Correlation function for a periodic box–ball system

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Abstract

We investigate correlation functions in a periodic box–ball system. For the two point functions of short distance, we give explicit formulae obtained by combinatorial methods. We give expressions for general N-point functions in terms of ultradiscrete theta functions.

1 Introduction

Quantum integrable systems such as quantum integrable spin chains and solvable lattice models are systems whose Hamiltonians or transfer matrices can be diagonalised and for which eigenstates or free energies can be explicitly obtained [1]. To investigate physical properties of these systems, such as *e.g.* the linear response to external forces, however, we further need to evaluate correlation functions for these systems. This is one of the main problems in the field of quantum integrable systems and in fact, obtaining correlation functions is even fairly difficult for the celebrated XXZ model or the 6 vertex model [2].

A periodic box-ball system (PBBS) is a soliton cellular automaton obtained by ultradiscretizing the KdV equation [3, 4]. It can also be obtained at the $q \to 0$ limit of the generalized 6 vertex model [5, 6]. Hence, from the view point of quantum integrable lattice models, it is interesting and may actually give some new insights into the correlation functions of the vertex models themselves, to obtain correlation functions of the PBBS. In this paper, we give expressions for N-point functions for the PBBS, using combinatorial methods and the solution for the PBBS expressed in terms of the ultradiscrete theta functions.

The PBBS can be defined in the following way. Let $L \geq 3$ and let $\Omega_L = \left\{ f \,|\, f : [L] \to \{0,1\} \text{ such that } \sharp f^{-1}(\{1\}) < L/2 \right\}$ where $[L] = \{1, 2, \ldots, L\}$. When $f \in \Omega_L$ is represented as a sequence of 0s and 1s, we write

$$f(1)f(2)\ldots f(L).$$

The mapping $T_L : \Omega_L \to \Omega_L$ is defined as follows (see Fig. 1):

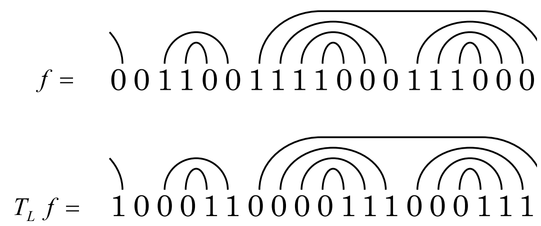

Figure 1: Definition of T_L for $f \in \Omega_L$

1. In the sequence f find a pair of positions n and $n+1$ such that $f(n) = 1$ and $f(n+1) = 0$, and mark them; repeat the same procedure until all such pairs are marked. Note that we always use the convention that the position n is defined in $[L]$, *i.e.* $n + L \equiv n$.

2. Skipping the marked positions we get a subsequence of f; for this subsequence repeat the same process of marking positions, so that we get another marked subsequence.

3. Repeat part 2 until one obtains a subsequence consisting only of 0s. A typical situation is depicted in Fig. 1. After these preparatory processes, change all values at the marked positions simultaneously; One thus obtains the sequence $T_L f$.

Sometimes we shall write $T_L^t f$ for $\underbrace{T_L(\cdots(T_L(T_L}_{t} f)))$. The pair (Ω_L, T_L) is called a PBBS of length L [4, 7]. An element of Ω_L is called a state, and the mapping T_L the time evolution.

An N-point function of the PBBS with M balls may be defined as follows.

$$\langle s_1, s_2, \ldots, s_N \rangle := \frac{1}{Z_H} \sum_{f \in \Omega_{L;M}} \mathrm{e}^{\sum_{k=1}^{L} H_k(f)} f(s_1) f(s_2) \cdots f(s_N)$$

where $\Omega_{L;M} := \left\{ f \in \Omega_L \,\middle|\, \sharp f^{-1}(\{1\}) = M \right\}$, $Z_H := \sum_{f \in \Omega_{L;M}} \mathrm{e}^{\sum_{k=1}^{L} H_k(f)}$ and $H_k(f)$ is the kth energy of the state f, which is proportional to the number of kth arc lines defined when determining the time evolution rule [4], or the kth conserved quantity of the PBBS [8]. (Note that $H_k(f)$ is essentially equal to the energy function for the transfer matrix of the crystal lattice models with $k+1$ states on a vertical link [5, 7].) Noticing the fact that $\Omega_{L;M} = \bigsqcup_Y \Omega_Y$,

$$\langle s_1, s_2, \ldots, s_N \rangle = \frac{1}{Z_H} \sum_{Y} \sum_{f \in \Omega_Y} \mathrm{e}^{\sum_{k=1}^{L} H_k(f)} f(s_1) f(s_2) \cdots f(s_N),$$

where Y are partitions of M corresponding to the conserved quantities of the PBBS. (See Section 2.) Since, for $f_i \in \Omega_{Y_i}$ $(i = 1, 2)$, ${}^{\forall}k$ $H_k(f_1) = H_k(f_2)$ $(k = 1, 2, 3, \ldots)$ implies $Y_1 = Y_2$ and vice versa, by choosing a state f_Y in Ω_Y we can write

$$\langle s_1, s_2, \ldots, s_N \rangle = \frac{1}{Z_H} \sum_{Y} \mathrm{e}^{\sum_{k=1}^{L} H_k(f_Y)} \sum_{f \in \Omega_Y} f(s_1) f(s_2) \cdots f(s_N).$$

