Dynamical evolution of quantum oscillators towards equilibrium

A. R. Usha Devi[1, 2, 3, *] and A. K. Rajagopal[3]

[1]Department of Physics, Bangalore University, Bangalore-560 056, India
[2]H. H. Wills Physics Laboratory, University of Bristol, Bristol BS8 1TL, UK
[3]Inspire Institute Inc., McLean, VA 22101, USA.

(Dated: July 10, 2009)

A pure quantum state of large number N of oscillators, interacting via harmonic coupling, evolves such that any small subsystem $n << N$ of the global state approaches equilibrium. This provides a novel example where stationarity emerges as a natural phenomena under quantum dynamics alone, with no necessity to bring in any additional statistical postulates. Mixedness of equilibrated subsystems consisting of $1, 2, \ldots, n << N$ clearly indicates that small subsystems are entangled with the rest of the state i.e., the bath. Every single mode oscillator is found to relax in a mixed density matrix of the Boltzmann canonical form. In two oscillator stationary subsystems, intra-entanglement within the 'system' oscillators is found to exist when the magnitude of the squeezing parameter of the bath is comparable in magnitude with that of the coupling strength.

PACS numbers: 05.30.-d, 03.65.-w

I. INTRODUCTION

Deducing the statistical distribution in many particle systems as an intrinsic property, resulting solely from quantum dynamics, has attracted much attention in the literature [1]. In this context, considerable interest has been evoked recently [2, 3, 4] on a deeper understanding of the basic mechanism of equilibration [5], occurring as a natural consequence of quantum dynamical evolution - without invoking any additional statistical assumptions. More specifically, equilibration is realized entirely in a quantum mechanical setting – with the key element ascribed to quantum entanglement between the system and the environment. Erstwhile statistical postulates on ensemble averaging over initial distributions are not required at all – as quantum dynamics of individual pure states of a many body physical system itself leads to equilibration of smaller subsystems. This features an exciting foundational development, where 'subjective' lack of knowledge in terms of statistical ensemble averaging is replaced by the 'objective' randomness due to entanglement [6].

Based on powerful general arguments, Linden et. al. established [4] that an overwhelming majority of pure quantum states of interacting large quantum systems evolve such that any small subsystem approaches a stationary state. This brings out an important implication: dynamics of almost every pure many body quantum state, envisages stationarity of any small subsystem as an inherent property, with all the statistical ingredients already built within the basic quantum framework itself.

In this paper, we present an explicit analysis of the quantum evolution of an initially uncoupled pure squeezed state of a large number N of oscillators, subjected to a harmonic interaction Hamiltonian, resulting eventually in equilibrium of any small subsystem of $n << N$ oscillators of the global pure state - with the rest acting as the bath. Note that in Ref. [4], the *smallness* of the system in relation to the size of the bath is described by their respective dimensions. In the present case, individual systems constituting the whole state are infinite dimensional (being harmonic oscillators) and so, the *smallness* of the subsystems of the global quantum system consisting of N oscillators is expressed legitimately in terms of the number $n << N$ of a subset of oscillators under consideration. Essentially, we find that every small subsystem of oscillators tend to relax in a stationary state, specified by a mixed density matrix - which is indeed a signature of quantum entanglement of the 'system' and the 'bath', emerging due to quantum dynamics. It may be worth pointing out here that due to its mathematical transparency and simplicity, the physical model of a linear assembly of coupled oscillators has played a paradigmatic role in understanding difficult formal principles underlying statistical mechanics [7]. A detailed investigation of quantum dynamics of pure states leading to equilibrated small subsystems in this model would therefore be illuminating.

