Wigner functions of thermo number state, photon subtracted and added thermo vacuum state at finite temperature

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Based on Takahashi-Umezawa thermo field dynamics and the order-invariance of Weyl ordered operators under similar transformations, we present a new approach to deriving the exact Wigner functions of thermo number state, photon subtracted and added thermo vacuum state. We find that these Wigner functions are related to the Gaussian-Laguerre type functions of temperature, whose statistical properties are then analysed.

I.   INTRODUCTION

In recent years photon subtracted and added quantum states have been paid much attention because these fields exhibit an abundant of nonclassical properties and may give access to a complete engineering of quantum states and to fundamental quantum phenomena [1-8]. However, all these discussions are restricted to the case at zero point temperature. In fact, most systems are not isolated, but are immersed in a "thermal reservoir", excitation and de-excitation processes of a system are influenced by its energy exchange with reservoirs. In this work we study field properties by photon subtracting and adding at finite temperature.

The Wigner function (WF) is a powerful tool to investigate the nonclassicality of optical fields [9,10]. Its partial negativity implies the highly nonclassical properties of quantum states and is often used to describe the decoherence of quantum states [7,8,11,12]. In one dimensional case, the WF of a density matrix ρ is defined as $\text{Tr}\left[\rho\Delta(\alpha)\right]$, where $\Delta(\alpha)$ is the single-mode Wigner operator, whose normally ordered form and Weyl ordered form are given as [13-15], respectively,

$$\Delta\left(\alpha\right) = \frac{1}{\pi} : e^{-(q-Q)^2-(p-P)^2} : \; = \frac{1}{\pi} : e^{-2(\alpha-a)\left(\alpha^*-a^\dagger\right)} : \, , \tag{1}$$

and

$$\Delta\left(\alpha\right) = \frac{1}{2} \genfrac{}{}{0pt}{}{:}{:} \delta\left(\alpha-a\right)\delta\left(\alpha^*-a^\dagger\right) \genfrac{}{}{0pt}{}{:}{:}, \tag{2}$$

where $\alpha = (q+\mathrm{i}p)/\sqrt{2}$, $a = (Q+\mathrm{i}P)/\sqrt{2}$, $[Q,P]=\mathrm{i}$, $\hbar=1$; a and $a^\dagger$ ($\left[a,a^\dagger\right]=1$) are Bose annihilation and creation operators, the symbols $:\;:$ and $\genfrac{}{}{0pt}{}{:}{:}$ denote the normal ordering and the Weyl ordering, respectively. Our main aim is to provide a new and direct approach to deriving the WFs of quantum states at finite temperature by using the order-invariance of Weyl ordered operators under similar transformations [13-15], which means

$$S \genfrac{}{}{0pt}{}{:}{:} (\circ\circ\circ) \genfrac{}{}{0pt}{}{:}{:} S^{-1} = \genfrac{}{}{0pt}{}{:}{:} S (\circ\circ\circ) S^{-1} \genfrac{}{}{0pt}{}{:}{:}, \tag{3}$$

as if the "fence" $\genfrac{}{}{0pt}{}{:}{:}$ did not exist, so S can pass through it. We also appeal to the Takahashi-Umezawa thermo field dynamics (TFD) [16-18], we consider it convenient to obtaining the explicit expressions of WFs.

II.   BRIEF REVIEW OF THERMO STATE

The main point of TFD lies in converting the evaluation of ensemble average at nonzero temperature into the equivalent expectation value with a pure state. This worthwhile convenience is at the expense of introducing a fictitious field (or a so-called tilde-conjugate field, denoted as operator $\tilde{a}^\dagger$) in the extending Hilbert space $\tilde{H}$, thus the original optical field state $|n\rangle$ in the Hilbert space $\mathcal{H}$ is accompanied by a tilde state $|\tilde{n}\rangle$ in $\tilde{H}$. A similar rule holds for operators: every annihilation operator a acting on $\mathcal{H}$ has an image $\tilde{a}$ acting on $\tilde{H}$. At finite temperature T the thermal vacuum

$|0(\beta)\rangle$ is defined by the requirement that the vacuum expectation value agrees with the statistical average [16-18], i.e.

$$\langle A \rangle = \text{Tr}\,(\rho_c A) = \langle 0(\beta)|\, A \,|0(\beta)\rangle = \text{Tr}\,\left(A e^{-\beta H}\right)/\text{Tr}\,\left(e^{-\beta H}\right), \tag{4}$$

where $\beta = \frac{1}{kT}$, k is the Boltzmann constant and H is the system's Hamiltonian. For the ensemble of free bosons with Hamiltonian $H_0 = \omega a^\dagger a$, the thermal vacuum state $|0(\beta)\rangle$ is

$$|0(\beta)\rangle = \text{sech}\theta \exp\left[a^\dagger \tilde{a}^\dagger \tanh\theta\right]|0,\tilde{0}\rangle = S(\theta)\,|0,\tilde{0}\rangle, \tag{5}$$

where $|0,\tilde{0}\rangle$ is annihilated by a and $\tilde{a}$, $\left[\tilde{a}, \tilde{a}^\dagger\right] = 1$, and

$$S(\theta) \equiv \exp\left[\theta\left(a^\dagger \tilde{a}^\dagger - a\tilde{a}\right)\right], \tag{6}$$