Thus, to obtain correlation functions of PBBS, we have only to evaluate those on the set Ω_Y:

$$\langle s_1, s_2, \ldots, s_N \rangle_Y := \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} f(s_1) f(s_2) \cdots f(s_N). \tag{1}$$

We also point out that if we put $^\forall k,\ ^\forall f,\ H_k(f) = 0$, N-point functions become trivial;

$$\langle s_1, s_2, \ldots, s_N \rangle = \frac{{}_{L-N}C_{M-N}}{{}_{L}C_M} = \frac{M(M-1)\cdots(M-N+1)}{L(L-1)\cdots(L-N+1)}.$$

In the following sections we shall evaluate (1).

First we summarize some useful properties of the PBBS. We say that f has (or that there is) a 10-wall at position n if $f(n-1) = 1$ and $f(n) = 0$. Let the number of the 10-walls be s and the positions be denoted by $a_1 > a_2 > \cdots > a_s$. Then, we have the following proposition:

Proposition 1 ([9])

$$\begin{aligned}
(T_L^t f)(n) &= \eta_{n+1}^{t-1} - \eta_{n+1}^{t} - \eta_{n}^{t-1} + \eta_{n}^{t}, \\
\eta_n^t &= \max_{\substack{m_i \in \mathbb{Z} \\ i \in [s]}} \left[\sum_{i=1}^{s} m_i \big(b_i + tW_i - n\big) - \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} m_i \Xi_{ij} m_j \right], & (2) \\
b_i &= a_i + \sum_{j=1}^{i-1} 2\min\{W_i, W_j\} + W_i + \frac{Z_i}{2}, & (3) \\
\Xi_{ij} &= \frac{Z_i}{2}\delta_{ij} + \min\{W_i, W_j\}, \\
Z_i &= L - \sum_{j=1}^{s} 2\min\{W_i, W_j\},
\end{aligned}$$

where W_i denotes the amplitude of the "soliton" corresponding to a_i obtained by the procedure explained in [9].

The set $\{W_i\}_{i=1}^{s}$ consists of quantities of the PBBS and η_n^t is the ultradiscrete theta function [10]. We shall use Proposition 1 for determining N-point functions in Section 3.

Next we introduce two procedures which are important in this paper. For a given $f \in \Omega_L$, a state $Ef = E(f)$ is defined to be

$$(Ef)(n) = \begin{cases}
\begin{cases}
f(n) & (1 \le n \le a_s - 2), \\
f(n+2k) & \begin{pmatrix} a_{s-k+1} - 2k + 1 \le n \le a_{s-k} - 2k - 2 \\ (k = 1, 2, \ldots, s-1) \end{pmatrix}, \\
f(n+2s) & (a_1 - 2s \le n \le L - 2s),
\end{cases} & (a_s > 1) \\[2ex]
\begin{cases}
f(n+1) & (1 \le n \le a_{s-1} - 3), \\
f(n+2k+1) & \begin{pmatrix} a_{s-k} - 2k \le n \le a_{s-k-1} - 2k - 3 \\ (k = 1, 2, \ldots, s-2) \end{pmatrix}, \\
f(n+2s-1) & (a_1 - 2s + 1 \le n \le L - 2s).
\end{cases} & (a_s = 1)
\end{cases}$$

The mapping $E : \Omega_L \to \Omega_{L-2s}$ is called the 10-elimination. Ef is a subsequence of f obtained by eliminating all 10-walls in f simultaneously. For example,

$$\begin{aligned} f &= 001111\underline{10}0001111\underline{10}0000000111\underline{10}01\underline{10}0001111\underline{10}011\underline{10}000111\underline{10}000000000, \\ Ef &= 001111 \quad 0001111 \quad 0000000111 \quad 01 \quad 0001111 \quad 011 \quad 000111 \quad 000000000 \\ &= 0011110001111000000011101000111101100011100000000\text{0}. \end{aligned}$$