The paper is organized as follows: In Sec. II, we describe the physical model of N harmonically coupled oscillators and discuss its exact solution using symplectic transformations – which offers a most natural elegant approach to the problem of interest. We then identify the $N \times N$ symplectic transformation corresponding to unitary time evolution in this model. This is followed by Sec. III, where we analyze the time evolution of a global pure uncoupled squeezed state (which is not an eigenstate of the Hamiltonian) of the system-bath oscillators. Under the assumptions of continuum limit $N \to \infty$ and weak coupling approximation we show that any small subsystem of the whole pure state of N oscillators exhibits a stationary long time behavior. Every single mode oscillator system is shown to relax in a mixed state of Boltzmann canonical form, with effective temperature related

*Electronic address: arutth@rediffmail.com

to the squeezing parameter of the bath oscillators.It is also shown that any two oscillator system approach a stationary mixed state with an intra-entanglement surviving whenever the magnitude of the bath squeezing parameter is comparable in magnitude with that of the coupling strength. Sec. IV has concluding remarks.

II. THE PHYSICAL MODEL

We consider a long chain of coupled harmonic oscillators, the Hamiltonian of which is given by,

$$\hat{H} = \frac{1}{2m}\sum_{i=1}^{N}\hat{p}_i^2 + \frac{K}{2}\sum_{i=1}^{N}\hat{q}_i^2 + \frac{k}{2}\sum_{i=1}^{N}(\hat{q}_{i+1}-\hat{q}_i)^2. \quad (1)$$

Here $\hat{q}_i,\ \hat{p}_i$ denote position, momentum operators of the oscillators, satisfying the canonical commutation relations $[\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{i,j}$. It is convenient to define a $2N$-component operator column $\hat{\xi}$ of dimensionless variables,

$$\hat{\xi} = \begin{pmatrix} \hat{Q}_i = \sqrt{\frac{m\omega}{\hbar}}\, q_i \\ \hat{P}_j = \sqrt{\frac{1}{m\omega\hbar}}\, p_j \end{pmatrix}, \quad \omega^2 = \frac{K}{m},\ i,j = 1,2,\ldots,N,$$

and express the commutation relations compactly as,

$$[\hat{\xi}_\alpha, \hat{\xi}_\beta] \;=\; i\,\Gamma_{\alpha\beta};\quad \alpha,\beta = 1,2,\ldots,2N, \quad (2)$$

where, $\Gamma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$; I denotes the $N\times N$ unit matrix. A general real homogeneous linear transformation on $\hat{\xi}$ preserving the canonical commutation relations (2) is a $2N\times 2N$ symplectic transformation [8] $S \in \mathrm{Sp}(2N, \mathrm{R})$ and there exists a corresponding unitary operator $\hat{U}(S)$ on the Hilbert space on which the operators $\hat{\xi}$ act:

$$\hat{U}^\dagger(S)\hat{\xi}_\alpha\hat{U}(S) = \hat{\xi}'_\alpha = \sum_{\alpha'} S_{\alpha\alpha'}\hat{\xi}_{\alpha'}$$
$$\text{such that } [\hat{\xi}'_\alpha, \hat{\xi}'_\beta] = i\,\Gamma_{\alpha\beta} \Rightarrow S\Gamma S^T = \Gamma.$$

Here, we restrict to Gaussian quantum states of the system and the bath. These are completely characterized by the first and second moments of $\hat{\xi}$, arranged conveniently in the form of $2N\times 2N$ covariance matrix V as,

$$V_{\alpha\beta} = \frac{1}{2}\,\langle\{\triangle\hat{\xi}_\alpha, \triangle\hat{\xi}_\beta\}\rangle,\quad \alpha,\beta = 1,2,\ldots,2N,$$

where $\triangle\hat{\xi} = \hat{\xi} - \langle\hat{\xi}\rangle$, $\{\hat{O}_1, \hat{O}_2\} = \hat{O}_1\hat{O}_2 + \hat{O}_2\hat{O}_1$ and $\langle\hat{O}\rangle = \mathrm{Tr}[\hat{\rho}\hat{O}]$ denotes the expectation value of the operator $\hat{O}$ in the quantum state $\hat{\rho}$. Under symplectic transformation, a Gaussian state is mapped to another Gaussian state characterized by the covariance matrix $V' = SVS^T$.