is the thermo squeezing operator which transforms the zero-temperature vacuum $|0,\tilde{0}\rangle$ into the thermo vacuum state $|0(\beta)\rangle$, and θ is related to the Bose distribution by

$$\tanh\theta = \exp\left(-\frac{\omega}{2kT}\right), \tag{7}$$

which is determined by comparing the Bose–Einstein distribution

$$n_c = \left[\exp\left(\frac{\omega}{kT}\right) - 1\right]^{-1} \tag{8}$$

and

$$\langle 0(\beta)|\, a^\dagger a \,|0(\beta)\rangle = \sinh^2\theta. \tag{9}$$

In particular, when operator A is the Wigner operator $\Delta(\alpha)$ itself, it is easy to see that

$$\begin{aligned}
\text{Tr}_a\left(\Delta(\alpha)\,e^{-\beta H}\right)/\text{Tr}_a\left(e^{-\beta H}\right) &= \langle 0(\beta)|\,\Delta(\alpha)\,|0(\beta)\rangle \\
&= \text{Tr}_{a,\tilde{a}}\left[\Delta(\alpha)\,|0(\beta)\rangle\,\langle 0(\beta)|\right],
\end{aligned} \tag{10}$$

which is just the WF of thermo vacuum state. From Eq.(10) one can see that the calculation of WF for thermo states is converted into the expectation value of Wigner operator in themo vacuum state $|0(\beta)\rangle$ ($\rho_c \rightarrow |0(\beta)\rangle\,\langle 0(\beta)|$), which is defined in the enlarged Fock space. This implies that it is convenient to deriving some WFs of density operators at finite temperature by doubly enlarging the original space.

III. NORMALLY ORDERED FORM OF $S^\dagger(\theta)\,\Delta(\alpha)\,S(\theta)$

In order to deriving conveniently the WFs of density operators at finite temperature, let's first calculate the normally ordered form of $S^\dagger(\theta)\,\Delta(\alpha)\,S(\theta)$. Recalling that for single-mode case the Weyl rule [13-15] is defined as

$$\hat{H}\left(a, a^\dagger\right) = 2\int \text{d}^2\alpha\, h\left(\alpha, \alpha^*\right)\Delta(\alpha), \tag{11}$$

where $h\left(\alpha, \alpha^*\right)$ is the classical function corresponding to operator $\hat{H}\left(a, a^\dagger\right)$. Eq.(11) expresses the Weyl correspondence rule, using (2) it can be expressed as

$$\begin{aligned}
\hat{H}\left(a, a^\dagger\right) &= \int \text{d}^2\alpha\, h\left(\alpha, \alpha^*\right) \vdots \delta\left(\alpha - a\right)\delta\left(\alpha^* - a^\dagger\right) \vdots \\
&= \vdots h\left(a, a^\dagger\right) \vdots,
\end{aligned} \tag{12}$$

which means that Weyl ordered of operator $\vdots h\left(a, a^\dagger\right)\vdots$, whose Weyl correspondence is $h\left(\alpha, \alpha^*\right)$, can be obtained by just respectively replacing α, α^* in $h\left(\alpha, \alpha^*\right)$ by a and $a^\dagger$ without disturbing the form of function h.

According to the Weyl ordering invariance under similar transformations [13] and the following transform relation

$$
\begin{aligned}
S^\dagger(\theta)\, a\, S(\theta) &= a\cosh\theta + \tilde{a}^\dagger \sinh\theta, \\
S^\dagger(\theta)\, \tilde{a}\, S(\theta) &= \tilde{a}\cosh\theta + a^\dagger \sinh\theta,
\end{aligned}
\tag{13}
$$

it is easily seen

$$
\begin{aligned}
S^\dagger(\theta)\,\Delta(\alpha)\,S(\theta) = \frac{1}{2}\,\overset{:}{\underset{:}{}}\,\delta\left(\alpha - a\cosh\theta - \tilde{a}^\dagger\sinh\theta\right) \\
\times\,\delta\left(\alpha^* - a^\dagger\cosh\theta - \tilde{a}\sinh\theta\right)\,\overset{:}{\underset{:}{}},
\end{aligned}
\tag{14}
$$

which is just the Weyl ordering of $S^\dagger(\theta)\,\Delta(\alpha)\,S(\theta)$ in the enlarged Fock space. Based on the Weyl rule, the classical correspondence $h\left(\beta,\beta^*;\tilde{\beta},\tilde{\beta}^*\right)$ of the operator $S^\dagger(\theta)\,\Delta(\alpha)\,S(\theta)$ can be obtained by replacing $(a,a^\dagger)$ and $(\tilde{a},\tilde{a}^\dagger)$ with (β,β^*) and $(\tilde{\beta},\tilde{\beta}^*)$, respectively, i.e.,

$$
\begin{aligned}
h\left(\beta,\beta^*;\tilde{\beta},\tilde{\beta}^*\right) = \frac{1}{2}\delta\left(\alpha - \beta\cosh\theta - \tilde{\beta}^*\sinh\theta\right) \\
\times\,\delta\left(\alpha^* - \beta^*\cosh\theta - \tilde{\beta}\sinh\theta\right).
\end{aligned}
\tag{15}
$$