Its inverse process is called the 10-insertion, $I(j_1, j_2, \ldots, j_d) = I_2 \circ I_1(j_1, j_2, \ldots, j_d) : \Omega_L \to \Omega_{L+2(d+s)}$ where s is the number of 10-walls in $f \in \Omega_L$. The 10-insertion is defined as follows: Shifting the origin if necessary, we can assume that $f(L) = 0$. For $\{j_1, j_2, \ldots, j_d\}$ $(1 < j_1 < j_2 < \cdots < j_d \leq L + d)$, the mapping $I_1(j_1, j_2, \ldots, j_d) : \Omega_L \to \Omega_{L+2d}$ is defined as

$$\begin{aligned} &(I_1(j_1, j_2, \ldots, j_d)f)(n) \\ &\quad = \begin{cases} 1 & (n = L + 2d - j_k - k + 1), \\ 0 & (n = L + 2d - j_k - k + 2), \\ f(n) & (1 \leq n \leq L + d - j_d), \\ f(n - 2(d - k + 1)) & (L + 2d - j_k - k + 3 \leq n \leq L + 2d - j_{k-1} - k + 1), \\ f(n - 2d) & (L + 2d - j_1 + 2 \leq n \leq L + 2d) \end{cases} \end{aligned}$$

where $k \in [d]$; furthermore, $I_2 : \Omega_{L+2d} \to \Omega_{L+2(d+s)}$ is defined to be

$$(I_2 f')(n) = \begin{cases} 1 & (n = g_k + 2(s - k) + 1), \\ 0 & (n = g_k + 2(s - k) + 2), \\ f'(n) & (1 \leq n \leq g_s), \\ f'(n - 2(s - k + 1)) & (g_k + 2(s - k) + 3 \leq n \leq g_{k-1} + 2(s - k) + 2), \\ f'(n - 2s) & (g_1 + 2s - 2 \leq n \leq L + 2(d + s)) \end{cases}$$

where $k \in [s]$, $f' \equiv I_1(j_1, j_2, \ldots, j_d)f \in \Omega_{L+2d}$ and

$$\begin{aligned} g'_k &= \max\Big\{ m \in [L + d] \,\Big|\, m = a_k - 1 + \sharp\{ r \in [d] \,|\, L + d - j_r + 1 < m \} \Big\}, \\ g_k &= g'_k + \sharp\{ r \in [d] \,|\, L + d - j_r + 1 < g'_k \}. \end{aligned} \tag{4}$$

For example,

$$\begin{aligned} f &= 001110011100000110100011100000\text{0}, \\ I_1(3, 11, 25)f &= 001110011 * 1000001101000 * 1110000 * 00 \\ &= 001110011\underline{10}1000001101000\underline{10}1110000\underline{10}00, \\ I(3, 11, 25)f &= 00111\boxed{10}0011\underline{10}1\boxed{10}0000011\boxed{10}01\boxed{10}000\underline{10}111\boxed{10}00000\underline{10}00 \end{aligned}$$

where $\underline{10}$ and $\boxed{10}$ denote the inserted 10 at $f \mapsto I_1(j_1, j_2, \ldots, j_d)f$ and $I_1(j_1, j_2, \ldots, j_d)f \mapsto I_2(I_1(j_1, j_2, \ldots, j_d)f)$ respectively.

2 One and two point functions obtained by combinatorial methods

We assume that Y denoting the conserved quantities of $f \in \Omega_Y$, is the partition

$$(\underbrace{P_1, P_1, \cdots, P_1}_{n_1}, \underbrace{P_2, P_2, \cdots, P_2}_{n_2}, \ldots \underbrace{P_\ell, P_\ell, \cdots, P_\ell}_{n_\ell})$$

where $P_1 > P_2 > \cdots > P_\ell \geq 1$. Note that Y is a partition of M, *i.e.* $M = \sum_{i=1}^{\ell} n_i P_i$. As mentioned in Section 1, we consider N-point functions (1) of the PBBS,

$$\langle s_1, s_2, \ldots, s_N \rangle_Y = \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} f(s_1) f(s_2) \cdots f(s_N).$$

The value of $|\Omega_Y|$ is already known:

Proposition 2 ([11])

$$\left|\Omega_Y\right| = \frac{L}{L_0} \binom{L_0 + n_1 - 1}{n_1} \binom{L_1 + n_2 - 1}{n_2} \cdots \binom{L_{\ell-1} + n_\ell - 1}{n_\ell}$$

where $L_0 = L - 2M,\ L_i = L_0 + \sum_{j=1}^{i} 2n_j(P_j - P_{i+1})$ *and* $P_{\ell+1} = 0$.

Since the N-point function $\langle s_1, s_1 + d_1, \ldots, s_1 + d_{N-1} \rangle_Y$ does not depend on the specific site s_1 (because of translational symmetry), we denote

$$C_Y(d_1, d_2, \ldots, d_{N-1}) \equiv \langle s_1, s_1 + d_1, \ldots, s_1 + d_{N-1} \rangle_Y$$

where $1 \leq d_1 < d_2 < \cdots < d_{N-1} < L$. Note that $C_Y(\emptyset)$ denotes the 1-point function $\langle s_1 \rangle_Y$.

