Classification and quantification of nonlocality in ensembles consisting of orthogonal bipartite states

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An ensemble of mutually orthogonal multipartite states may not be distinguishable by means of local operations and classical communication (LOCC) and could exhibit a kind of nonlocality different from quantum entanglement. We here introduce a measure to quantify and classify the nonlocality of ensembles consisting of mutually orthogonal bipartite states, which is the entanglement cost in addition to LOCC to distinguish the states with vanishing error in the asymptotic limit. We estimate various upper and lower bounds for the nonlocality measure and evaluate the exact values for ensembles consisting of mutually orthogonal maximally entangled bipartite states.

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Introduction.–Although mutually orthogonal multipartite states are always distinguishable through joint measurements, it was found by Bennett *et al.* in 1999 that there are ensembles of mutually orthogonal bipartite product states that cannot be distinguished by means of local operations and classical communication (LOCC); this phenomenon was referred to as the nonlocality without entanglement [1]. The essence of this nonlocality is that the maximal information about the system state achievable through LOCC is strictly less than that achievable through joint measurements. Since the discovery of this phenomenon, substantial efforts have been devoted to search the conditions under which a given ensemble can exhibit such a kind of nonlocality [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. As is known, the ensemble consisting of the four Bell states can exhibit the nonlocality [11], while the ensembles consisting of only two orthogonal states cannot [12]. As a latest result, it was indicated recently that the nonlocality exists in almost all ensembles consisting of more than d mutually orthogonal $d^{\otimes n}$ states [13].

LOCC distinguishing the states in the ensemble $\varepsilon = \left\{p_X, \rho_X^{AB}\right\}$ can be conceived as a game. Suppose that there is a classical information source producing symbol X with probability p_X. If the source outputs symbol X, Alice and Bob will be given a quantum state ρ_X^{AB}. They know the ensemble ε and their task is to determine the value of X via a measurement implemented through LOCC. How much information they have gained about the value of X can be described by the mutual information between X and the measurement result Y

$$I\left(X;Y\right) = H\left(X\right) + H\left(Y\right) - H\left(XY\right), \qquad (1)$$

where $H\left(\cdot\right)$ is the Shannon entropy of the random variables. The maximal mutual information achievable through LOCC is called locally accessible information and it will be denoted by $I^{LOCC}\left(\varepsilon\right)$. The value of X can be determined through LOCC if and only if $I^{LOCC}\left(\varepsilon\right) = H\left(X\right)$ [14]. Similarly we can define $I^{Global}\left(\varepsilon\right)$ which is the maximal mutual information achievable through joint measurements. When the states are mutually orthogonal, there is always $I^{Global}\left(\varepsilon\right) = H\left(X\right)$; the value of X can be determined through joint measurements. Generally there is $I^{LOCC}\left(\varepsilon\right) \leq I^{Global}\left(\varepsilon\right) \leq H\left(X\right)$.

Ensembles consisting of mutually orthogonal multipartite states can be divided into two categories according to whether the states can be distinguished *via* LOCC. This division is too coarse. For an ensemble whose states are LOCC indistinguishable, one can ask how much entanglement in addition to LOCC is needed to distinguish the states [15]. While for an ensemble whose states are LOCC distinguishable, one can ask how much entanglement can be distilled through distinguishing the states.

In this paper, we focus on ensembles consisting of mutually orthogonal bipartite states and consider the tensor power $\varepsilon^{\otimes n}$ instead of the ensemble ε itself, such that some results from information theory can be used directly. Since quantum entanglement can be consumed in distinguishing the states of $\varepsilon^{\otimes n}$, we introduce the entanglement cost (defined below) as a measure to quantify and classify the nonlocality of the ensemble ε. Here we do not require the states to be distinguished without error, but require that the error is vanished in the asymptotic limit. To our knowledge, quantification of the nonlocality of an ensemble has only been addressed in Ref. [16], though how to quantify quantum entanglement has been intensively studied.