Time evolution of the elements of the variance matrix under $\hat{\mathcal{U}}(t) = \exp\{-it\hat{H}/\hbar\}$ may be identified as a symplectic transformation (as the Hamiltonian of Eq. (1) is a quadratic in the canonical operators):

$$\begin{aligned} V_{\alpha\beta}(t) &= \frac{1}{2}\mathrm{Tr}[\hat{\rho}(t)\,\{\triangle\hat{\xi}_\alpha(0), \triangle\hat{\xi}_\beta(0)\}] \\ &= \frac{1}{2}\mathrm{Tr}[\hat{\rho}(0)\,\hat{\mathcal{U}}^\dagger(t)\{\triangle\hat{\xi}_\alpha(0), \triangle\hat{\xi}_\beta(0)\}\hat{\mathcal{U}}(t)] \\ &= \left(\mathcal{S}(t)V(0)\mathcal{S}^T(t)\right)_{\alpha\beta} \end{aligned}$$

or $V(t) = \mathcal{S}(t)V(0)\mathcal{S}^T(t)$, where $\mathcal{S}(t)$ denotes the $2N\times 2N$ symplectic matrix corresponding to the unitary time evolution on the Hilbert space of the quantum state. The explicit structure of the symplectic transformation matrix $\mathcal{S}(t)$ associated with the dynamical evolution $\hat{\mathcal{U}}(t)$ in the present model is readily identified, as will be outlined in the following.

We first express the Hamiltonian (1) in the following quadratic form

$$\hat{H} = \frac{\omega\hbar}{2}\,\hat{\xi}^T\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\hat{\xi} \quad (3)$$

where the elements of the $N\times N$ bolck matrix A are given by,

$$A_{i,j} = (2\epsilon+1)\,\delta_{i,j} - \epsilon\,[\delta_{i,j+1} + \delta_{i+1,j}],\quad \epsilon = \frac{k}{K}. \quad (4)$$

Identifying the real orthogonal transformation σ which diagonalizes the real symmetric matrix A i.e.,

$$\begin{aligned} \sigma A\sigma^T &= \Lambda = \mathrm{diag}(\lambda(\phi_1), \lambda(\phi_2), \ldots, \lambda(\phi_N)), \\ \lambda(\phi_l) &= 1 + 2\epsilon\,(1-\cos\phi_l);\ \phi_l = \frac{l\pi}{N+1} \quad (5) \\ \sigma_{s,l} &= \sqrt{\frac{2}{N+1}}\,\sin(s\phi_l), \end{aligned}$$

we express the Hamiltonian (3) in its decoupled structure:

$$\begin{aligned} \hat{H} &= \frac{\omega\hbar}{2}\,\left(\mathcal{S}\hat{\xi}\right)^T\left(\Lambda^{\frac{1}{2}}\oplus\Lambda^{\frac{1}{2}}\right)\left(\mathcal{S}\hat{\xi}\right) \\ &= \frac{\omega\hbar}{2}\,\hat{U}^\dagger(\mathcal{S})\left[\sum_{l=1}^{N}\lambda_l^{\frac{1}{2}}\left(\hat{P}_l^2 + \hat{Q}_l^2\right)\right]\hat{U}(\mathcal{S}), \quad (6) \end{aligned}$$

where $\mathcal{S} = \Lambda^{\frac{1}{4}}\sigma\oplus\Lambda^{-\frac{1}{4}}\sigma$ is a symplectic transformation [9] on the $2N$ component operator column $\hat{\xi}$.