Proposition 3

$$C_Y(\emptyset) = \frac{M}{L}.$$

Proof Since $\sum_{n=1}^{L} f(n) = M$,

$$LC_Y(\emptyset) = \sum_{s_1=1}^{L} \langle s_1 \rangle_Y = \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} \sum_{n=1}^{L} f(n) = \frac{1}{|\Omega_Y|} |\Omega_Y| M = M.$$

□

Next we consider the 2-point functions.

Proposition 4

$$C_Y(1) = \frac{M - s}{L}$$

where $s = \sum_{i=1}^{\ell} n_i$.

Proof Since $\sum_{n=1}^{L} f(n)f(n+1) = M - s$,

$$LC_Y(1) = \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} \sum_{n=1}^{L} f(n)f(n+1) = M - s.$$

$\square$

In order to investigate $C_Y(2)$, let us put

$$\begin{aligned} k_i &:= \begin{cases} n_j & (i = P_j), \\ 0 & \text{otherwise}, \end{cases} \\ \hat{k}_i &:= \sum_{j=i}^{P_1} k_i, \\ \tilde{L} &:= L - 2\hat{k}_1 \quad (= L - 2s), \\ N_Y(2) &:= \sum_{i=3}^{P_1} k_i(i-2). \end{aligned}$$

We also define

$$\begin{aligned} V_{f_0} &:= \left\{ f \in \Omega_Y \,\middle|\, Ef = f_0 \right\}, \\ G_2(f) &:= \sharp \left\{ n \in [L] \,\middle|\, f(n)f(n+2) = 1 \right\}. \end{aligned}$$

The following lemma is the key to evaluating $C_Y(2)$.

Lemma 1
Let

$$V_{f_0}^{(j)} := \left\{ f \in V_{f_0} \,\middle|\, G_2(f) = N_Y(2) + j \right\}.$$

Then, if $V_{f_0} \neq \phi$, $V_{f_0} = \bigsqcup_{j=0}^{k_1} V_{f_0}^{(j)}$ *and*

$$\left| V_{f_0}^{(k_1 - j)} \right| = \frac{\nu_j}{k_1!} \tag{5}$$

where

$$\begin{aligned} \nu_j \quad := \quad & \left(\prod_{i=0}^{j-1} (\tilde{L} - 2\hat{k}_2 - i) \right) \left(\prod_{i=0}^{k_1+j-1} (2\hat{k}_2 + i) \right) \\ & \times \left(\sum_{\substack{0 \le i_1 < \cdots < i_j < k_1+j-1 \\ i_h+1 < i_{h+1}}} \prod_{h=1}^{j} \frac{1}{(2\hat{k}_2 + i_h)(2\hat{k}_2 + i_h + 1)} \right). \end{aligned}$$

Proof When $f \in V_{f_0}$, there exists a set of positive numbers $\{j_i\}_{i=1}^{k_1}$ $(1 < j_1 < j_2 < \ldots < j_{k_1} \le \tilde{L} + k_1)$ such that

$$f = I(j_1, j_2, \ldots, j_{k_1}) f_0.$$

By examining the positions of 101 and 111, we find that

$$G_2(f) = N_Y(2) + \gamma + J$$

where $\gamma = \gamma\big(f_0; \{j_i\}_{i=1}^{k_1}\big)$ is the number of $\underline{10}$s inserted into the positions adjacent to consecutive 1s, and $J = \sharp\big\{\, i \in [d-1] \,\big|\, j_i + 1 = j_{i+1} \,\big\}$. (See the table below.) For example,

$$f_0 = 001110000100110000$$

and $f = I(5, 6, 14, 15, 18)f_0$, then

$$\begin{aligned} f &= 00111101000101000110001110101000000 \\ &(= 00111\boxed{10}\underline{10}000\underline{10}\,\underline{10}0001\boxed{10}00111\boxed{10}\underline{10}\,\underline{10}0000\ ). \end{aligned}$$

In this example, $k_1 = 5$, $\hat{k}_2 = 3$, $N_Y(2) = 3$, $\gamma = 2$ and $J = 2$. Since $0 \le \gamma + J \le k_1$, we have the decomposition $V_{f_0} = \bigsqcup_{j=0}^{k_1} V_{f_0}^{(j)}$.

f_0	00111000			
$G_2(f_0)$	1			
$f = I(k)f_0$	$00\underline{10}111\boxed{10}000$ $(k=7)$	$001\underline{10}11\boxed{10}000$ $(k=6)$	$00111\boxed{10}\underline{10}000$ $(k=4)$	$00111\boxed{10}00\underline{10}0$ $(k=2)$
$G_2(f)$	3 $(\gamma = 1,\ J = 0)$	2 $(\gamma = 0,\ J = 0)$	3 $(\gamma = 1,\ J = 0)$	2 $(\gamma = 0,\ J = 0)$

To know $\left|V_{f_0}^{(j)}\right|$, we have only to count the number of states with $\gamma + J = j$.