It then follows from Eqs.(11) and (15) that

$$
S^\dagger(\theta)\,\Delta(\alpha)\,S(\theta) = 4\int \mathrm{d}^2\beta\,\mathrm{d}^2\tilde{\beta}\,\Delta\left(\beta,\beta^*;\tilde{\beta},\tilde{\beta}^*\right) h\left(\beta,\beta^*;\tilde{\beta},\tilde{\beta}^*\right),
\tag{16}
$$

where $\Delta\left(\beta,\beta^*;\tilde{\beta},\tilde{\beta}^*\right)$ is the two-mode Wigner operator, whose normally ordering form is

$$
\Delta\left(\beta,\beta^*;\tilde{\beta},\tilde{\beta}^*\right) = \frac{1}{\pi^2}:\exp\left[-2\left(a^\dagger - \beta^*\right)\left(a - \beta\right) - 2\left(\tilde{a}^\dagger - \tilde{\beta}^*\right)\left(\tilde{a} - \tilde{\beta}\right)\right]:.
\tag{17}
$$

On substituting Eq.(17) into Eq.(16) and using the integral formula [19]

$$
\int \frac{\mathrm{d}^2 z}{\pi} e^{\zeta|z|^2 + \xi z + \eta z^*} = -\frac{1}{\zeta}e^{-\frac{\xi\eta}{\zeta}},\ \ \mathrm{Re}\left(\zeta\right) < 0,
\tag{18}
$$

we can derive the normally ordered form of (16) as follows

$$
\begin{aligned}
S^\dagger(\theta)\,\Delta(\alpha)\,S(\theta) &= 2\int \frac{\mathrm{d}^2\beta\,\mathrm{d}^2\tilde{\beta}}{\pi^2}\delta\left(\alpha - \beta\cosh\theta - \tilde{\beta}^*\sinh\theta\right) \\
&\quad\times\delta\left(\alpha^* - \beta^*\cosh\theta - \tilde{\beta}\sinh\theta\right) \\
&\quad\times:\exp\left[-2\left(a^\dagger - \beta^*\right)\left(a - \beta\right) - 2\left(\tilde{a}^\dagger - \tilde{\beta}^*\right)\left(\tilde{a} - \tilde{\beta}\right)\right]: \\
&= \frac{\mathrm{sech}2\theta}{\pi}e^{-2|\alpha|^2\mathrm{sech}2\theta}:\exp\left\{-\left(a\tilde{a} + a^\dagger\tilde{a}^\dagger\right)\tanh 2\theta\right. \\
&\quad+2\mathrm{sech}2\theta\left[\sinh\theta\left(\alpha^*\tilde{a}^\dagger + \alpha\tilde{a}\right) + \cosh\theta\left(\alpha^* a + \alpha a^\dagger\right)\right. \\
&\quad\left.\left.-\left(\tilde{a}^\dagger\tilde{a}\sinh^2\theta + a^\dagger a\cosh^2\theta\right)\right]\right\}:,
\end{aligned}
\tag{19}
$$

which is just the normally ordered form of (16). Eq.(19) directly leads to the WF of the thermo vacuum state $|0(\beta)\rangle$,

$$
\begin{aligned}
\langle 0(\beta)|\,\Delta(\alpha)\,|0(\beta)\rangle &= \langle 0,\tilde{0}|\,S^\dagger(\theta)\,\Delta(\alpha)\,S(\theta)\,|0,\tilde{0}\rangle = \frac{\mathrm{sech}2\theta}{\pi}e^{-2|\alpha|^2\mathrm{sech}2\theta} \\
&= \frac{1 - e^{-\beta\omega}}{\pi(1 + e^{-\beta\omega})}e^{-2|\alpha|^2\frac{1 - e^{-\beta\omega}}{1 + e^{-\beta\omega}}}.
\end{aligned}
\tag{20}
$$

IV. WIGNER FUNCTION OF PHOTON-SUBTRACTED THERMO VACUUM STATE

At finite temperature, the photon-subtracted thermo vacuum state can be expressed as [20]

$$\rho_1 = C_1 a^n |0(\beta)\rangle \langle 0(\beta)| a^{\dagger n}, \tag{21}$$

where C_1 is the normalized factor, defined by

$$C_1^{-1} = \text{Tr}\left[a^n S(\theta) |0,\tilde{0}\rangle \langle 0,\tilde{0}| S^\dagger(\theta) a^{\dagger n}\right], \tag{22}$$

which can be calculated as follows. Using Eq.(5) and the binomial formula

$$\sum_{l=0}^{\infty} \frac{(n+l)!}{n!l!} x^l = (1-x)^{-n-1}, \tag{23}$$

we have

$$\begin{aligned}
C_1^{-1} &= \langle 0,\tilde{0}| S^\dagger(\theta) a^{\dagger n} a^n S(\theta) |0,\tilde{0}\rangle \\
&= \text{sech}^2\theta \langle 0,\tilde{0}| e^{a\tilde{a}\tanh\theta} a^{\dagger n} a^n e^{a^\dagger \tilde{a}^\dagger \tanh\theta} |0,\tilde{0}\rangle \\
&= \text{sech}^2\theta \sum_{k,l=0}^{\infty} \tanh^{l+k}\theta \left\langle k,\tilde{k}\right| a^{\dagger n} a^n \left|l,\tilde{l}\right\rangle \\
&= \text{sech}^2\theta \sum_{l=n}^{\infty} \frac{l!}{(l-n)!} \tanh^{2l}\theta = n!\sinh^{2n}\theta.
\end{aligned} \tag{24}$$

