{i=1}^n u^i$, it is clear that one can show by induction that the partial derivatives $\partial_j K_{(1,\alpha)}(u)$ are homogeneous functions of degree α , so that $K_{(1,\alpha)}(u)$ is homogeneous of degree $(\alpha + 1)$. Using this fact, again from (5.27) we have that

$$\sum_{j=1}^n \partial_j K_{(1,\alpha+1)} = \sum_{j=1}^n (u^j \partial_j K_{(1,\alpha)} - \epsilon K_{(1,\alpha)}) = (\alpha + 1 - n\epsilon) K_{(1,\alpha)},$$

so that

$$\sum_{j=1}^n \partial_j Y_{(1,\alpha+1)}^i = (\alpha + 1 - n\epsilon) Y_{(1,\alpha)}^i, \quad i = 1, \dots, n, \quad (5.28)$$

and relations (5.23) for $X_{(1,\alpha)}^i$ follow.

The case $p = 2, \dots, n$ can be treated in the same way. The only difference is that the degree of homogeneity of $\partial_j K_{(p,\alpha)}$ is $(\alpha + n\epsilon)$, so that $K_{(p,\alpha)}$ is homogeneous of degree $(\alpha + 1 + n\epsilon)$ if $\alpha \neq -1 - n\epsilon$. \square

As a consequence of the above proposition we have that, if $\epsilon \neq \frac{k}{n}$ for all $k \in \mathbb{N}$, the flows

$$u_{t_{(1,\alpha)}}^i = X_{(1,\alpha)}^i u_x^i = \frac{Y_{(1,\alpha)}^i}{\prod_{j=1}^{\alpha} (j - n\epsilon)} u_x^i = -\frac{\partial_i K_{(1,\alpha)}}{\epsilon \prod_{j=1}^{\alpha} (j - n\epsilon)} u_x^i, \quad (5.29)$$

$$u_{t_{(p,\alpha)}}^i = X_{(p,\alpha)}^i u_x^i = \frac{Y_{(p,\alpha)}^i}{\alpha!} u_x^i = -\frac{\partial_i K_{(p,\alpha)}}{\epsilon \alpha!} u_x^i, \quad p \neq 1, \quad (5.30)$$

(with $i = 1, \dots, n$ and $\alpha \geq 0$) define the principal hierarchy of the ϵ -system.

If $\epsilon = \frac{k}{n}$ for some $k \in \mathbb{N}$, all the flows (5.30) and the flows (5.29) with $\alpha = 0, \dots, k-1$ still belong to the principal hierarchy. Even though the latter is well defined, relations (5.29) do not make sense for $\alpha \geq k$, since the denominator vanishes. The point is that the vector field $Y_{(1,k)}$ is flat, as one can immediately check using (5.28), and its components are homogeneous of degree $k = \epsilon n$. Therefore $Y_{(1,k)}$ is a linear combination (with constant coefficients) of the flat homogeneous vector fields $X_{(2,0)}, \dots, X_{(n,0)}$. This means that $Y_{(1,\alpha)}$ is, for $\alpha \geq k$, a linear combinations of the vector fields $Y_{(p,\alpha-k)}$, with $p = 2, \dots, n$. In order to obtain the missing flows of the principal hierarchy, associated to the vector fields $X_{(1,\alpha)}$ with $\alpha \geq k$, one has to solve the system (5.21,5.22) with $p = 1, \alpha \geq k$ and $X_{(1,k-1)} = \frac{1}{\prod_{j=1}^{k-1} (j-k)} Y_{(1,k-1)}$.