Thus, we obtain the $2N\times 2N$ symplectic matrix $\mathcal{S}(t)$ corresponding to the unitary time evolution operator $e^{-it\hat{H}/\hbar}$ as [10]:

$$\mathcal{S}(t) = \begin{pmatrix} \cos(\omega t\,A^{\frac{1}{2}}) & A^{-\frac{1}{2}}\sin(\omega t\,A^{\frac{1}{2}}) \\ -A^{\frac{1}{2}}\sin(\omega t\,A^{\frac{1}{2}}) & \cos(\omega t\,A^{\frac{1}{2}}) \end{pmatrix}. \quad (7)$$

We proceed now to investigate the quantum evolution of a pure uncoupled squeezed state of oscillators.

III. TIME EVOLUTION OF PURE UNCOUPLED SQUEEZED STATE

First, we decompose the global quantum state of N-oscillators into two parts: $n << N$ 'system' oscillators i.e., a n oscillator subsystem and the rest of the whole state, the bath. We consider an initial state of the whole system to be pure product states of individual oscillators,

$$
\begin{aligned}
|\Psi(\eta,\mu,t=0)\rangle &= |\phi_b(\eta,t=0)\rangle\otimes|\phi_b(\eta,t=0)\rangle\otimes\ldots \\
&\otimes\underbrace{|\chi_s(\mu,t=0)\rangle\otimes\ldots\otimes|\chi_s(\mu,t=0)\rangle}_{n\text{ system oscillators}} \\
&\otimes|\phi_b(\eta,t=0)\rangle\otimes\ldots\otimes|\phi_b(\eta,t=0)\rangle, \quad (8)
\end{aligned}
$$

where the oscillators in the bath are in the squeezed state

$$|\phi_b(\eta,t=0)\rangle = \hat{U}(S(\eta))|0\rangle. \quad (9)$$

Here, $\hat{U}((S(\eta))$ denotes the squeezing operator (with associated 2×2 symplectic matrix given by [8], $S(\eta) = \mathrm{diag}(e^{-\eta/2},\ e^{\eta/2})$. The initial state of each of the system oscillators [11],

$$|\chi_s(\mu,t=0)\rangle = \hat{U}(S(\mu))|0\rangle, \quad (10)$$

is characterized by the squeezing parameter μ. (Here, $|0\rangle$ denotes the ground state of the oscillator.)

It may be noted that initially, the whole system-bath state is a product state of oscillators and the subsystems are not already in a stationary state, when the couplings are switched on at $t=0^+$. In other words, the global initial pure state (8) is *not* an energy eigenstate of the total Hamiltonian – this being a trivial case leaving the subsystems stationary under dynamical evolution. Also, in contrast to the case where the bath is initially in thermal state characterized by a temperature T, here the bath is in a pure state specified by a squeezing parameter η.

The initial variance matrix of the system-bath pure state is given by,

$$V(\eta,\mu;t=0) \;=\; \frac{1}{2}\left(D_Q(\eta,\mu)\oplus D_P(\eta,\mu)\right) \quad (11)$$

where the blocks $D_Q(\eta,\mu)$, and $D_P(\eta,\mu)$ are $N\times N$ diagonal matrices,

$$
\begin{aligned}
D_Q(\eta,\mu) &= \mathrm{diag}(e^{-\eta},\ldots,\underbrace{e^{-\mu},\ldots,e^{-\mu}}_{r_1,r_2,\ldots r_n},\ \ldots,e^{-\eta}) \\
&= D_P^{-1}(\eta,\mu). \quad (12)
\end{aligned}
$$

It is convenient to split the variance matrix $V(\eta,\mu;0)$ as,

$$
\begin{aligned}
V(\eta,\mu;0) &= v(\eta;0)+\sum_{i=1}^{n} v_i(\eta,\mu;0) \\
\text{where,}\ \ v(\eta;0) &= \frac{1}{2}\left(e^{-\eta}\, I\oplus e^{\eta}\, I\right) \\
\sum_{i=1}^{n} v_i(\eta,\mu;0) &= V(\eta,\mu;0)-v(\eta;0). \quad (13)
\end{aligned}
$$