By using Eqs. (21) and (19), we calculate the WF of photon-subtracted thermal state ρ_1

$$\begin{aligned}
W_1(\alpha) &= C_1 \langle 0,\tilde{0}| S^\dagger(\theta) a^{\dagger n} \Delta(\alpha) a^n S(\theta) |0,\tilde{0}\rangle \\
&= \langle 0,\tilde{0}| \left[S^\dagger(\theta) a^{\dagger n} S(\theta)\right] S^\dagger(\theta) \Delta(\alpha) S(\theta) \left[S^\dagger(\theta) a^n S(\theta)\right] |0,\tilde{0}\rangle.
\end{aligned} \tag{25}$$

Noticing Eq.(13) we see

$$\begin{aligned}
\left[S^\dagger(\theta) a^n S(\theta)\right] |0,\tilde{0}\rangle &= \left(a\cosh\theta + \tilde{a}^\dagger \sinh\theta\right)^n |0,\tilde{0}\rangle \\
&= \sqrt{n!}\sinh^n\theta |0,\tilde{n}\rangle,
\end{aligned} \tag{26}$$

then substituting (26) into Eq.(25) and using Eq.(19) yields

$$\begin{aligned}
W_1(\alpha) &= \frac{e^{-2|\alpha|^2 \text{sech}2\theta}}{\pi\cosh 2\theta} \langle\tilde{n}| e^{\frac{2\sinh\theta}{\cosh 2\theta}\alpha^* \tilde{a}^\dagger} (\text{sech}2\theta)^{\tilde{a}^\dagger \tilde{a}} e^{\frac{2\sinh\theta}{\cosh 2\theta}\tilde{a}\alpha} |\tilde{n}\rangle \\
&= \frac{e^{-2|\alpha|^2 \text{sech}2\theta}}{\pi\cosh 2\theta} \sum_{k,l=0}^{n} \frac{\alpha^{*k}\alpha^l}{k!l!} \left(\frac{2\sinh\theta}{\cosh 2\theta}\right)^{k+l} \langle\tilde{n}| \tilde{a}^{\dagger k} (\text{sech}2\theta)^{\tilde{a}^\dagger \tilde{a}} \tilde{a}^l |\tilde{n}\rangle \\
&= \frac{e^{-2|\alpha|^2 \text{sech}2\theta}}{\pi\cosh^{n+1} 2\theta} \sum_{l=0}^{n} \frac{n!}{l!l!(n-l)!} \left(\frac{4\sinh^2\theta}{\cosh 2\theta}|\alpha|^2\right)^l,
\end{aligned} \tag{27}$$

where we have used the identity operator [21]

$$\exp\left[\lambda\tilde{a}^\dagger \tilde{a}\right] =: \exp\left[\left(e^\lambda - 1\right)\tilde{a}^\dagger \tilde{a}\right] : . \tag{28}$$

Recalling that the definition of Laguerre polynomials [22],

$$L_n(x) = \sum_{l=0}^{n} \frac{n!}{(l!)^2 (n-l)!} (-x)^l, \tag{29}$$
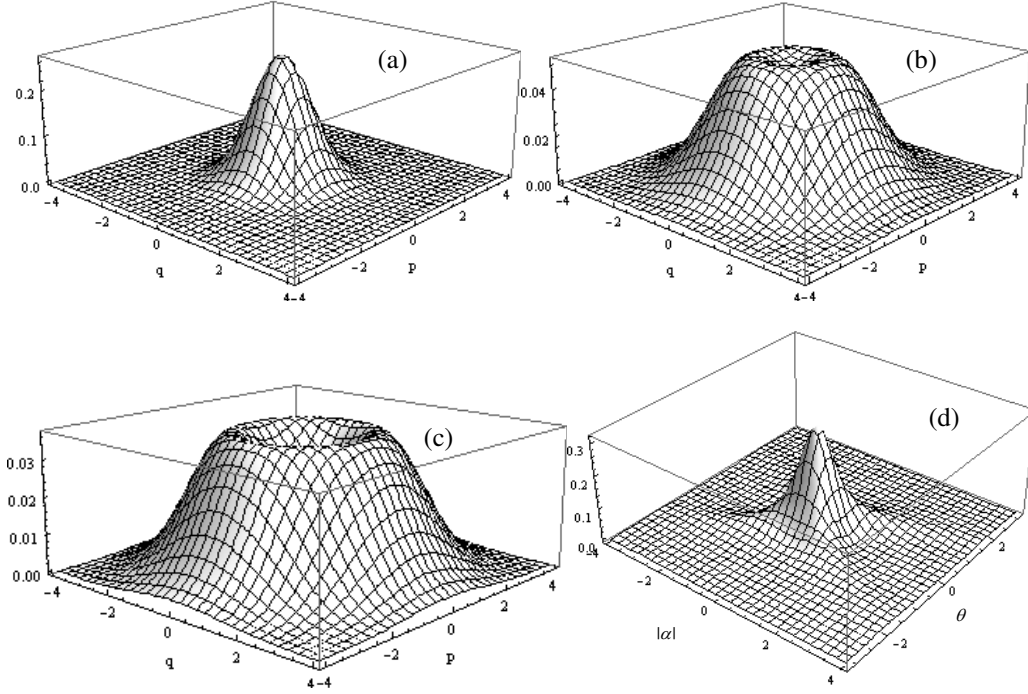
FIG. 1: Wigner function distributions of photon-subtracted thermo state in (q,p) phase space with (a) $n = 1, \theta = 0.2$, (b) $n = 1, \theta = 0.8$, (c) $n = 2, \theta = 0.8$, and in $(|\alpha|, \theta)$ space with (d) $n = 1$.