For instance, in the case $\epsilon = \frac{1}{2}$, $n = 2$ one can immediately check that the vector field

$$\begin{aligned} Y_{(1,1)}^1 &= u^1 - \epsilon \text{Tr} L = \frac{u^1 - u^2}{2} \\ Y_{(1,1)}^2 &= u^2 - \epsilon \text{Tr} L = \frac{u^2 - u^1}{2} \end{aligned}$$

is flat, unlike the vector field

$$\begin{aligned} X_{(1,1)}^1 &= \frac{1}{2}(u^1 - u^2) \ln(u^1 - u^2) + \frac{3}{2}u^2 - \frac{1}{2}u^1 + c_1 Y_{(1,1)}^1 + c_2 \\ X_{(1,1)}^2 &= -\frac{1}{2}(u^1 - u^2) \ln(u^1 - u^2) + \frac{3}{2}u^1 - \frac{1}{2}u^2 + c_1 Y_{(1,1)}^2 + c_2 \end{aligned}$$

(c_1 and c_2 are arbitrary constants), obtained solving the system

$$\begin{aligned} \partial_2 X_{(1,1)}^1 + \frac{1}{2} \frac{X_{(1,1)}^1 - X_{(1,1)}^2}{u^1 - u^2} &= 0 \\ \partial_1 X_{(1,1)}^2 + \frac{1}{2} \frac{X_{(1,1)}^1 - X_{(1,1)}^2}{u^1 - u^2} &= 0 \\ \partial_1 X_{(1,1)}^1 - \frac{1}{2} \frac{X_{(1,1)}^1 - X_{(1,1)}^2}{u^1 - u^2} &= 1 \\ \partial_2 X_{(1,1)}^2 - \frac{1}{2} \frac{X_{(1,1)}^1 - X_{(1,1)}^2}{u^1 - u^2} &= 1. \end{aligned}$$

To conclude, we observe that, under suitable assumptions, the recursion relations (5.25) can be written in a more explicit form. Indeed, we have the following

Proposition 5.7 *The recursion relations (5.25) are algebraically solved by*

$$K_{(1,\alpha)} = \frac{1}{\alpha + 1} \left(\sum_{j=1}^n (u^j)^2 \partial_j K_{(1,\alpha-1)} - \epsilon \left(\sum_{j=1}^n u^j \right) K_{(1,\alpha-1)} \right) \quad (5.31)$$

and, for $\alpha \neq -1 - n\epsilon$, by

$$K_{(p,\alpha)} = \frac{1}{\alpha + 1 + n\epsilon} \left(\sum_{j=1}^n (u^j)^2 \partial_j K_{(p,\alpha-1)} - \epsilon \left(\sum_{j=1}^n u^j \right) K_{(p,\alpha-1)} \right), \quad p = 2, \dots, n. \quad (5.32)$$

Proof. It suffices to multiply (5.27) by u^j and to sum over j , taking into account the homogeneity of the functions $K_{(p,\alpha)}$. \square

6 Appendix

Let us consider the system

$$\partial_i f = \theta_i, \quad (6.1)$$

$$\partial_j \theta_i - \epsilon \frac{\theta_i - \theta_j}{u^i - u^j} = 0, \quad i = 1, 2, 3, j \neq i \quad (6.2)$$

$$\theta_1 + \theta_2 + \theta_3 = 0, \quad (6.3)$$

$$u^1 \theta_1 + u^2 \theta_2 + u^3 \theta_3 = (1 - 3\epsilon)f, \quad (6.4)$$

providing the homogeneous flat coordinates f for the natural connection of the ϵ -system in the case $n = 3$, $\epsilon \neq \frac{1}{3}$. Using (6.3) and (6.4) we can write two of the components of θ , for instance θ_1 and θ_3 , in terms of the remaining one and of the flat coordinate f :

$$\theta_1 = \frac{(u^3 - u^2)\theta_2 + (1 - 3\epsilon)f}{u^1 - u^3} \quad (6.5)$$

$$\theta_3 = \frac{(u^2 - u^1)\theta_2 - (1 - 3\epsilon)f}{u^1 - u^3}. \quad (6.6)$$

