Complete proof of Gisin's theorem for three qubits

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Gisin's theorem assures that for any pure bipartite entangled state, there is violation of Bell-CHSH inequality revealing its contradiction with local realistic model. Whether, similar result holds for three-qubit pure entangled states, remained unresolved. We show analytically that all three-qubit pure entangled states violate a Bell-type inequality, derived on the basis of local realism, by exploiting the Hardy's non-locality argument.

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Not all measurement correlations in some state of a composite quantum system can be described by local hidden variable theory (LHVT) [1], a fact which is said to be "the most profound discovery of science" [2]. Experimental verification of this fact (*i.e.*, whether measurement correlations in Nature obey quantum rules or LHVT) goes also in favour of Qunatum Theory, modulo some loopholes [3]. Every LHVT description [4] of the measurement correlations of a composite system (assumed to be finite dimensional in the present paper) gives rise to one (or, more than one) linear inequality (or, inequalities) involving these correlations [5]. There are states of a composite quantum system, which violate some or all these inequalities for suitable choices of the subsystem observables.

Gisin's theorem assures that for any pure entangled state of two-qudits, the above-mentioned violation is generic for two settings per site (*i.e.*, for the choice of one between two non-commuting observables per qudit) [6]. In other words, *all* pure entangled states of two d-dimensional quantum systems violate a *single* Bell-type inequality with two settings per site, where the choice of the observables depends on that of the state. However, validity of this theorem for multi-partite systems is still not guaranteed. For example, for odd N, there is a family of entangled pure states of N qubits, each of which satisfies *all* Bell-type inequalities involving correlation functions, arising out of measurement of one between two non-commuting dichotomic observables per qubit [7]. Later, Chen et al. [8] provided a Bell-type inequality involving joint probabilities, associated to measurement of one between two non-commuting dichotomic observables per qubit, which is violated by all the states of the above-mentioned family [9]. But a single Bell-type inequality is not guaranteed to be violated by *all* pure entangled states of three-qubits, although there are claims of having numerical evidences in favour of this violation [8, 10, 11].

Quantum Theory also shows contradiction with LHVT via 'non-locality without inequalities' (NLWI) [12]. In this case, a set of values of joint probabilities of outcomes of measurements of one between two non-commuting observables per site, has contradiction with LHVT but can be realized in Quantum Theory. Unfortunately, NLWI is weaker than Bell-type inequalities, as *no* maximally entangled state of two-qudits seems to show NLWI (in the case of Hardy-type NLWI, this has been shown in [13]) even though each of them violates a Bell-type inequality. This situation changes drastically when we consider Hardy-type NLWI for three two-level systems [14, 15], where all but one of the joint probabilities in the above-mentioned set are zero. *Every* maximally entangled state of three qubits [16] satisfies Hardy-type NLWI for suitably chosen pairs of non-commuting dichotomic observables per qubit [17]. Moreover, an attempt was made in ref. [17] towards achieving the result that *every* pure state of three qubits, having genuine tri-partite entanglement, satisfies Hardy-type NLWI. But in absence of the discovery of canonical form for three-qubit pure states (which was done later in ref. [18]), it did not yield a complete proof.

Now, from the set of joint probabilities in any NLWI argument, one can, in principle, construct a linear inequality involving these joint probabilities by using local realistic assumption. This inequality is automatically violated by every quantum state which satisfies the corresponding NLWI argument. In the case of Hardy-type NLWI argument for two two-level systems, this inequality (given in equation (11) of ref. [19], equation (11) of ref. [20], and equation (26) of ref. [21]) is nothing but the corresponding CH inequality [22]. So, by Gisin's theorem, *every* two-qubit pure entangled state (irrespective of its amount of entanglement) will violate the former inequality. In this letter, we show *analytically* that *every* three-qubit pure entangled state violates a linear inequality of the above-mentioned type (see eqn. (3) below) involving joint probabilities associated with the Hardy-type NLWI, irrespective of whether the state has genuine tripartite entanglement or pure bi-partite entanglement.