Eq. (27) can be further put into the following neat form,

$$W_1(\alpha) = \frac{e^{-2|\alpha|^2 \text{sech} 2\theta}}{\pi \cosh^{n+1} 2\theta} L_n \left(-\frac{4 \sinh^2 \theta}{\cosh 2\theta} |\alpha|^2 \right), \tag{30}$$

which is just the WF of photon-subtracted thermo vacuum state, a Gaussian-Laguerre type function of temperature, since $\tanh \theta = \exp\left(-\frac{\omega}{2kT}\right)$. Due to $\cosh 2\theta > 0$ and $L_n(-\frac{4 \sinh^2 \theta}{\cosh 2\theta} |\alpha|^2) \geqslant 0$, for the photon-subtracted case, $W_1(\alpha)$ has no chance to present the negative value in phase space, which can be seen from Fig.1. On the other hand, the amplitude value of WF in $(|\alpha|, \theta)$ space decreases with the increasing temperature (corresponding to θ). In appendix A, in order to check the result in Eq. (30), we have derived the WF of photon-subtracted thermo vacuum state by using the coherent state representation of Wigner operator. Comparing with the result in Ref.[20], Eq.(30) seems *more concise and convenient* for further discussion.

V. WIGNER FUNCTION OF PHOTON-ADDED THERMO VACUUM STATE

At finite temperature, the photon-added thermo vacuum state is expressed as [23]

$$\rho_2 = C_2 a^{\dagger n} |0(\beta)\rangle \langle 0(\beta)| a^n. \tag{31}$$

By the same procedures as deriving Eqs. (22) and (26), we have

$$C_2^{-1} = n! \cosh^{2n} \theta, \tag{32}$$

and

$$S^{\dagger}(\theta) a^{\dagger n} S(\theta) |0, \tilde{0}\rangle = \sqrt{n!} \cosh^n \theta |n, \tilde{0}\rangle. \tag{33}$$
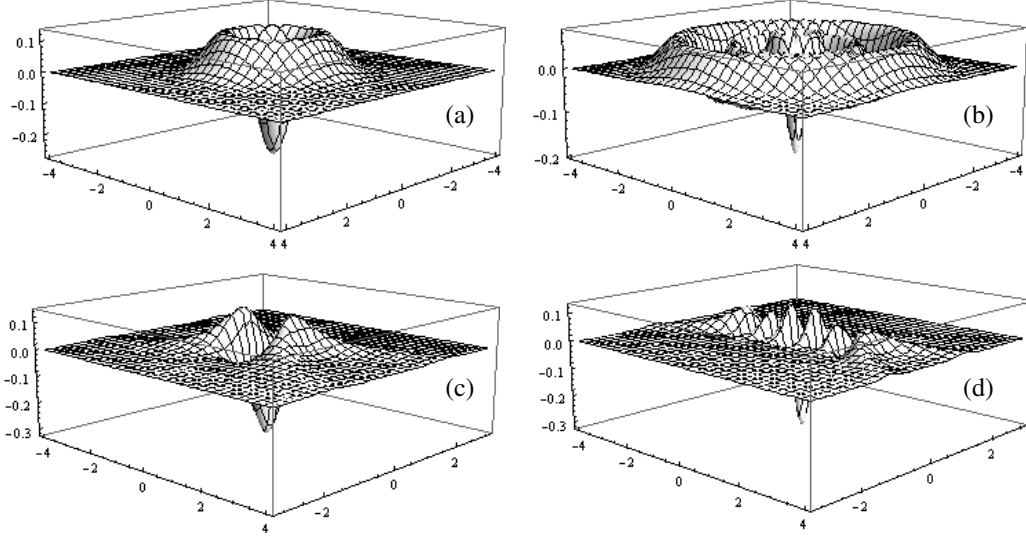
FIG. 2: Wigner function distributions of photon-added thermo state in (q, p) phase space with $\theta = 0.2$ for (a) $n = 1$, (b) $n = 2$, and in $(|\alpha|, \theta)$ space with (c) $n = 1$ and (d) $n = 5$.