For $k_1 = 1$, $\left|V_{f_0}\right| = \tilde{L}$. Since there are $\hat{k}_2$ sets of consecutive 1s, $2\hat{k}_2$ states have $\gamma + J = 1$ $(\gamma = 1,\ J = 0)$ and the other $\tilde{L} - 2\hat{k}_2$ states have $\gamma + J = 0$ $(\gamma = 0,\ J = 0)$.

For $k_1 = 2$, let $f = I(j_1, j_2)f_0$. As was seen in case $k_1 = 1$, there are $2\hat{k}_2$ positions at which $\gamma+J$ can be increased by one. If one 10 pair is inserted in one of these positions, then there are $2\hat{k}_2+1$ positions for the other pair to increase $\gamma+J$ by one, and $\tilde{L}-2\hat{k}_2$ positions not to increase it. On the other hand, if one 10 pair is inserted at one of the $\tilde{L}-2\hat{k}_2$ non-increasing positions, then there are $2\hat{k}_2+2$ positions for the other pair to increase $\gamma+J$ by one, and $\tilde{L}-2\hat{k}_2-1$ positions not to increase it. Hence, considering duplication of insertion, there are $(2\hat{k}_2)(2\hat{k}_2+1)/2!$ states with $\gamma + J = 2$, $\left[(2\hat{k}_2)(\tilde{L} - 2\hat{k}_2) + (\tilde{L} - 2\hat{k}_2)(2\hat{k}_2 + 2)\right]/2!$ states with $\gamma + J = 1$, and $(\tilde{L} - 2\hat{k}_2)(\tilde{L} - 2\hat{k}_2 - 1)/2!$ states with $\gamma + J = 0$.

In general, we can proceed in a similar manner and, referring to the chart in Fig. 2, we obtain (5). □

Proposition 5

$$C_Y(2) = \frac{\displaystyle\sum_{j=0}^{k_1} \nu_j \left(\sum_{i=3}^{P_1} k_i(i-2) + (k_1 - j) \right)}{\displaystyle L \sum_{j=0}^{k_1} \nu_j}.$$

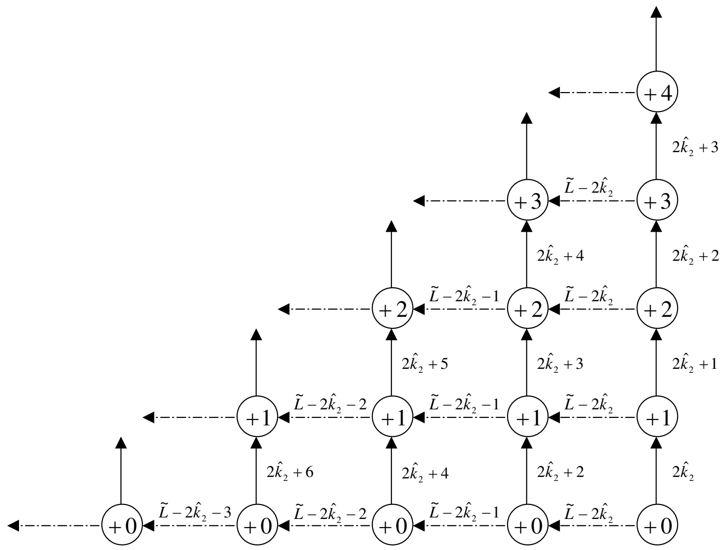

Figure 2: A chart corresponding to $\gamma + J$ in the proof of Lemma 1.

Proof From Lemma 1, we see that if $V_{f_0} \neq \phi$,

$$\sum_{f \in V_{f_0}} \sum_{n=1}^{L} f(n)f(n+2) = \sum_{j=0}^{k_1} \frac{\nu_j}{k_1!}\Big(N_Y(2) + (k_1 - j)\Big)$$

and

$$\left|V_{f_0}\right| = \sum_{j=0}^{k_1} \frac{\nu_j}{k_1!}.$$

Since the right hand side of the last equation does not depend on f_0, and since any state $f \in \Omega_Y$ belongs to some V_{f_0}, we obtain

$$LC_Y(2) = \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} \sum_{n=1}^{L} f(n)f(n+2) = \frac{\displaystyle\sum_{j=0}^{k_1} \nu_j \Big(N_Y(2) + (k_1 - j)\Big)}{\displaystyle\sum_{j=0}^{k_1} \nu_j}.$$

$\square$

For $C_Y(d)$ $(d \geq 3)$ we can use similar arguments based on elementary combinatorics. However, the expressions become more and more complicated when the difference d increases. Instead in the next section we shall use Proposition 1 to obtain expressions for general N-point functions.