Hardy-type NLWI argument starts from the following set of five joint probability conditions for three two-level systems:

$$\begin{array}{rcl} P(D_1=+1,U_2=+1,U_3=+1) & = & 0, \\ P(U_1=+1,D_2=+1,U_3=+1) & = & 0, \\ P(U_1=+1,U_2=+1,D_3=+1) & = & 0, \\ P(D_1=-1,D_2=-1,D_3=-1) & = & 0, \\ P(U_1=+1,U_2=+1,U_3=+1) & > & 0, \end{array} \tag{1}$$

where each U_j (as well as D_j) is a $\{+1,-1\}$-valued random variable. This set of conditions can not be satisfied by a local realistic theory, and hence it has contradiction with LHVT [14, 17].

To show that in quantum theory there are states which exhibit this kind of non-locality, we replace U_j and D_j by the

$\{+1,-1\}$-valued observables $\hat{U}_j$ and $\hat{D}_j$ respectively with $[\hat{U}_j, \hat{D}_j] \neq 0$. The probabilities appearing in (1) are expectation values of one dimensional projectors corresponding to the following five product vectors:

$$
\begin{aligned}
&|\hat{D}_1 = +1\rangle|\hat{U}_2 = +1\rangle|\hat{U}_3 = +1\rangle,\\
&|\hat{U}_1 = +1\rangle|\hat{D}_2 = +1\rangle|\hat{U}_3 = +1\rangle,\\
&|\hat{U}_1 = +1\rangle|\hat{U}_2 = +1\rangle|\hat{D}_3 = +1\rangle,\\
&|\hat{D}_1 = -1\rangle|\hat{D}_2 = -1\rangle|\hat{D}_3 = -1\rangle,\\
&|\hat{U}_1 = +1\rangle|\hat{U}_2 = +1\rangle|\hat{U}_3 = +1\rangle.
\end{aligned}
$$

One can easily check that these five vectors are linearly independent and hence span five dimensional subspace of the eight dimensional Hilbert space associated to the total system. Hence one can choose any one (among infinitely many) vector which is orthogonal to the first four vectors and non-orthogonal to the last one. Actually this result shows that for any choice of observables in the above-mentioned non-commuting fashion, one can always find a quantum state which exhibits contradiction with local realism [23].

But, in this letter, our purpose is to find the converse. We would like to see whether every three-qubit pure entangled state exhibits contradiction with local realism. In this context, it has to be mentioned that Gisin's theorem could provide a single prescription for finding the observables for any bipartite pure state to show violation of the Bell-CHSH inequality, due to the existence of Schmidt decomposition. Schmidt decomposition, in its strict sense [24], is absent for systems comprising of three and more subsystems. This gives rise to complications and one needs to find the observables for each inequivalent case (depending upon the values of the parameters describing the state) separately. In this direction, we start with an arbitrary three-qubit pure state $|\psi\rangle$, which can always be taken in the canonical form [18]:

$$
|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \tag{2}
$$

where $0 \leq \lambda_j$ (for $j = 0,1,2,3,4$), $\sum_{j=0}^{4} \lambda_j^2 = 1$, and $0 \leq \phi \leq \pi$.

We now fully classify the above-mentioned three-qubit state $|\psi\rangle$ into four major classes: (A) $|\psi\rangle$ is a fully product state, (B) $|\psi\rangle$ has pure two-qubit non-maximal entanglement, (C) $|\psi\rangle$ has pure two-qubit maximal entanglement, and (D) $|\psi\rangle$ has genuine pure three-qubit entanglement. Depending on the values of $\lambda_i's$ and ϕ, in table I, we further classify each of these four classes into several sub-classes: (A) consists of (A.1) - (A.3); (B) consists of (B.1) - (B.5); (C) consists of (C.1) - (C.3); (D) consists of (D.1) - (D.14).