Uisng Eq.(32) and (33), the WF $W_2(\alpha)$ of ρ_2 is given by

$$
\begin{aligned}
W_2(\alpha) &= C_2 \langle 0, \tilde{0} | S^{\dagger}(\theta) a^n S(\theta) \left[S^{\dagger}(\theta) \Delta(\alpha) S(\theta) \right] S^{\dagger}(\theta) a^{\dagger n} S(\theta) | 0, \tilde{0} \rangle \\
&= \langle n, \tilde{0} | S^{\dagger}(\theta) \Delta(\alpha) S(\theta) | n, \tilde{0} \rangle \\
&= \frac{(-1)^n e^{-2|\alpha|^2 \mathrm{sech} 2\theta}}{\pi \cosh^{n+1} 2\theta} \sum_{l=0}^{n} \frac{n!}{l! l! (n-l)!} \left(-\frac{4 \cosh^2 \theta}{\cosh 2\theta} |\alpha|^2 \right)^l \\
&= \frac{(-1)^n e^{-2|\alpha|^2 \mathrm{sech} 2\theta}}{\pi \cosh^{n+1} 2\theta} L_n \left(\frac{4 \cosh^2 \theta}{\cosh 2\theta} |\alpha|^2 \right),
\end{aligned}
\tag{34}
$$

a Gaussian-Laguerre type function which may present negative region in phase space (see Fig.2). In particular, when $n = 1$, Eq.(34) reduces to

$$
W_2(\alpha) = -\frac{e^{-2|\alpha|^2 \mathrm{sech} 2\theta}}{\pi \cosh^2 2\theta} \left(1 - \frac{4 \cosh^2 \theta}{\cosh 2\theta} |\alpha|^2 \right).
\tag{35}
$$

In Fig. 2, the behaviour of WF distributions of photon-added thermo state are plotted in (q, p) phase space and $(|\alpha|, \theta)$ space. From Fig.2, one can see clearly the modulation action of photon-added number and temperature. The "oscillating frequency" of WF increases with the increasing photon-added number; while the amplitude value of WF in $(|\alpha|, \theta)$ space decreases with the increasing temperature (corresponding to θ), which indicates that the nonclassicality is weakened at finite temperature.

VI. WIGNER FUNCTION OF THERMO NUMBER STATE

At finite temperature, according to TFD, the number state $|n\rangle$ is replaced by $|n, \tilde{n}\rangle$, thus the thermo number state (i.e., number states at finite temperature) is $S(\theta) |n, \tilde{n}\rangle$ in the enlarged Fock space. Using the un-normalized coherent state representation of number state,

$$
|n, \tilde{n}\rangle = \frac{1}{n!} \frac{d^{2n}}{dz^n d\tilde{z}^n} |z, \tilde{z}\rangle |_{z=\tilde{z}=0}, \quad \langle z' | z \rangle = e^{z'^* z},
\tag{36}
$$

where $|z, \tilde{z}\rangle = \exp[za^\dagger + \tilde{z}\tilde{a}^\dagger]|0, \tilde{0}\rangle$ is the non-normalized two-mode coherent state, and employing Eq.(19), we calculate the WF $W_3(\alpha)$ of thermo number state as

$$
\begin{aligned}
W_3(\alpha) &= \langle n, \tilde{n}| S^\dagger \Delta(\alpha) S |n, \tilde{n}\rangle \\
&= \frac{1}{n!^2} \frac{d^{2n}}{df^n dr^n} \frac{d^{2n}}{dz^n dt^n} \langle f^*, r^*| S^\dagger \Delta(\alpha) S |z, t\rangle|_{f=r=z=t=0} \\
&= \mathcal{A} \frac{d^{2n}}{df^n dr^n} \frac{d^{2n}}{dz^n dt^n} \exp\{-(tz + fr)\tanh 2\theta \\
&\quad + (rt - fz)\operatorname{sech}2\theta + zE^* + fE + rF^* + tF\}_{f=r=z=t=0},
\end{aligned}
\tag{37}
$$

where we have set

$$
\mathcal{A} = \frac{e^{-2|\alpha|^2 \operatorname{sech}2\theta}}{\pi n!^2 \cosh 2\theta}, \quad E = 2\alpha\operatorname{sech}2\theta \cosh\theta, \quad F = 2\alpha\operatorname{sech}2\theta \sinh\theta.
\tag{38}
$$

Expanding the exponential term $\exp[(rt - fz)\operatorname{sech}2\theta]$ as series, we have

$$
\begin{aligned}
W_3(\alpha) &= \mathcal{A} \frac{d^{2n}}{df^n dz^n} \frac{d^{2n}}{dr^n dt^n} \exp[-(fr + tz)\tanh 2\theta] \\
&\quad \times \sum_{l,k=0}^{\infty} \frac{(-1)^k \operatorname{sech}^{l+k}2\theta}{l!k!} (rt)^l (fz)^k \exp[zE^* + fE + tF + rF^*]_{z=t=f=r=0} \\
&= \mathcal{A} \sum_{l,k=0}^{\infty} \frac{(-1)^k \operatorname{sech}^{l+k}2\theta}{l!k!} \frac{\partial^{2l}}{\partial F^l \partial F^{*l}} \frac{\partial^{2k}}{\partial E^k \partial E^{*k}} \\
&\quad \times \frac{d^{2n}}{df^n dz^n} \frac{d^{2n}}{dr^n dt^n} \exp[-(fr + tz)\tanh 2\theta + fE + rF^* + zE^* + tF]_{z=t=f=r=0}.
\end{aligned}
\tag{39}
$$