3 N-point correlation functions for the PBBS

Let the state f_0 and the set $\mathcal{X}_Y \subset \mathbb{Z}_+^{n_1} \times \mathbb{Z}_+^{n_2} \times \cdots \times \mathbb{Z}_+^{n_\ell}$ $(= \mathbb{Z}_+^s)$ be

$$f_0 = \underbrace{000\cdots 00}_{L_0},$$

and

$$\mathcal{X}_Y := \left\{ \left\{x_i(k)\right\}_{i=1,k=1}^{\ell,\ \ n_i} \ \middle|\ \begin{array}{r} 1 < x_i(1) < x_i(2) < \cdots < x_i(n_i) \le L_{i-1} + n_i \\ (i = 1, 2, \ldots, \ell) \end{array} \right\}. \tag{6}$$

We define the state f_X recursively as

$$\begin{aligned} f_j &:= \underbrace{I(\emptyset)\cdots I(\emptyset)}_{P_{\ell-j+1}-P_{\ell-j+2}-1}\ I(X_j) f_{j-1} \quad (\, j = 1, 2, \ldots, \ell\,), \\ f_X &:= f_\ell \end{aligned}$$

where $X_j = \left\{x_j(k)\right\}_{k=1}^{n_j} \subset X \in \mathcal{X}_Y$. Note that, from the definition of an 10-insertion, $I(\emptyset)$ is the procedure needed to insert $\boxed{10}$s between 10:

$$\begin{aligned} f &= 00111001110000011010001110000000, \\ I(\emptyset) f &= 00111\boxed{10}00111\boxed{10}0000011\boxed{10}01\boxed{10}000111\boxed{10}000000 \end{aligned}$$

and $f_X \in \Omega_Y$ by construction. We also define $\tilde{\Omega}_Y$ by

$$\tilde{\Omega}_Y := \left\{ f_X \,\middle|\, X \in \mathcal{X}_Y \right\}.$$

Lemma 2

$$\langle s_1, s_2, \ldots, s_N \rangle_Y = \frac{1}{L|\tilde{\Omega}_Y|} \sum_{f \in \tilde{\Omega}_Y} \sum_{k=1}^{L} f(k+s_1) f(k+s_2) \cdots f(k+s_N). \tag{7}$$

Proof By virtue of the definition of f_X, $\tilde{\Omega}_Y$ is the set of states with conserved quantities Y and the last entry of the 10 sequence is one of the 0s that are not marked in the time evolution rule, *i.e.*, $f_X(L) = (T_L f_X)(L) = 0$. By defining the shift operator S by $(Sf)(n) := f(n+1)$, and $(S^k f) := S(S^{k-1} f)$ $(k = 1, 2, \ldots)$ with $S^0 f := f$ and for sets

$$S^k \tilde{\Omega}_Y := \left\{ S^k f_X \,\middle|\, X \in \mathcal{X}_Y \right\} \qquad (k = 1, 2, \ldots, L),$$

we find

$$^\forall f \in \Omega_Y, \quad \sharp \left\{ k \,\middle|\, f \in S^k \tilde{\Omega}_Y \ (k = 1, 2, \ldots, L) \right\} = L_0.$$

Note that $S^L f = f$. Since $|\Omega_Y| = \frac{L}{L_0}|\tilde{\Omega}_Y|$,

$$
\begin{aligned}
\langle s_1, s_2, \ldots, s_N \rangle_Y &= \frac{1}{|\Omega_Y|} \sum_{f \in \Omega_Y} f(s_1) f(s_2) \cdots f(s_N) \\
&= \frac{1}{L|\tilde{\Omega}_Y|} \sum_{k=1}^{L} \sum_{f \in S^k \tilde{\Omega}_Y} f(s_1) f(s_2) \cdots f(s_N) \\
&= \frac{1}{L|\tilde{\Omega}_Y|} \sum_{k=1}^{L} \sum_{f \in \tilde{\Omega}_Y} f(s_1 + k) f(s_2 + k) \cdots f(s_N + k).
\end{aligned}
$$

Thus we obtain (7). □

Proposition 6
For $X \in \mathcal{X}_Y$, f_X is explicitly given as

$$
f_X(n) = u_n^0(X)
$$

where

$$
\begin{aligned}
u_n^t(X) &:= \eta_{n+1}^{t-1}(X) - \eta_{n+1}^{t}(X) - \eta_n^{t-1}(X) + \eta_n^t(X), \\
\eta_n^t(X) &:= \max_{\substack{m_{ij} \in \mathbb{Z}, \\ i \in [\ell];\ j \in [n_i]}} \Bigg[\sum_{i=1}^{\ell} \sum_{k=1}^{n_i} m_{ik} \big(tP_i - n - x_i(k) + L + k + 1 + \frac{Z_i}{2} \big) \\
&\qquad - \sum_{i=1}^{\ell} \sum_{k=1}^{n_i} \sum_{j=1}^{\ell} \sum_{h=1}^{n_j} m_{ik} \Xi_{ikjh} m_{jh} \Bigg], \qquad (8) \\
\Xi_{ikjh} &:= \frac{Z_i}{2} \delta_{ij} \delta_{kh} + P_{\max[i,j]}, \\
Z_i &:= L - 2 \left(P_i \sum_{j=1}^{i} n_j + \sum_{j=i+1}^{\ell} n_j P_j \right).
\end{aligned}
$$