If $|\psi\rangle$ has only bi-partite non-maximal entanglement we then first consider the situation where $|\psi\rangle = |\eta\rangle \otimes |\chi\rangle$, with $|\eta\rangle$ being a two-qubit non-maximally entangled state of the first and the second qubits, while $|\chi\rangle$ is a state of the third qubit. Hardy [12] has shown that for all two-qubit non-maximally entangled pure states, one can choose observables for both the

TABLE I: Classification of $|\psi\rangle$.

Condition	case
$\lambda_0\lambda_1 \neq 0,\ \lambda_2 = \lambda_3 = \lambda_4 = 0$	(A.1)
$\lambda_0 \neq 0,\ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$	(A.2)
$\lambda_0 = 0,\ \lambda_1\lambda_4 e^{i\phi} = \lambda_2\lambda_3$	(A.3)
$\lambda_0\lambda_1\lambda_2 \neq 0,\ \lambda_3 = \lambda_4 = 0$	(B.1)
$\lambda_0\lambda_1\lambda_3 \neq 0,\ \lambda_2 = \lambda_4 = 0$	(B.2)
$0 < \lambda_0\lambda_2 < 1/2,\ \lambda_1 = \lambda_3 = \lambda_4 = 0$	(B.3)
$0 < \lambda_0\lambda_3 < 1/2,\ \lambda_1 = \lambda_2 = \lambda_4 = 0$	(B.4)
$\lambda_0 = 0$ and $\sqrt{2}\begin{pmatrix} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}$ is neither a singular matrix nor a unitary matrix	(B.5)
$\lambda_0\lambda_2 = 1/2,\ \lambda_1 = \lambda_3 = \lambda_4 = 0$	(C.1)
$\lambda_0\lambda_3 = 1/2,\ \lambda_1 = \lambda_2 = \lambda_4 = 0$	(C.2)
$\lambda_0 = 0$ and $\sqrt{2}\begin{pmatrix} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}$ is a unitary matrix	(C.3)
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0,\ \phi > 0$	(D.1)
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0,\ \phi = 0,\ \lambda_2\lambda_3 \neq \lambda_1\lambda_4$	(D.2)
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0,\ \phi = 0,\ \lambda_2\lambda_3 = \lambda_1\lambda_4$	(D.3)
$\lambda_0\lambda_1\lambda_2\lambda_3 \neq 0,\ \lambda_4 = 0$	(D.4)
$\lambda_0\lambda_1\lambda_2\lambda_4 \neq 0,\ \lambda_3 = 0$	(D.5)
$\lambda_0\lambda_1\lambda_3\lambda_4 \neq 0,\ \lambda_2 = 0,\ \lambda_0 \neq \lambda_4$	(D.6)
$\lambda_0\lambda_1\lambda_3\lambda_4 \neq 0,\ \lambda_2 = 0,\ \lambda_0 = \lambda_4$	(D.7)
$\lambda_0\lambda_1\lambda_4 \neq 0,\ \lambda_2 = \lambda_3 = 0$	(D.8)
$\lambda_0\lambda_3\lambda_4 \neq 0,\ \lambda_1 = \lambda_2 = 0$	(D.9)
$\lambda_0\lambda_2\lambda_3\lambda_4 \neq 0,\ \lambda_1 = 0,\ \lambda_2 \neq \lambda_4$	(D.10)
$\lambda_0\lambda_2\lambda_3\lambda_4 \neq 0,\ \lambda_1 = 0,\ \lambda_2 = \lambda_4$	(D.11)
$\lambda_0\lambda_2\lambda_3 \neq 0,\ \lambda_1 = \lambda_4 = 0$	(D.12)
$\lambda_0\lambda_2\lambda_4 \neq 0,\ \lambda_1 = \lambda_3 = 0$	(D.13)
$\lambda_0\lambda_4 \neq 0,\ \lambda_1 = \lambda_2 = \lambda_3 = 0$	(D.14)