The non-zero elements of $v_i(\eta,\mu;0)$ are readily identified as (see Eqs. (11),(12), (13)),

$$
\begin{aligned}
\left[v_i(\eta,\mu;0)\right]_{r_i,r_i} &= \frac{1}{2}(e^{-\mu}-e^{-\eta}), \\
\left[v_i(\eta,\mu;0)\right]_{n+r_i,n+r_i} &= \frac{1}{2}(e^{\mu}-e^{\eta})
\end{aligned}
$$

Temporal evolution of the quantum state (8) is entirely determined by the symplectic transformation $\mathcal{S}(t)$ (given by Eq. (7)) on the variance matrix,

$$
\begin{aligned}
V(\eta,\mu;t) &= \mathcal{S}(t)V(\eta,\mu;0)\mathcal{S}^T(t) \\
&= v(\eta;t)+\sum_{i=1}^{n} v_i(\eta,\mu;t) \\
&= \begin{pmatrix} V_{QQ}(\eta,\mu;t) & V_{QP}(\eta,\mu;t) \\ V_{QP}^T(\eta,\mu;t) & V_{PP}(\eta,\mu;t)\end{pmatrix}, \quad (14)
\end{aligned}
$$

where $V_{QQ}(\eta,\mu;t), V_{PP}(\eta,\mu;t)$ and $V_{QP}(\eta,\mu;t)$ respectively denote the $N\times N$ diagonal and off-diagonal blocks of the variance matrix.

In an infinitely long chain ($N\to\infty$), closed form analytical expressions are obtained for the elements of the variance matrix $V(\eta,\mu;t)$ (by replacing the discrete variable '$\phi_l=\frac{l\pi}{N+1}$' of (5) by a continuous parameter 'ϕ' and the sum '$\frac{1}{N+1}\sum_{l=1}^{N}$' by the integral '$\frac{1}{\pi}\int_0^{\pi}\mathrm{d}\phi$ '):

$$
\begin{aligned}
\left[V_{QQ}(\eta,\mu;t)\right]_{s,l} &= e^{-\eta}\, C_{s,l}^{(2,0)}(t)+e^{\eta}S_{s,l}^{(2,-1)}(t)+\frac{e^{-\mu}-e^{-\eta}}{2}\sum_{i=1}^{n} C_{s,r_i}^{(1,0)}(t)C_{l,r_i}^{(1,0)}(t)+\frac{e^{\mu}-e^{\eta}}{2}\sum_{i=1}^{n} S_{s,r_i}^{(1,-\frac{1}{2})}(t)S_{l,r_i}^{(1,-\frac{1}{2})}(t), \\
\left[V_{PP}(\eta,\mu;t)\right]_{s,l} &= e^{\eta}\, C_{s,l}^{(2,0)}(t)+e^{-\eta}S_{s,l}^{(2,1)}(t)+\frac{e^{\mu}-e^{\eta}}{2}\sum_{i=1}^{n} C_{s,r_i}^{(1,0)}(t)C_{l,r_i}^{(1,0)}(t)+\frac{e^{\mu}-e^{\eta}}{2}\sum_{i=1}^{n} S_{s,r_i}^{(1,\frac{1}{2})}(t)S_{l,r_i}^{(1,\frac{1}{2})}(t), \\
\left[V_{QP}(\eta,\mu;t)\right]_{s,l} &= -\frac{e^{-\eta}}{2}\, S_{s,l}^{(1,\frac{1}{2})}(2t)+\frac{e^{\eta}}{2}\, S_{s,l}^{(1,-\frac{1}{2})}(2t)-\frac{e^{-\mu}-e^{-\eta}}{2}\sum_{i=1}^{n} C_{s,r_i}^{(1,0)}(t)S_{l,r_i}^{(1,\frac{1}{2})}(t) \\
&\quad +\frac{e^{-\mu}-e^{-\eta}}{2}\sum_{i=1}^{n} S_{s,r_i}^{(1,-\frac{1}{2})}(t)C_{l,r_i}^{(1,0)}(t), \quad (15)
\end{aligned}
$$