Then making the variable replacement for f, r, t, z we can rewrite Eq.(39) as

$$
\begin{aligned}
W_3(\alpha) &= \mathcal{A} \tanh^{2n} 2\theta \sum_{l,k=0}^{\infty} \frac{(-1)^k \operatorname{sech}^{l+k}2\theta}{l!k!} \frac{\partial^{2l}}{\partial F^l \partial F^{*l}} \frac{\partial^{2k}}{\partial E^k \partial E^{*k}} \\
&\quad \times \frac{d^{2n}}{df^n dr^n} \frac{d^{2n}}{dz^n dt^n} \exp\left[-fr + fE + \frac{rF^*}{\tanh 2\theta} - tz + zE^* + \frac{tF}{\tanh 2\theta}\right]_{z=t=f=r=0} \\
&= \mathcal{A} \tanh^{2n} 2\theta \sum_{l,k=0}^{\infty} \frac{(-1)^k \operatorname{sech}^{l+k}2\theta}{l!k!} \\
&\quad \times \frac{\partial^{k+l}}{\partial E^k \partial F^{*l}} \frac{\partial^{k+l}}{\partial E^{*k} \partial F^l} H_{n,n}\left(E, \frac{F^*}{\tanh 2\theta}\right) H_{n,n}\left(E^*, \frac{F}{\tanh 2\theta}\right).
\end{aligned}
\tag{40}
$$

Noticing the formula

$$
\frac{\partial^{l+k}}{\partial\xi^l \partial\eta^k} H_{m,n}(\xi, \eta) = \frac{m!n!}{(m-l)!(n-k)!} H_{m-l,n-k}(\xi, \eta),
\tag{41}
$$

we have

$$
\begin{aligned}
W_3(\alpha) &= \frac{n!^2 e^{-2|\alpha|^2 \operatorname{sech}2\theta}}{\pi \cosh 2\theta} \sum_{l,k=0}^{n} \frac{(-1)^k \operatorname{sech}^{l+k}2\theta \tanh^{2(n-l)} 2\theta}{l!k!\left[(n-l)!(n-k)!\right]^2} \\
&\quad \times \left|H_{n-k,n-l}\left(E, \frac{F^*}{\tanh 2\theta}\right)\right|^2.
\end{aligned}
\tag{42}
$$

From Eq.(42) one can see clearly that the WF of thermo number state is a real number.

In particular, when $n = 0$, noticing that $\tanh \theta = e^{-\frac{1}{2}\omega\beta}$, $\cosh^2 \theta = \frac{1}{1-e^{-\beta\omega}}$, $\sinh^2 \theta = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}$, Eq.(42) reduces to the WF of thermo vacuum state $|0(\beta)\rangle$ in Eq.(20). On the other hand, when $T \to 0$,(i.e., finite temperature case reduces to zero temperature case) $e^{-\beta\omega} \to e^{-\infty} \to 0$, $\sinh \theta \to 0$, $\cosh \theta \to 1$, $E \to 2\alpha$, $\frac{F^*}{\tanh 2\theta} \to \alpha^*$, and noticing Eq.(29) and the definition of two-variable Hermite polynomials [24,25],

$$H_{m,n}(\xi, \kappa) = \sum_{l=0}^{\min(m,n)} \frac{m! n! (-1)^l \xi^{m-l} \kappa^{n-l}}{l! (n-l)! (m-l)!}, \tag{43}$$

which leads to $H_{n-k,0}(2\alpha, \alpha^*) = (2\alpha)^{n-k}$, then Eq.(42) becomes

$$
\begin{aligned}
W_3(\alpha) &= \frac{1}{\pi} e^{-2|\alpha|^2} \sum_{k=0}^{n} \frac{(-1)^k n!}{k! [(n-k)!]^2} |H_{n-k,0}(2\alpha, \alpha^*)|^2 \\
&= \frac{(-1)^n}{\pi} e^{-2|\alpha|^2} \sum_{k=0}^{n} \frac{n!}{k! [(n-k)!]^2} \left(-4|\alpha|^2\right)^{n-k} \\
&= \frac{(-1)^n}{\pi} e^{-2|\alpha|^2} L_n(4|\alpha|^2),
\end{aligned} \tag{44}
$$

which is just the WF of number state $|n\rangle$ at zero temperature.

In sum, by using TFD and Weyl ordered operators' order-invariance under similar transformations, we present a new approach to deriving the exact expressions of Wigner functions for thermo number state, photon subtracted and added thermo vacuum state. These WF are related to the Gaussian-Laguerre type functions, which are easily to be further analysed. The affection of temperature to nonclassical behaviour of the fields is manifestly shown. For discussions about the decoherence at finite temperature, we refer to [30,31].