qubits in such a way that the condition of non-locality without inequality holds. Now in our three-qubit case, we first choose $|\hat{U}_3 = +1\rangle = (1/\sqrt{2})(|\chi\rangle + |\chi^\perp\rangle)$ and $|\hat{D}_3 = +1\rangle = |\chi^\perp\rangle$, where $\langle\chi^\perp|\chi\rangle = 0$. We can then choose two pairs of non-commuting dichotomic observables $(\hat{U}_1, \hat{D}_1)$ and $(\hat{U}_2, \hat{D}_2)$ in such a way that the state $|\eta\rangle$ satisfies Hardy's NLWI conditions for two two-level systems corresponding to these observables . This immediately shows that the state $|\psi\rangle$ satisfies the Hardy-type NLWI condition (1). As condition (1) is symmetric with respect to the qubits, we see that for each of the cases (B.1) - (B.5), the state $|\psi\rangle$ will satisfy the Hardy-type NLWI argument (1).

Again, let $|\psi\rangle = |\eta\rangle \otimes |\chi\rangle$, where $|\eta\rangle$ is a two-qubit maximally entangled state of the first and the second qubits, while $|\chi\rangle$ is a state of the third qubit. If we now demand $|\psi\rangle$ to satisfy (1), it will immediately follow that $|\eta\rangle$ must satisfy Hardy's NLWI conditions for two two-level systems – an impossibility [12]. As above, we see that in none of the cases (C.1) - (C.3), state $|\psi\rangle$ will satisfy the Hardy-type NLWI argument.

If $|\psi\rangle$ have genuine tri-partite entanglement then to show that it satisfies the Hardy-type NLWI argument (1), one can choose the three pairs of $\{+1,-1\}$-valued non-commutating observables $(\hat{U}_j, \hat{D}_j)$ (where j is associated with j-th system

($j = 1, 2, 3$)) as follows:

$$|\hat{U}_j = +1\rangle = k_j(\alpha_j|0\rangle+\beta_j|1\rangle),\ |\hat{D}_j = +1\rangle = l_j(\gamma_j|0\rangle+\delta_j|1\rangle),$$

where $0 < |k_j l_j(\alpha_j\gamma_j^* + \beta_j\delta_j^*)|, |k_j l_j(\alpha_j\delta_j - \beta_j\gamma_j)| < 1$, $|k_j\alpha_j|^2 + |k_j\beta_j|^2 = |l_j\gamma_j|^2 + |l_j\delta_j|^2 = 1$, and k_j's, l_j's are the normalization constants (for $j = 1, 2, 3$). The values of $\alpha_j,\ \beta_j,\ \gamma_j,\ \delta_j$ are given in table II for all the cases (D.1) - (D.14).

We now derive a linear inequality (mentioned in equation (7) of ref. [25] for n qubits) involving the joint probabilities in equation (1), starting from local realistic theory. For this, we first assume that all the experimental probabilities $P(A_k = i_k)$, $P(A_k = i_k, A_l = i_l)$, $P(A_k = i_k, A_l = i_l, A_m = i_m)$ (with $A_k \in \{U_k, D_k\}$, and $i_k \in \{+1, -1\}$ for $k, l, m = 1, 2, 3$, and $k \neq l \neq m$) can be described by a local hidden variable ω, defined on the probability space Ω with probability density $\rho(\omega)$. For local realistic theory, the probabilities would satisfy the following conditions:
i) $P_\omega(A_k = i_k)$ (for $i_k \in \{+1, -1\}$ with $k = 1, 2, 3$) can only take values 1 or 0.
ii)$P_\omega(A_k = i_k, A_l = i_l) = P_\omega(A_k = i_k)P_\omega(A_l = i_l)$, $P_\omega(A_k = i_k, A_l = i_l, A_m = i_m) = P_\omega(A_k = i_k)P_\omega(A_l = i_l)P_\omega(A_m = i_m)$.