where we have denoted,

$$C_{s,l}^{(a,\kappa)}(t) \;=\; \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi\, \sin(s\phi)\sin(l\phi)\,\lambda^{\kappa}(\phi)\,\cos^{a}[\omega t\lambda^{\frac{1}{2}}(\phi)]$$
$$S_{s,l}^{(a,\kappa)}(t) \;=\; \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi\, \sin(s\phi)\sin(l\phi)\,\lambda^{\kappa}(\phi)\sin^{a}[\omega t\lambda^{\frac{1}{2}}(\phi)] \qquad (16)$$

with $\lambda(\phi) = \epsilon\gamma^{-1}(1+2\gamma\cos\phi)$, $\gamma = \frac{k}{K+2k}$. These results are formally exact in the long chain limit. To make their meaning evident, one resorts to the weak coupling approximation $\epsilon \approx \gamma << 1$, in which case the following standard form

$$\frac{1}{\pi}\int_0^{\pi}\mathrm{d}\phi\cos(s\phi)\cos[x(1-\gamma\cos\phi)] = J_s(\gamma x)\cos(x-\frac{s\pi}{2})$$

(where $J_s(x)$ denotes Bessel function of integral order) can be employed to simplify $C_{s,l}^{(a,\kappa)}(t), S_{s,l}^{(a,\kappa)}(t)$ of (16) up to order $O(\gamma)$ – thus reducing the elements of the variance matrix $V(\eta,\mu;t)$ (given by Eq. (15)) to time dependent $\cos(\Omega t)$ or $\sin(\Omega t)$ functions, oscillating rapidly with Bessel functions $J_s(\gamma\Omega t)$ being the amplitudes (with $\Omega = \sqrt{\frac{K+2k}{m}}$). The long time behavior of the system gets specified by the asymptotic decay of Bessel functions i.e., $\lim\limits_{x\to\infty} J_s(x) \to |x|^{-\frac{1}{2}}$ for $x >> s$. More specifically, we find,

$$C_{s,l}^{(2,0)} \;\to\; \frac{1}{4}\,\delta_{s,l}, \qquad S_{s,l}^{(1,\pm\frac{1}{2})}, C_{s,l}^{(1,0)} \to 0,$$
$$S_{s,l}^{(2,\pm 1)} \;\to\; \frac{1}{4}\left(\delta_{s,l} \mp \gamma\,[\delta_{s,l+1}+\delta_{s,l-1}]\right), \qquad (17)$$

in the limit $\gamma\Omega t \to \infty$.

It is thus evident that a subsystem of $n << N$ oscillator relaxes in a steady state, specified by the $2n\times 2n$ variance matrix, $V^{(n)} = V_Q^{(n)} \oplus V_P^{(n)}$, with elements,

$$\left[V_Q^{(n)}\right]_{sl} \;=\; \frac{1}{2}\left(\cosh\eta + \frac{e^{-\eta}\gamma}{2}[\delta_{s,l+1}+\delta_{s,l-1}]\right)$$
$$\left[V_P^{(n)}\right]_{sl} \;=\; \frac{1}{2}\left(\cosh\eta - \frac{e^{\eta}\gamma}{2}[\delta_{s,l+1}+\delta_{s,l-1}]\right) \quad (18)$$

(which exhibit a correlation $\langle\hat{Q}_j\hat{Q}_{j\pm1}\rangle = \frac{\gamma}{2}\,e^{-\eta}$ and an anticorrelation $\langle\hat{P}_j\hat{P}_{j\pm1}\rangle = -\frac{\gamma}{2}\,e^{\eta}$ between neighbors). Evidently, the equilibrium state of the 'system' is independent of its initial form (i.e., it does not contain the squeezing parameter μ of the initial system oscillators), but depends on the squeezing parameter η of the bath.