Appendix A Checking Eq.(30)

In fact, in original Fock space, the photon-subtracted thermo state is expressed as [20]

$$\rho_1 = C_1 \mathrm{Tr}_{\tilde{a}} \left[a^n |0(\beta)\rangle \langle 0(\beta)| a^{\dagger n} \right] = C_1 a^n \rho_c a^{\dagger n}, \tag{A1}$$

where ρ_c is the thermo state

$$\rho_c = \sum_{l=0}^{\infty} \frac{n_c^l}{(n_c+1)^{l+1}} |l\rangle \langle l| = \frac{1}{n_c+1} e^{a^\dagger a \ln \frac{n_c}{n_c+1}}, \quad n_c = \sinh^2 \theta. \tag{A2}$$

Using the the coherent state representation of Wigner operator [26],

$$\Delta(\alpha) = e^{2|\alpha|^2} \int \frac{d^2 z}{\pi^2} |z\rangle \langle -z| \exp \left[-2(z\alpha^* - z^*\alpha) \right], \tag{A3}$$

where $|z\rangle$ is the coherent state [27,28], we have

$$
\begin{aligned}
W_1(\alpha) &= \mathrm{Tr}(\Delta(\alpha) \rho_1) \\
&= \frac{C_1 e^{2|\alpha|^2}}{n_c+1} \int \frac{d^2 z}{\pi^2} \langle -z| a^n e^{a^\dagger a \ln \frac{n_c}{n_c+1}} a^{\dagger n} |z\rangle \exp \left[-2(z\alpha^* - z^*\alpha) \right].
\end{aligned} \tag{A4}
$$

Note that

$$e^{a^\dagger a \ln \frac{n_c}{n_c+1}} a^{\dagger n} e^{-a^\dagger a \ln \frac{n_c}{n_c+1}} = \frac{n_c^n}{(n_c+1)^n} a^{\dagger n}, \tag{A5}$$

and

$$e^{a^\dagger a \ln \frac{n_c}{n_c+1}} |z\rangle = e^{-\frac{2n_c+1}{2(n_c+1)^2}} \left| \frac{n_c z}{n_c+1} \right\rangle, \tag{A6}$$

Eq.(A4) can be rewritten as

$$W_1 (\alpha) = \frac{n_c^n C_1 e^{2|\alpha|^2}}{(n_c + 1)^{n+1}} \int \frac{d^2 z}{\pi^2} \left\langle -z \middle| a^n a^{\dagger n} \middle| \frac{n_c z}{n_c + 1} \right\rangle$$
$$\times \exp \left[-\frac{2n_c + 1}{2 (n_c + 1)^2} |z|^2 - 2 (z\alpha^* - z^*\alpha) \right]. \tag{A7}$$

Further using the operator identity [29]

$$a^n a^{\dagger n} = (-1)^n : H_{n,n} (ia^\dagger, ia) : , \tag{A8}$$

where $H_{m,n} (x, y)$ is the two-variable Hermite polynomials, whose generating function is

$$H_{m,n} (x, y) = \left. \frac{\partial^{m+n}}{\partial t^m \partial t'^n} \exp \left[-tt' + tx + t'y \right] \right|_{t=t'=0}, \tag{A9}$$

we have

$$W_1 (\alpha) = \frac{(-1)^n e^{2|\alpha|^2}}{n! (n_c + 1)^{n+1}} \frac{\partial^{2n}}{\partial t^n \partial \tau^n} e^{-t\tau}$$
$$\times \int \frac{d^2 z}{\pi^2} \exp \left\{ -\frac{2n_c + 1}{n_c + 1} |z|^2 + \left(\frac{i\tau n_c}{n_c + 1} - 2\alpha^* \right) z + (2\alpha - it) z^* \right\}_{t=\tau=0}$$
$$= \frac{(-1)^n}{n! \pi (n_c + 1)^n} \frac{e^{2|\alpha|^2}}{2n_c + 1} \frac{\partial^{2n}}{\partial t^n \partial \tau^n} \exp \left[-t\tau \right]$$
$$\exp \left[\frac{n_c + 1}{2n_c + 1} \left(\frac{i\tau n_c}{n_c + 1} - 2\alpha^* \right) (2\alpha - it) \right]_{t=\tau=0}$$
$$= \frac{(-1)^n}{n! \pi (n_c + 1)^n} \frac{e^{-\frac{2|\alpha|^2}{2n_c+1}}}{2n_c + 1} \frac{\partial^{2n}}{\partial t^n \partial \tau^n} \exp \left\{ -\frac{n_c + 1}{2n_c + 1} t\tau \right.$$
$$+ 2i\alpha^* \frac{n_c + 1}{2n_c + 1} t + 2i\alpha \frac{n_c}{2n_c + 1} \tau \Big\}_{t=\tau=0}$$
$$= \frac{e^{-\frac{2|\alpha|^2}{2n_c+1}}}{(2n_c + 1)^{n+1}} \frac{(-1)^n}{n! \pi} H_{n,n} \left(\frac{2i n_c \alpha}{\sqrt{(2n_c + 1)(n_c + 1)}}, 2i\sqrt{\frac{n_c + 1}{2n_c + 1}} \alpha^* \right), \tag{A10}$$

then using the relation

$$\frac{(-1)^n}{n!} H_{n,n} (x, y) = L_n (xy), \tag{A11}$$

and noticing that $n_c = \sinh^2 \theta$, $2n_c + 1 = \cosh 2\theta$, Eq.(A10) can be put into

$$W_1 (\alpha) = \frac{e^{-\frac{2|\alpha|^2}{2n_c+1}}}{\pi (2n_c + 1)^{n+1}} L_n \left(-\frac{4n_c |\alpha|^2}{2n_c + 1} \right), \tag{A12}$$

which is just the Eq.(30). Thus we have checked the result using a new appraoch.

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