Condition (i) is equivalent to assigning definite values to the observables. In any LHVT, the experimental probabilities would be reproduced in the following way:
$P(A_k = i_k) = \int_\Omega \rho(\omega)d\omega P_\omega(A_k = i_k)$, $P(A_k = i_k, A_l = i_l) = \int_\Omega \rho(\omega)d\omega P_\omega(A_k = i_k, A_l = i_l)$, $P(A_k = i_k, A_l = i_l, A_m = i_m) = \int_\Omega \rho(\omega)d\omega P_\omega(A_k = i_k, A_l = i_l, A_m = i_m)$, where $\int_\Omega \rho(\omega)d\omega = 1$.

Now consider the following quantity

$$\begin{aligned} B(\omega) &= P_\omega(D_1 = -1)P_\omega(D_2 = -1)P_\omega(D_3 = -1) \\ &+ P_\omega(D_1 = +1)P_\omega(U_2 = +1)P_\omega(U_3 = +1) \\ &+ P_\omega(U_1 = +1)P_\omega(D_2 = +1)P_\omega(U_3 = +1) \\ &+ P_\omega(U_1 = +1)P_\omega(U_2 = +1)P_\omega(D_3 = +1) \\ &- P_\omega(U_1 = +1)P_\omega(U_2 = +1)P_\omega(U_3 = +1). \end{aligned}$$

One can easily check that $B(\omega) \geq 0$ for all $\omega \in \Omega$. Then obviously

$$\int_\Omega \rho(\omega)d\omega B(\omega) \geq 0,$$

which, in turn, gives rise to the following Bell-type inequality:

$$\begin{aligned} P(D_1 = -1, D_2 = -1, D_3 = -1) + P(D_1 = +1, U_2 = +1, U_3 = +1) + P(U_1 = +1, D_2 = +1, U_3 = +1) \\ +P(U_1 = +1, U_2 = +1, D_3 = +1) - P(U_1 = +1, U_2 = +1, U_3 = +1) \geq 0. \end{aligned} \tag{3}$$

Thus we see that every LHVT satisfies the inequality (3).

From our above-mentioned discussion on Hardy-type NLWI, it follows that *every* three-qubit pure state will violate the inequality (3) unless it is a fully product state or it has pure bi-partite maximal entanglement. We now show that this inequality is even violated when $|\psi\rangle$ has pure bi-partite maximal entanglement, although, in this case, $|\psi\rangle$ does not satisfy the Hardy-type NLWI condition (1). Without loss of generality, we can take $|\psi\rangle$ in this case as: $|\psi\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle) \otimes |0\rangle$. Choose $|\hat{U}_1 = +1\rangle = (\sqrt{0.96}|0\rangle + 0.2|1\rangle)$, $|\hat{D}_1 = +1\rangle = |0\rangle$, $|\hat{U}_2 = +1\rangle = (0.2|0\rangle + \sqrt{0.96}|1\rangle)$, $|\hat{D}_2 = +1\rangle = |1\rangle$, $|\hat{U}_3 = +1\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ $|\hat{D}_3 = +1\rangle = |1\rangle$. With this choice, $|\psi\rangle$ will violate the inequality (3).

Thus we have established the desired result that *every pure entangled state of three qubits violates the Bell-type inequality* (3).