Entanglement cost and information nonlocality.– Consider the ensemble $\varepsilon = \left\{p_X, \rho_X^{AB}\right\}$ consisting of mutually orthogonal bipartite states. Its tensor power is defined as $\varepsilon^{\otimes n} = \left\{p_{X^n}, \rho_{X^n}^{A^nB^n}\right\}$, where $p_{X^n} = p_{X_1}p_{X_2}\cdots p_{X_n}$, $\rho_{X^n}^{A^nB^n} = \rho_{X_1}^{A_1B_1} \otimes \rho_{X_2}^{A_2B_2} \cdots \otimes \rho_{X_n}^{A_nB_n}$ and X_i are independent and identically distributed classical variables as X. Suppose that Alice holds A^n and Bob holds B^n. To determine the value of X^n, they make a measurement that satisfies the conditions: (1) it is implemented through LOCC plus $n \times \alpha_n$ ebits of entanglement (a bipartite pure state with the quantum entropy of the reduced state being $n \times \alpha_n$); (2) the mutual information between X^n and the measurement result Y satisfies $I(X^n; Y) \geq I^{Global}(\varepsilon^{\otimes n}) - \delta_n$ with $\lim_{n\to\infty} \delta_n = 0$; (3) when the measurement result Y with the probability p_Y is obtained, $n \times \beta_{nY}$ ebits of entanglement is distilled at the same time, i.e, distilled a bipartite states whose distance to some pure state with $n \times \beta_{nY}$ ebits of entanglement is η_{nY} and there is $\lim_{n\to\infty} \eta_{nY} = 0$. The second condition ensures that the value of X^n can be inferred from the measurement result Y with vanishing error since the states of $\varepsilon^{\otimes n}$ are mutually orthogonal such that $I^{Global}(\varepsilon^{\otimes n}) = H(X^n)$. We now introduce the entanglement cost defined as

$$N(\varepsilon) = \inf \lim_{n\to\infty} \left(\alpha_n - \sum_Y p_Y \times \beta_{nY} \right), \tag{2}$$

where the infimum operation is taken over all measurements satisfying the above conditions.

The entanglement cost $N(\varepsilon)$ defined for ensemble $\varepsilon = \left\{p_X, \rho_X^{AB}\right\}$ may be positive, negative, and zero. It is not hard to know that it can be positive only for ensembles whose states are LOCC indistinguishable, i.e., $I^{LOCC}(\varepsilon) < I^{Global}(\varepsilon) = H(X)$. These ensembles have the mentioned nonlocality that is different from quantum entanglement. When $N(\varepsilon)$ is positive, we refer to the corresponding nonlocality of the ensemble as *information nonlocality.* The positive $N(\varepsilon)$ can be used as a measure for this nonlocality since it quantifies the minimal nonlocal resources (entanglement) that is needed asymptotically to get the full information $I^{Global}(\varepsilon) = H(X)$ for identifying the system's state in addition to LOCC. We can manifest the meaning of $N(\varepsilon)$ through the symbolic expression

$$N(\varepsilon)[qq] + LOCC\,|_{\varepsilon} => I^{Global}(\varepsilon), \tag{3}$$

where $[qq]$ means an ebit of quantum entanglement.

When $N(\varepsilon)$ is negative, asymptotically no nonlocal resources is needed to get the full information $I^{Global}(\varepsilon)$ for identifying the system's state (if the entanglement is still needed to assist the process, it could be viewed as a kind of catalyst), and additionally $|N(\varepsilon)|$ ebits of entanglement can be distilled. Similarly the meaning of $N(\varepsilon)$ may be manifested through the expression

$$LOCC\,|_{\varepsilon} => I^{Global}(\varepsilon) + |N(\varepsilon)|[qq]. \tag{4}$$

In this case, the ensemble ε has no information nonlocality, however it still has another kind of nonlocality since certain entanglement can be distilled. Hereafter we may refer to such kind of nonlocality of the ensemble as *entanglement nonlocality*, and employ $|N(\varepsilon)|$ as a measure to quantify it.

Interestingly, when $N(\varepsilon)$ is zero, the ensemble ϵ has neither information nonlocality nor entanglement nonlocality mentioned above. As a typical example, $N(\varepsilon) = 0$ for ensembles that consist of LOCC distinguishable product states. Notably, a single quantum state may also be regarded as a special ensemble, and $N(\varepsilon) = -D(\varepsilon)$ in this case, where $D(\varepsilon)$ denotes the distillable entanglement of the state.

Bounds for the entanglement cost.–Although the entanglement cost $N(\varepsilon)$ is usually hard to compute, some useful bounds of it can be obtained. Bennett *et al.* actually gave the first upper bound for $N(\varepsilon)$, which is the quantum entropy of the ensemble state seen by Alice [1], i.e., $N(\varepsilon) \leq S\left(\rho^A\right)$ where $\rho^A = Tr_B\left(\sum_X p_X \rho_X^{AB}\right)$ and $S(\cdot)$ is the quantum entropy. This is obtained through the protocol that Alice first compresses her state [17] and teleports it to Bob [18] and Bob then distinguishes the states locally. Actually, there are more tight upper bounds.