Proof From Proposition 1, f_X is determined by the parameters W_n and a_n ($n = 1, 2, \ldots, s$). Here W_n is the amplitude of the nth soliton and a_n is its position, *i.e.* the position of the nth 10-wall, counting from the right. From the definition of the position and of the amplitude of a soliton, it follows that both can be determined from 10 insertions. Because of the way f_X was constructed, the set $\{x_j(k)\}_{k=1}^{n_j}$ corresponds to the position of n_j solitons with amplitude P_j, though it does not directly gives their position. Hereafter we shall refer to a soliton with amplitude P as a P-soliton. By considering the relation between the position of a soliton and 10-insertions, we find that the position of the kth P_j-soliton counting from the right is $L - x_j^{(\ell)}(k) + 2$, where $x_j^{(\ell)}(k)$ is determined recursively: we define $x_j^{(i)}(k)$ $\big(i \in [\,\ell\,],\ j \in [\,i\,],\ k \in [n_j]\big)$ as

$$\begin{aligned} x_j^{(i)}(k) &:= x_j(k) + (P_j - P_{i+1})(2\beta_j(k) + 2k - 1) \\ &\qquad + \sum_{s=j+1}^{i} 2(P_s - P_{i+1})\alpha_j^{(s)}(k) - k + 1 \end{aligned}$$

where

$$\alpha_j^{(i)}(k) := \sharp\big\{\, r \in [n_i] \,\big|\, L_{i-1} + n_i - x_i(r) + 1 > g_j^{(i)}(k) \,\big\},$$

$$\beta_1(k) := 0, \qquad \beta_i(k) := \sum_{s=1}^{i-1} \sharp\Big\{\, r \in [n_s] \,\Big|\, g_s^{(i)}(r) > L_{i-1} + n_i - x_i(k) + 1 \,\Big\},$$

$$g_j^{(i)}(k) := \max\left\{ m \in [L_{i-1} + n_i] \;\middle|\; \begin{array}{l} m = L_{i-1} - x_j^{(i-1)}(k) + 1 \\ \qquad + \sharp\big\{\, r \in [n_i] \,\big|\, L_{i-1} + n_i - x_i(r) + 1 < m \,\big\} \end{array} \right\}.$$

Note that $x_j^{(i)}(1) < x_j^{(i)}(2) < \cdots < x_j^{(i)}(n_j)$.

Recalling the fact that $\sharp\big\{\, r \in [d] \,\big|\, L + d - j_r + 1 < g'_k \,\big\}$ in (4) is the number of inserted $\underline{10}$s, on the left of the kth soliton (here we do not count the inserted $\underline{10}$s as solitons), the concrete meaning of these variables becomes clear: $\alpha_j^{(i)}(k)$ denotes the number of P_i-solitons on the right of the kth P_j-soliton, and $\beta_j(k)$ denotes the number of solitons with amplitudes less than P_j, to the right of the kth P_j-soliton.

Since $\{L - x_j^{(\ell)}(k) + 2\}_{j=1,k=1}^{\ell,\quad n_j}$ is the complete set of positions of the solitons, there exists a one to one mapping $\rho : \big\{\, (j,k) \,\big|\, j \in [\,\ell\,],\, k \in [n_j] \,\big\} \to [\,s\,]$ such that

$$a_{\rho(j,k)} = L - x_j^{(\ell)}(k) + 2.$$

From these recursion relations we have

$$\begin{aligned} x_j^{(\ell)}(k) &= x_j(k) + P_j\big(2\beta_j(k) + 2k - 1\big) + \sum_{i=j+1}^{\ell} 2P_i\alpha_j^{(i)}(k) - k + 1 \\ &= x_j(k) + 2\left\{ P_j\big(\beta_j(k) + (k-1)\big) + \sum_{i=j+1}^{\ell} P_i\alpha_j^{(i)}(k) \right\} + P_j - k + 1. \end{aligned}$$