Hence, using (6.2) with $i = 2, j = 1$, we obtain f in terms of θ_2 :

$$f = \frac{(u^3 - u^2)(u^3 - u^1)\partial_1 \theta_2 + \epsilon(2u^3 + u^1 + u^2)\theta_2}{\epsilon(1 - 3\epsilon)}. \quad (6.7)$$

In this way equation (6.2) with $i = 2, j = 3$ reduces to a PDE involving only the unknown function θ_2 ,

$$(u^2 - u^3)\partial_3 \theta_2 + (u^2 - u^1)\partial_1 \theta_2 - 3\epsilon\theta_2 = 0, \quad (6.8)$$

whose solution is given by

$$\theta_2 = G(u^2, \nu)(u^1 - u^2)^{-3\epsilon} \quad (6.9)$$

where

$$\nu = \frac{u^3 - u^2}{u^1 - u^2}$$

and $G(u^2, \nu)$ is an arbitrary function. Substituting (6.9) in (6.7) and the result in the equation

$$(u^1 - u^3)\partial_1\partial_3 f + \epsilon(\partial_1 f - \partial_3 f) = 0$$

we obtain the third order ODE

$$G''' + \frac{(4\nu+5\epsilon\nu-4\epsilon-2)}{\nu(\nu-1)}G'' + \frac{(9\epsilon\nu^2-13\epsilon^2\nu+2\nu^2-2\nu-2\epsilon+7\epsilon^2\nu^2+4\epsilon^2-9\epsilon\nu)}{\nu^2(\nu-1)^2}G' + \frac{3\epsilon^2(2\epsilon-\epsilon\nu-\nu)}{\nu^2(\nu-1)^2}G = 0,$$

where G''' , G'' , G' are the derivatives of $G(u^2, \nu)$ with respect to ν . The above equation can be explicitly solved in terms of hypergeometric functions:

$$\begin{aligned} G = & G_1(u^2)(\nu-1)^\epsilon\nu^{-2\epsilon} + G_2(u^2)(\nu-1)^{-\epsilon}\nu^{-2\epsilon} {}_2F_1\left(\epsilon; 1-\epsilon; 1+2\epsilon; \frac{\nu}{\nu-1}\right) \\ & + G_3(u^2)(\nu-1)^{-\epsilon} {}_2F_1\left(\epsilon; 1-\epsilon; 1-2\epsilon; \frac{\nu}{\nu-1}\right), \end{aligned}$$

where G_1, G_2, G_3 are arbitrary functions of a single variable. The choice $G_2 = G_3 = 0$ gives rise to $f = 0$, while the choices ($G_2=\text{constant}, G_1 = G_3 = 0$) and ($G_3=\text{constant}, G_1 = G_2 = 0$) give rise to the homogeneous flat coordinates

$$\begin{aligned} f_\epsilon^2 = & (1-3\epsilon)(2u^2 - u^3 - u^1)[(u^3 - u^1)(u^1 - u^2)^2]^{-\epsilon} {}_2F_1\left(\epsilon; 1-\epsilon; 1+2\epsilon; \frac{u^2 - u^3}{u^1 - u^3}\right) + \\ & (1+\epsilon)[(u^3 - u^1)(u^1 - u^2)^2]^{-\epsilon}(u^1 - u^3) {}_2F_1\left(2-\epsilon; \epsilon-1; 1+2\epsilon; \frac{u^2 - u^3}{u^1 - u^3}\right) \\ f_\epsilon^3 = & (2u^2 - u^1 - u^3)[(u^3 - u^1)(u^3 - u^2)^2]^{-\epsilon} {}_2F_1\left(\epsilon; 1-\epsilon; 1-2\epsilon; \frac{u^3 - u^2}{u^3 - u^1}\right) + \\ & -[(u^3 - u^1)(u^3 - u^2)^2]^{-\epsilon}(u^3 - u^1) {}_2F_1\left(\epsilon-1; 2-\epsilon; 1-2\epsilon; \frac{u^3 - u^2}{u^3 - u^1}\right). \end{aligned}$$

as one can check by a straightforward computation, substituting in the equations (6.1,6.2,6.3,6.4).