Theorem 1 *Suppose $\varepsilon = \left\{p_X, \rho_X^{AB}\right\}$ is an ensemble consisting of mutually orthogonal bipartite states. The entanglement cost satisfies*

$$N(\varepsilon) \leq S(A|B) = S\left(\rho^{AB}\right) - S\left(\rho^B\right), \tag{5}$$

$$N(\varepsilon) \leq S(B|A) = S\left(\rho^{AB}\right) - S\left(\rho^A\right), \tag{6}$$

where $\rho^{AB} = \sum_X p_X \rho_X^{AB}$, $\rho^B = Tr_A \rho^{AB}$, $\rho^A = Tr_B \rho^{AB}$ and $S(\cdot)$ is the quantum entropy.

Proof. The theorem can be derived from the quantum state merging [19, 20]. To distinguish the states in the ensemble $\varepsilon^{\otimes n}$, the part A^n on Alice's side can be merged to Bob and then he distinguishes the states locally. In the process of merging [19, 20], the net entanglement consumed can be $S(A|B)$ ebits per ensemble, so Eq. (5) is obtained. If the part B^n on Bob's side is first merged to Alice and then she distinguishes the states locally, similarly the net entanglement consumed can be $S(B|A)$ ebits per ensemble, so Eq. (6) is obtained. ■

The above upper bounds depend only on the ensemble state ρ^{AB}, so different ensembles may have the same upper bound. It is possible to get a smaller bound if we examine the states in the ensemble carefully since it is possible that only a part of A needs to be merged to Bob and then the states become LOCC distinguishable.

Theorem 2 *Suppose $\varepsilon = \left\{p_X, \rho_X^{AB}\right\}$ is an ensemble consisting of mutually orthogonal bipartite pure states. The entanglement cost satisfies*

$$N(\varepsilon) \geq \sum p_X S\left(\rho_X^A\right) - I_{\rho^{AB}}(A;B), \tag{7}$$

where $S(\cdot)$ is the quantum entropy and $I_{\rho^{AB}}(A;B) = S\left(\rho^A\right) + S\left(\rho^B\right) - S\left(\rho^{AB}\right)$ is the quantum mutual information with $\rho^{AB} = \sum_X p_X \rho_X^{AB}$, $\rho^A = Tr_B \rho^{AB}$, $\rho^B = Tr_A \rho^{AB}$ and $\rho_X^A = Tr_B \rho_X^{AB}$.

The quantum mutual information $I_{\rho^{AB}}(A;B)$ is always nonnegative and it can be regarded as a quantification of the total correlation between A and B, so Eq. (7) means that the entanglement cost is not smaller than the average entanglement of the states in the ensemble minus the total correlation between A and B. When $\rho^{AB}=\rho^{A}\otimes\rho^{B}$ there is $I_{\rho^{AB}}(A;B)=0$ and $N(\varepsilon)\geq\sum p_{X}S(\rho_{X}^{A})$; the average entanglement of the states in the ensemble is a lower bound of the entanglement cost. This case will happen when we consider the nonlocality of an ensemble consisting of a full basis states with equal probability.

Proof. According to the definition of $N(\varepsilon)$ we should consider distinguishing the states of the ensemble $\varepsilon^{\otimes n}=\left\{p_{X^{n}},\rho_{X^{n}}^{A^{n}B^{n}}\right\}$ using LOCC plus $n\times\alpha_{n}$ ebits of entanglement. It is equivalent to distinguish the states of the ensemble $\left\{p_{X^{n}},\rho_{X^{n}}^{A^{n}B^{n}}\otimes\Phi_{n}^{A_{0}B_{0}}\right\}$ using LOCC only, where $\Phi_{n}^{A_{0}B_{0}}$ is a bipartite pure state with $S\left(\Phi_{n}^{A_{0}}\right)=S\left(\Phi_{n}^{B_{0}}\right)=n\times\alpha_{n}$. The mutual information between X^{n} and the measurement result Y will satisfy [21]

$$I(X^{n};Y)\leq n\left(S\left(\rho^{B}\right)+S\left(\rho^{A}\right)-\sum p_{X}S\left(\rho_{X}^{A}\right)\right)+n\left(\alpha_{n}-\sum p_{Y}\beta_{nY}\right). \quad (8)$$

Note that $I(X^{n};Y)\geq I^{Global}(\varepsilon^{\otimes n})-\delta_{n}$ is required and there is $I^{Global}(\varepsilon^{\otimes n})=nS\left(\rho^{AB}\right)$, we can get

$$\alpha_{n}-\sum p_{Y}\beta_{nY}\geq\sum p_{X}S\left(\rho_{X}^{A}\right)-I_{\rho^{AB}}(A;B)-\delta_{n}/n. \quad (9)$$