Since the position of the kth P_j-soliton is $a_{\rho(j,k)}$, $W_{\rho(j,k)} = P_j$ and the set of amplitudes of the solitons on the right of the kth P_j-soliton is nothing but $\big\{W_h\big\}_{h=1}^{\rho(j,k)-1}$. From the definition of $\alpha_j^{(i)}(k)$, $\beta_j(k)$,

$$\begin{aligned} \alpha_j^{(i)}(k) &= \sharp\left\{ W \in \big\{W_h\big\}_{h=1}^{\rho(j,k)-1} \,\middle|\, W = P_i \right\}, \\ \beta_j(k) &= \sharp\left\{ W \in \big\{W_h\big\}_{h=1}^{\rho(j,k)-1} \,\middle|\, W > P_j \right\} \end{aligned}$$

and

$$\sharp\left\{ W \in \big\{W_h\big\}_{h=1}^{\rho(j,k)-1} \,\middle|\, W = P_j \right\} = k - 1.$$

Thus we obtain

$$x_j^{(\ell)}(k) = x_j(k) + \sum_{h=1}^{\rho(j,k)-1} 2\min\big\{W_{\rho(j,k)}, W_h\big\} + W_{\rho(j,k)} - k + 1.$$

Therefore we find a concrete expression of $a_{\rho(j,k)}$, and (8) is immediately obtained from (2) and (3). $\square$

From Lemma 2 and Proposition 6, we immediately obtain the following theorem:

Theorem 1
Let $\mathcal{X}$ be the set defined in (6) *we have*

$$C_Y(d_1, d_2, \ldots, d_{N-1}) = \frac{1}{L|\mathcal{X}|} \sum_{X \in \mathcal{X}} \sum_{n=1}^{L} u_n(X) \prod_{i=1}^{N-1} u_{n+d_i}(X),$$

for $u_n(X) \equiv u_n^0(X)$ as given in (8).

4 Concluding remarks

In this article, we investigated correlation functions for the PBBS and obtained explicit forms for 1-point and 2-point functions at short distances. We also give expressions in terms of ultradiscrete theta functions for general N-point functions. Investigating their asymptotic properties and to clarify the relation to correlation functions for quantum integrable systems are problems that will be addressed in the future.

Finally we should comments on the time averages of quantities in the PBBS. The time average:

$$C_f(d_1, d_2, \ldots, d_{N-1}) = \frac{1}{L|\mathcal{T}_f|} \sum_{t=1}^{\mathcal{T}_f} \sum_{n=1}^{L} (T_L^t f)(n) \prod_{j=1}^{N-1} (T_L^t f)(n + d_j)$$

where $\mathcal{T}_f$ is the fundamental cycle of $f \in \Omega_L$ depends not only on the conserved quantities of the state but, in general, also on the initial state f itself. For example, the conserved quantities of the states $f_1 = 0100100$ and $f_2 = 0101000$ are the same, but $C_{f_1}(3) = \frac{1}{7}$ and $C_{f_2}(3) = 0$. Hence, in general, $C_f(d_1, d_2, \ldots, d_{N-1}) \neq C_Y(d_1, d_2, \ldots, d_{N-1})$ even for $f \in \Omega_Y$. Note that, for the 1-point function $C_f(\emptyset)$, we can easily show that

$$^{\forall} f \in \Omega_Y, \quad C_f(\emptyset) = C_Y(\emptyset) = \frac{M}{L}.$$

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A Example of values for the correlation function

From Theorem 1, we obtain the following examples.

(a) $L = 12;\ P_1 = 3,\ n_1 = 1;\ P_2 = 1,\ n_2 = 2:$

$$C_Y(\emptyset) = \frac{5}{12}, \quad C_Y(1) = \frac{1}{6}, \quad C_Y(2) = \frac{13}{84}, \quad C_Y(3) = \frac{19}{126}, \quad C_Y(1,2) = \frac{5}{84};$$

(b) $L = 14;\ P_1 = 2,\ n_1 = 2;\ P_2 = 1,\ n_2 = 2:$

$$C_Y(\emptyset) = \frac{3}{7}, \quad C_Y(1) = \frac{1}{7}, \quad C_Y(2) = \frac{5}{49}, \quad C_Y(3) = \frac{82}{441}, \quad C_Y(1,2) = 0;$$

(c) $L = 14;\ P_1 = 3,\ n_1 = 1;\ P_2 = 1,\ n_2 = 3:$

$$C_Y(\emptyset) = \frac{3}{7}, \quad C_Y(1) = \frac{1}{7}, \quad C_Y(2) = \frac{5}{28}, \quad C_Y(3) = \frac{69}{392}, \quad C_Y(1,2) = \frac{5}{112};$$

(d) $L = 14;\ P_1 = 3,\ n_1 = 1;\ P_2 = 2,\ n_2 = 1;\ P_2 = 1,\ n_2 = 1:$

$$C_Y(\emptyset) = \frac{3}{7}, \quad C_Y(1) = \frac{3}{14}, \quad C_Y(2) = \frac{3}{28}, \quad C_Y(3) = \frac{13}{112}, \quad C_Y(1,2) = \frac{1}{16}.$$

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