In particular, the variance matrix $V^{(1)}(t)$ of any single mode subsystem of the dynamically evolving global quantum state eventually converges, in the limit $t >> (\gamma\Omega)^{-1}$, to an 'equilibrium' structure,

$$V^{(1)} = \frac{1}{2}\begin{pmatrix}\cosh\eta & 0\\ 0 & \cosh\eta\end{pmatrix}, \qquad (19)$$

which corresponds to a mixed density matrix of the familiar Boltzmann form,

$$\hat{\rho}^{(1)} \;=\; \frac{e^{-\beta\hbar\omega\hat{H}_r}}{\mathrm{Tr}[e^{-\beta\hbar\omega\hat{H}_r}]}, \quad \hat{H}_r = \frac{\hat{p}_r^2}{2m} + \frac{K}{2}\hat{q}_r^2 \quad . \qquad (20)$$

The inverse temperature β is related to the squeezing parameter η of the bath via, $\beta = \frac{2}{\hbar\omega}\coth^{-1}[\cosh\eta]$. The purity [12] of the single oscillator equilibrium state $\nu^{(1)} = \mathrm{Tr}[(\rho^{(1)})^2] = [2\sqrt{\det(V^{(1)})}]^{-1} = [\cosh\eta]^{-1} < 1$ captures the system-bath entanglement. This also reflects in the increase of the von Neumann entropy [12] of the state from its initial value zero to the equilibrium value

$$S(\hat{\rho}^{(1)}) \;=\; -\mathrm{Tr}[\hat{\rho}^{(1)}\ln\hat{\rho}^{(1)}]$$
$$\;=\; \left(\frac{1-\nu^{(1)}}{2\nu^{(1)}}\right)\ln\left(\frac{1+\nu^{(1)}}{1-\nu^{(1)}}\right) - \ln\left(\frac{2\nu^{(1)}}{1+\nu^{(1)}}\right).$$

Any two oscillator subsystem is found to eventually relax in a 'stationary' state specified by the two mode variance matrix,

$$V^{(2)} = \frac{1}{2}\begin{pmatrix}\cosh\eta & \frac{e^{-\eta}\gamma}{2} & 0 & 0\\ \frac{e^{-\eta}\gamma}{2} & \cosh\eta & 0 & 0\\ 0 & 0 & \cosh\eta & -\frac{e^{\eta}\gamma}{2}\\ 0 & 0 & -\frac{e^{\eta}\gamma}{2} & \cosh\eta\end{pmatrix}. \qquad (21)$$

The stationary density matrix of the two oscillator system has its purity, $\nu^{(2)} = [4\sqrt{\det(V^{(2)})}]^{-1} \approx [\cosh\eta]^{-2}$ which is clearly less than 1, and reveals the system-bath entanglement. One finds internal entanglement between the two oscillators if [13] $(\cosh^2\eta - \frac{e^{-2\eta}\gamma^2}{4})(\cosh^2\eta - \frac{e^{2\eta}\gamma^2}{4}) - \cosh 2\eta < 0$, and this happens when the squeezing parmeter η of the bath is comparable in magnitude with the coupling γ. Thus, one finds a tradeoff of entanglement within the system oscillators and that between the system-bath – as survival of internal entanglement implies nearly vanishing mixedness of the two oscillator system (i.e., $\nu^{(2)} \approx 1$ for small squeezing parameter).

IV. CONCLUSIONS

In summary, we have shown, through an explicit analysis under the assumptions of continuum limit $N\to\infty$ and weak coupling approximation $\epsilon \approx \gamma << 1$, that any small subsystem of the whole pure squeezed state of N

oscillators evolving under hamonically coupled Hamiltonian, approaches equilibrium. This provides an excellent example in which stationary behaviour of any small subsystem of a large global pure state is shown to emerge as a natural phenomena consequent to quantum dynamical evolution - without the aid of any additional statistical postulates [4]. Equilibrated subsystems consisting of $1, 2, \ldots, n << N$ oscillators are found to be mixed, revealing their quantum entanglement with the rest of the system i.e., the bath. A single mode oscillator 'system' is shown to relax in a mixed density matrix in the Boltzmann canonical form. The connection between the squeezing parameter of the bath and the temperature is a new feature of our work. We also find (in the case of $n = 2$) that entanglement within the 'system' oscillators in the steady state survives only when the squeezing parameter and the coupling strength are of comparable magnitude. These features on two and more subsystem oscillators, to the best of our knowledge, have not been recorded in the literature.