Since δ_{n}/n will go to zero in the asymptotic limit, we can get Eq. (7) from Eq. (9). ■

The above two theorems give upper and lower bounds on $N(\varepsilon)$. It is valuable to know when the upper and the lower bounds will be close for ensembles consisting of mutually orthogonal bipartite pure states. We first rewrite the lower bound expression in theorem 2 as

$$N(\varepsilon)\geq S(A|B)-\chi^{A}(\varepsilon)=S(B|A)-\chi^{B}(\varepsilon), \quad (10)$$

where $\chi^{A}(\varepsilon)=S\left(\rho^{A}\right)-\sum p_{X}S\left(\rho_{X}^{A}\right)$ and $\chi^{B}(\varepsilon)=S\left(\rho^{B}\right)-\sum p_{X}S\left(\rho_{X}^{B}\right)$ are the Holevo information of the ensembles seen by Alice and Bob, respectively. It is not hard to find that the contents of the two theorems can be summarized as

$$S(A|B)-\chi^{A}(\varepsilon)\leq N(\varepsilon)\leq S(A|B) \quad (11)$$

when $\chi^{A}(\varepsilon)\leq\chi^{B}(\varepsilon)$ and

$$S(B|A)-\chi^{B}(\varepsilon)\leq N(\varepsilon)\leq S(B|A) \quad (12)$$

when $\chi^{B}(\varepsilon)\leq\chi^{A}(\varepsilon)$. From Eqs. (11) and (12), we know that the upper and the lower bounds will be closer if ever $\chi^{A}(\varepsilon)$ or $\chi^{B}(\varepsilon)$ is smaller. Noting that $\chi^{A}(\varepsilon)$ is the upper bound of the information about the value of X that Alice can gain solely [14], Eq. (11) means that the difference between the bounds is small if Alice can gain little information about the value of X without cooperation with Bob. When $\chi^{A}(\varepsilon)=0$ or $\chi^{B}(\varepsilon)=0$, the exact value of $N(\varepsilon)$ can be obtained. This occurs only when all ρ_{X}^{A} (or ρ_{X}^{B}) are the same. Consequently, we have the following result.

Corollary 3 *Suppose that $\varepsilon=\left\{p_{X},\rho_{X}^{AB}\right\}$ is an ensemble consisting of mutually orthogonal $d\times d$ maximally entangled pure states. The entanglement cost satisfies*

$$N(\varepsilon)=S\left(\rho^{AB}\right)-S\left(\rho^{B}\right)=H(X)-\log d, \quad (13)$$

where $H(\cdot)$ is the Shannon entropy.

The corollary is true since all ρ_{X}^{A} and ρ_{X}^{B} are the same maximally mixed state, so there are both $\chi^{A}(\varepsilon)=0$ and $\chi^{B}(\varepsilon)=0$; the upper and the lower bounds of $N(\varepsilon)$ becomes the same value. As is known, the four Bell states are LOCC indistinguishable [11], however the corollary indicates that an ensemble consisting of the four Bell states may not have the information nonlocality since its entanglement cost $N(\varepsilon)$ may be non-positive. The reason lies in that whether an ensemble has the information nonlocality depends not only on its states but also on the probabilities of the states, while the LOCC indistinguishability depends solely on the states. So the introduced information nonlocality is different from the LOCC indistinguishability.

There are other ways to obtain upper bounds for entanglement cost $N(\varepsilon)$. For any ensemble $\varepsilon=\left\{p_{X},\rho_{X}^{AB}\right\}$ whose states are mutually orthogonal, there exist nonlocal unitary operations U^{AB} such that the states of the ensemble $\bar{\varepsilon}=\left\{p_{X},U^{AB}\rho_{X}^{AB}U^{\dagger AB}\right\}$ are LOCC distinguishable. The operation U^{AB} can be implemented through LOCC plus some entanglement. The average entanglement needed to implement U^{AB} is an upper bound of the entanglement cost $N(\varepsilon)$. For an example we consider the ensemble ε consisting of the following states:

$$U(-\theta)\left|0\right\rangle_{A}\left|0\right\rangle_{B}, \quad U(-\theta)\left|0\right\rangle_{A}\left|1\right\rangle_{B}, \quad (14)$$
$$U(-\theta)\left|1\right\rangle_{A}\left|0\right\rangle_{B}, \quad U(-\theta)\left|1\right\rangle_{A}\left|1\right\rangle_{B}, \quad (15)$$

where $U(-\theta)=\exp\left\{-i\theta\sigma_{x}^{A}\sigma_{x}^{B}\right\}$. The similar example has appeared in Ref. [22] where the entanglement cost for nonlocal measurements is discussed. The states have the same entanglement

$$H\left(\cos^{2}\theta\right)=-\cos^{2}\theta\log_{2}\cos^{2}\theta-\sin^{2}\theta\log_{2}\sin^{2}\theta. \quad (16)$$