References

- [1] A. Dimakis, F. Müller-Hoissen, *Bi-differential calculi and integrable models*, J. Phys. A: Math. Gen. **33** (2000), 957–974.
- [2] B.A. Dubrovin, *Geometry of 2D topological field theories*, in: Integrable Systems and Quantum Groups, Montecatini Terme, 1993. Editors: M. Francaviglia, S. Greco. Springer Lecture Notes in Math. **1620** (1996), pp. 120–348.

- [3] B.A. Dubrovin, S.P. Novikov, *Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory*, Uspekhi Mat. Nauk **44** (1989), 29–98. English translation in Russ. Math. Surveys **44** (1989), 35–124.
- [4] E.V. Ferapontov, *Integration of weakly nonlinear hydrodynamic systems in Riemann invariants*, Phys. Lett. A **158** (1991), no. 3-4, 112–118.
- [5] E.V. Ferapontov, *Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type*, Funct. Anal. Appl. **25** (1991), no. 3, 195–204 (1992).
- [6] E.V. Ferapontov, M.V. Pavlov, *Quasiclassical limit of Coupled KdV equations. Riemann invariants and multi-Hamiltonian structure*, Phys. D **52** (1991), 211-219.
- [7] A. Frölicher, A. Nijenhuis, *Theory of vector-valued differential forms*, Proc. Ned. Acad. Wetensch. Ser. A **59** (1956), 338–359.
- [8] C. Hertling, *Multiplication on the tangent bundle*, arXiv:math/9910116
- [9] C. Hertling, Y. Manin, *Weak Frobenius manifolds*, Internat. Math. Res. Notices **1999**, no. 6, 277–286.
- [10] P. Lorenzoni, *Flat bidifferential ideals and semi-Hamiltonian PDEs*, J. Phys. A **39** (2006), no. 44, 13701–13715.
- [11] P. Lorenzoni, F. Magri, *A cohomological construction of integrable hierarchies of hydrodynamic type*, Int. Math. Res. Not. **2005**, no. 34, 2087–2100.
- [12] P. Lorenzoni, M. Pedroni, A. Raimondo, *F-manifolds and integrable systems of hydrodynamic type*, arXiv:0905.4054.
- [13] F. Magri, *Lenard chains for classical integrable systems*, Theoret. and Math. Phys. **137** (2003), no. 3, 1716–1722.
- [14] Y. Manin, *F-manifolds with flat structure and Dubrovin’s duality*, Adv. Math. **198** (2005), no. 1, 5–26.
- [15] M.V. Pavlov, *Hamiltonian formalism of the equations of electrophoresis. Integrable equations of hydrodynamics*, Preprint No. 17, Landau Inst. Theor. Phys., Acad. Sci. USSR (1987), Moscow.
- [16] M.V. Pavlov, *Hamiltonian formalism of weakly nonlinear systems in hydrodynamics*, Teor. Math. Phys. **73** (1987), No. 2, 316–320.
- [17] M.V. Pavlov, *Elliptic coordinates and multi-Hamiltonian structures of hydrodynamic type systems*, Reports of RAS **339** (1994), No. 1, 21–23.
- [18] M.V. Pavlov, *Integrable hydrodynamic chains*, J. Math. Phys. **44** (2003), no. 9, 4134–4156.

- [19] S.P. Tsarev, *Semihamiltonian formalism for diagonal systems of hydrodynamic type and integrability of the equations of chromatography and isotachopheresis*, LIIAN preprint No. 106 (1989).
- [20] S.P. Tsarev, *The geometry of Hamiltonian systems of hydrodynamic type. The generalised hodograph transform*, USSR Izv. **37** (1991) 397–419.