The long chain limit and the weak coupling approximation made our analysis amenable to analytical results. One has to resort to numerical approach to evaluate the integrals (16) in the continuum, strong coupling limits. Also, the finite N limit may be addressed with the help of numerical investigations both in the strong and weak coupling limits. These issues would be of interest from a foundational point of view and we plan to address this issue in a separate communication.

ACKNOWLEDGEMENTS

ARU acknowledges financial support of the Commonwealth Commission, UK and thanks Professor Sandu Popescu and Paul Skrzypczyk for insightful comments.

---

[1] J. von Neumann, Z. Physik **57**, 30 (1929); W. Pauli and M. Fierz, Z. Physik **106**, 572 (1937); P. Bocchiere and L. Loinger, Phys. Rev. **114**, 948 (1959).

[2] H. Tasaki, Phys. Rev. Lett. **80**, 1373 (1998).

[3] P. Reimann, Phys. Rev. Lett. **101**, 190403 (2008); M. Rigol, V. Dunjko, and M. Olshanii, Nature (London) **452**, 854 (2008).

[4] N. Linden, S. Popescu, A. J. Short, and A. Winter, Phys. Rev. E **79**, 061103 (2009).

[5] The terminology 'equilibrium' is used here in a rather non-technical sense, to imply stationary long time behaviour.

[6] Even when the individual pure state of the whole system has zero entropy, subsystems are in a mixed state, exhibiting non-zero entropy, thus revealing 'objective' randomness. This is in stark contrast with classical statistical mechanics, where the probability distributions that define entropy reflect our 'subjective' lack of knowledge.

[7] Several investigations have been carried out on the statistical processes leading to equilibration phenomena in the linearly coupled oscillator system (see for e.g., G. Klein and I. Prigogine, **19**, 74 (1953); **19**, 89 (1953); **19**, 1053 (1953) for an extensive discussion in the classical context). The classical framework, however, requires statistical averaging with respect to initial probability distributions. Our investigation departs radically from that of the conventional approaches. It reveals that quantum dynamics of individual pure squeezed states of large number of oscillators inherits the required statistical ingredients such that smaller subsystems of $n = 1, 2, .. << N$ oscillators equilibrate.

[8] R. Simon, N. Mukunda and B. Dutta, Phys. Rev. A **49**, 1567 (1994); Arvind, B. Dutta, N. Mukunda, R. Simon, Phys. Rev. A **52**, 1609 (1995).

[9] The $2N \times 2N$ matrix $\mathcal{S} = \Lambda^{\frac{1}{4}}\sigma \oplus \Lambda^{-\frac{1}{4}}\sigma$ is a symplectic transformation belonging to Sp($2n$,R) as can be explicitly verified via the defining relation $\mathcal{S}\Gamma\mathcal{S}^T = \Gamma$.

[10] The 2×2 symplectic transformation corresponding to the unitary operation $\hat{U}(S(\phi)) = e^{\frac{-i\phi}{2}(\hat{P}^2+\hat{Q}^2)}$ on the single oscillator Hilbert space is given by [8] $S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$. Employing this in Eq. (6) for N-mode oscillators and simplifying yields $\mathcal{S}(t)$ of Eq. (7).

[11] The 'system' could be isolated oscillators spread throughout the whole quantum state. It is for the sake of mathematical convenience that they are grouped next to each other.

[12] A. Serafini, F. Illuminati, S. De Siena, J. Phys. B **37**, L21 (2004).

[13] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).