We first consider the case that the states have the equal probabilities. In this case the upper bounds obtained from theorem 1 can be expressed as $N(\varepsilon)\leq1$, which is not satisfactory. Note that if Alice and Bob implement $U(\theta)$ on AB the changed states will be LOCC distinguishable. The average entanglement $\bar{E}(\theta)$ needed to implement $U(\theta)$ is an upper bound of the entanglement cost $N(\varepsilon)$, i.e., $N(\varepsilon)\leq\bar{E}(\theta)$. Several expressions for $\bar{E}(\theta)$ are given [22, 23, 24, 25], and the one given in Ref. [23] shows that $\bar{E}(\theta)$ will be smaller than unit when $2\theta\leq0.75$. It means when $2\theta\leq0.75$ the upper bound expression obtained by calculating the average entanglement to implement $U(\theta)$ will be better than that in theorem 1. However this may not be true when the states have different probabilities p_{X}. To see this we note that

$S\left(\rho^A\right) \geq \sum_X p_X S\left(\rho_X^A\right) = H\left(\cos^2\theta\right)$, so from theorem 1 there is

$$N(\varepsilon) \leq S\left(\rho^{AB}\right) - S\left(\rho^A\right) \leq H(X) - H\left(\cos^2\theta\right). \quad (17)$$

The above upper bound for $N(\varepsilon)$ depends on the probabilities p_X and surely it will be smaller than $\bar{E}(\theta)$ when $H(X)$ is smaller than $\bar{E}(\theta) + H\left(\cos^2\theta\right)$.

Discussion.–The entanglement cost $N(\varepsilon)$ is defined for all ensembles consisting of mutually orthogonal bipartite states, so it is applicable not only to ensembles whose states are LOCC indistinguishable but also to ensembles whose states are LOCC distinguishable. Clearly, the ensembles may be quantitatively classified in terms of the value of $N(\varepsilon)$. Comparing with the coarse classification based on the LOCC distinguishability, this new classification is likely more precise. In addition, the entanglement cost $N(\varepsilon)$ has an operational meaning.

The nonlocality measure $N(\varepsilon)$ is defined for ensembles whose states are mutually orthogonal while its definition is also applicable to an ensemble $\varepsilon = \left\{p_X, \rho_X^{AB}\right\}$ whose states are not mutually orthogonal. In this case the states of ε cannot be distinguished even through joint measurements since $I^{Global}(\varepsilon) < H(X)$. However the entanglement cost $N(\varepsilon)$ still has an operational meaning. The entanglement cost $N(\varepsilon)$ can be positive only when $I^{LOCC}(\varepsilon) < I^{Global}(\varepsilon)$, the same as the case that the states are mutually orthogonal, and it means that joint measurements can get more information about the value of X than that can be obtained through LOCC. To make up the difference between LOCC and joint measurements in obtaining information about the value of X, quantum entanglement in addition to LOCC can be consumed. The positive $N(\varepsilon)$ just quantifies the minimal entanglement that is needed asymptotically to make up this difference. When $N(\varepsilon)$ is negative, asymptotically LOCC can get the information $I^{Global}(\varepsilon)$, and additionally $|N(\varepsilon)|$ ebits of entanglement can be distilled. The upper bound expressions in theorem 1 are also applicable but the lower bound expression in theorem 2 needs to be changed into

$$N(\varepsilon) \geq \sum p_X S\left(\rho_X^A\right) - I_{\rho^{AB}}(A;B) - \Delta(\varepsilon), \quad (18)$$

where $\Delta(\varepsilon) = S\left(\rho^{AB}\right) - I^{Global}(\varepsilon)$, which can be obtained in the same way.

Conclusion.–We have introduced the entanglement cost as a measure to quantify and classify the nonlocality of ensembles consisting of mutually orthogonal bipartite states. It can be applied not only to ensembles whose states are LOCC indistinguishable but also to ensembles whose states are LOCC distinguishable. The nonlocality measure has an operational meaning even when the states of the ensemble are not mutually orthogonal. The present work is expected to evoke more profound understandings of nonlocality in ensembles.

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