If a three-qubit state $|\psi\rangle$ violates the inequality (3) maximally corresponding to the set $\mathcal{S}(\psi)$ of three pairs of non-commuting observables $(\hat{U}_1, \hat{D}_1)$, $(\hat{U}_2, \hat{D}_2)$, and $(\hat{U}_3, \hat{D}_3)$, then the minimum value of the coefficient $v \in [0, 1]$ for which the state $\rho(\psi, v) \equiv v|\psi\rangle\langle\psi| + ((1 - v)/8)I$ (I being the 8×8 identity matrix) also violates the inequality (3), is called here as the 'threshold visibility' of the state $|\psi\rangle$. Lower the amount of threshold visibility, higher the amount of noise the inequality can sustain. The maximum negative violation of the inequality (3) by the GHZ state is numerically found to be -0.175459 (approx.), and so the threshold visibility v_{GHZ}^{thr} of this state turns out to be 0.68125 (approx.), which is approximately same as that found in [11]. On the other hand, the maximum negative violation of the inequality (3) by the W-state $(1/\sqrt{3})(|001\rangle + |010\rangle + |100\rangle)$ is numerically found to be -0.192608 (approx.), and so the threshold visibility v_W^{thr} of this state turns out to be 0.6606676 (approx.), which is also approximately same with the value 0.660668 of v_W^{thr}, found in [11]. It is to be noted that so far as the values of v_{GHZ}^{thr}, v_W^{thr} are concerned, although the probabilistic Bell-type inequality (18) of ref. [11] and the above-mentioned inequality (3) provide approximately the same values, unlike inequality (3), neither inequality (18) of [11] nor any other inequality, mentioned in the literature till date (see, for example, [8, 10, 11]), is analytically guaranteed to be violated by all pure entangled states of three qubits. By considering a modified form of the inequality (3) (e.g., inequality (11) of [20]), one may get a lower value of the threshold visibility for the states.

One may also try to find similar feature (*i.e.*, violation of

TABLE II: Observables for genuine tri-partite pure entanglement.

Case	Set of observables for different cases
(D.1), (D.2), (D.4), (D.5)	$\alpha_1 = \lambda_1,\ \beta_1 = -\lambda_0 e^{i\phi},\ \gamma_1 = 0,\ \delta_1 = 1$; $\alpha_2 = 1,\ \beta_2 = 0,\ \gamma_2 = \lambda_2\lambda_3 e^{i\phi} - \lambda_1\lambda_4,\ \delta_2 = \lambda_1\lambda_2$; $\alpha_3 = \lambda_2 e^{i\phi},\ \beta_3 = -\lambda_1,\ \gamma_3 = 1,\ \delta_3 = 0$
(D.3)	$\alpha_1 = 0,\ \beta_1 = 1,\ \gamma_1 = \lambda_0\lambda_1,\ \delta_1 = (1-\lambda_o^2)$; $\alpha_2 = \lambda_1\tau - \lambda_3\epsilon,\ \beta_2 = \lambda_3\tau + \lambda_1\epsilon,\ \gamma_2 = \lambda_3,\ \delta_2 = -\lambda_1$; $\alpha_3 = \lambda_1 + \lambda_2,\ \beta_3 = \lambda_2 - \lambda_1,\ \gamma_3 = \lambda_2,\ \delta_3 = -\lambda_1$, where, $\tau = \lambda_0^2\lambda_3(\lambda_1 + \lambda_2),\ \epsilon = \lambda_0^2\lambda_1(\lambda_1+\lambda_2) + (1-\lambda_0^2)$
(D.6)	$\alpha_1 = 0,\ \beta_1 = 1,\ \gamma_1 = \lambda_1 e^{-i\phi}(\lambda_4^2 - \lambda_0^2),\ \delta_1 = -\lambda_0(1-\lambda_0^2)$; $\alpha_2 = \lambda_3(1-\lambda_0^2),\ \beta_2 = -\lambda_1 e^{-i\phi}(1-\lambda_4^2),\ \gamma_2 = \lambda_3,\ \delta_2 = -\lambda_1 e^{-i\phi}$; $\alpha_3 = 1,\ \beta_3 = 0,\ \gamma_3 = \lambda_4(1-\lambda_4^2),\ \delta_3 = \lambda_3(\lambda_4^2 - \lambda_0^2)$
(D.7)	$\alpha_1 = \lambda_1 e^{-i\phi},\ \beta_1 = -\lambda_0,\ \gamma_1 = 0,\ \delta_1 = 1$; $\alpha_2 = \lambda_3,\ \beta_2 = -\lambda_1 e^{-i\phi},\ \gamma_2 = 1,\ \delta_2 = 0$; $\alpha_3 = 1,\ \beta_3 = 0,\ \gamma_3 = \lambda_0,\ \delta_3 = -\lambda_3$
(D.8)	$\alpha_1 = 0,\ \beta_1 = 1,\ \gamma_1 = \lambda_1 e^{-i\phi}(\epsilon + \lambda_4),\ \delta_1 = -\lambda_o\epsilon$; $\alpha_2 = 1,\ \beta_2 = 1,\ \gamma_2 = \lambda_4,\ \delta_2 = -\epsilon$; $\alpha_3 = \epsilon,\ \beta_3 = \lambda_1 e^{-i\phi},\ \gamma_3 = \lambda_4,\ \delta_3 = -\lambda_1 e^{-i\phi}$; with ϵ being a solution of $z^2(1-\lambda_4^2) + z\lambda_4(1-\lambda_0^2) + \lambda_4^4 = 0$
(D.9), (D.10)	$\alpha_1 = \lambda_2(\lambda_2^2 + \lambda_4^2) + \lambda_4(1-\lambda_0^2),\ \beta_1 = -\lambda_0\lambda_3\lambda_4,\ \gamma_1 = 1,\ \delta_1 = 0$; $\alpha_2 = 1,\ \beta_2 = 1,\ \gamma_2 = \lambda_4,\ \delta_2 = -\lambda_2$; $\alpha_3 = 0,\ \beta_3 = 1,\ \gamma_3 = \lambda_3\lambda_4,\ \delta_3 = \lambda_2^2 + \lambda_4^2$
(D.11)	$\alpha_1 = 0,\ \beta_1 = 1,\ \gamma_1 = \lambda_2^2\lambda_3,\ \delta_1 = \lambda_0(\lambda_2^2 + \lambda_3^2)$; $\alpha_2 = \lambda_2^2 + \lambda_3^2,\ \beta_2 = -\lambda_2^2,\ \gamma_2 = 1,\ \delta_2 = 0$; $\alpha_3 = 1,\ \beta_3 = 0,\ \gamma_3 = \lambda_3,\ \delta_3 = \lambda_2$
(D.12)	$\alpha_1 = 0,\ \beta_1 = 1,\ \gamma_1 = \delta\lambda_0\lambda_2\lambda_3,\ \delta_1 = \lambda_2^3\delta + \lambda_3^3$; $\alpha_2 = 1,\ \beta_2 = 1,\ \gamma_2 = \lambda_3,\ \delta_2 = -\lambda_2\delta$; $\alpha_3 = 1,\ \beta_3 = \delta,\ \gamma_3 = \lambda_2,\ \delta_3 = -\lambda_3$; with δ being a solution of $z^2\lambda_2^4 + z\lambda_2\lambda_3 + \lambda_3^4 = 0$
(D.13)	$\alpha_1 = 1,\ \beta_1 = 1,\ \gamma_1 = \lambda_2,\ \delta_1 = -\lambda_0\epsilon$; $\alpha_2 = 1,\ \beta_2 = 0,\ \gamma_2 = \lambda_4,\ \delta_2 = -(\lambda_0\epsilon + \lambda_2)$; $\alpha_3 = \epsilon,\ \beta_3 = 1,\ \gamma_3 = \lambda_2,\ \delta_3 = -\lambda_0$; with ϵ being a solution of $z^2\lambda_0^4 + z\lambda_0\lambda_2(\lambda_0^2 + \lambda_2^2) + \lambda_2^2(\lambda_2^2 + \lambda_4^2) = 0$
(D.14)	$\alpha_1 = 1,\ \beta_1 = 1,\ \gamma_1 = i\lambda_0,\ \delta_1 = -\lambda_4$; $\alpha_2 = 1,\ \beta_2 = 1,\ \gamma_2 = i\lambda_0,\ \delta_2 = -\lambda_4$; $\alpha_3 = \lambda_4^2,\ \beta_3 = i\lambda_0^2,\ \gamma_3 = \lambda_4,\ \delta_3 = -\lambda_0$

Bell-type inequality, derived from Hardy-type NLWI argument, by all pure entangled states) in the case of n-partite quantum systems.

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