Darboux transformation for two component derivative nonlinear Schrödinger equation

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Abstract

In this paper, we consider the two component derivative nonlinear Schrödinger equation and present a simple Darboux transformation for it. By iterating this Darboux transformation, we construct a compact representation for the $N-$soliton solutions.

Key words: Darboux transformation, solitons, DNLS

1 Introduction

The nonlinear partial differential equations with multi-soliton solutions have been studied extensively. They are often widely applicable in physics and thus constitute very important equations in mathematical physics. The celebrated examples include Korteweg-de Vries equation, sine-Gordon equation and nonlinear Schrödinger (NLS) equation and many others. These systems, named as soliton or integrable equations, are also very rich in mathematical properties and whole subject is closely related to other mathematical branches such as differential geometry, algebraic geometry, combinatorics, Lie algebras, etc [2].

Since integrable systems have remarkable mathematical properties and numerous physical applications, their generalizations or extensions have attracted attention of many researchers. One possible direction is multi-component generalization. This sort of extensions may also be physically interested. The most famous example might be the Manakov's two component NLS equation, which now is one of the most important equations in theory of pulse propagation along the optical fiber.

Another interesting soliton equation is the derivative nonlinear Schrödinger (DNLS) equation

$$iq_t = -q_{xx} + \frac{2}{3}i\epsilon(|q|^2q)_x,$$

which appeared in Plasma physics (see [3][4]), describing Alfvén wave propagation along the magnetic field. This equation was solved by inverse scattering transformation by Kaup and Newell [5]. Much research has been conducted for it and many results have been achieved. We mention here a simple looked Darboux transform, obtained independently by Imai [11] and Steudel [12], enables one to get its explicit $N-$soliton solution. The two component extension of DNLS equation was constructed by Morris and Dodd [6]. It reads as

$$iq_{1t} = -q_{1xx} + \frac{2}{3}i\epsilon\left[(|q_1|^2 + |q_2|^2)q_1\right]_x, \tag{1}$$

$$iq_{2t} = -q_{2xx} + \frac{2}{3}i\epsilon\left[(|q_1|^2 + |q_2|^2)q_2\right]_x, \tag{2}$$

where $\epsilon = \pm 1$. This system was studied by means of inverse scattering transformation. For convenience, we take $\epsilon = -1$ in the subsequent discussion. The zero-curvature representation in this case reads as

$$\Phi_x = U\Phi, \tag{3}$$

$$\Phi_t = V\Phi, \tag{4}$$

where $\Phi = \left(\phi_1, \phi_2, \phi_2\right)^T$, ζ is the spectral parameter and

$$U = U_2\zeta^2 + U_1\zeta,\ V = \zeta^4 V_4 + \zeta^3 V_3 + \zeta^2 V_2 + \zeta V_1$$

with

$$U_2 = \begin{pmatrix} -2i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix},\ U_1 = \begin{pmatrix} 0 & q_1 & q_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix},\ V_4 = \begin{pmatrix} -9i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 & 3q_1 & 3q_2 \\ 3r_1 & 0 & 0 \\ 3r_2 & 0 & 0 \end{pmatrix},\ V_2 = \begin{pmatrix} -i(r_1q_1 + r_2q_2) & 0 & 0 \\ 0 & ir_1q_1 & ir_1q_2 \\ 0 & ir_2q_1 & ir_2q_2 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & iq_{1x} + \frac{2}{3}(r_1q_1 + r_2q_2)q_1 & iq_{2x} + \frac{2}{3}(r_1q_1 + r_2q_2)q_2 \\ -ir_{1x} + \frac{2}{3}(r_1q_1 + r_2q_2)r_1 & 0 & 0 \\ -ir_{2x} + \frac{2}{3}(r_1q_1 + r_2q_2)r_2) & 0 & 0 \end{pmatrix}.$$

Then a straightforward calculation shows that the compatibility condition of (3)-(4) leads to a system which reduces to (1) and (2) under condition $r_k^* = -q_k$ $(k = 1, 2)$.

The purpose of this paper is to construct a compact representation of the $N-$soliton solution for the two component DNLS equation. We shall take Darboux transformation approach. Indeed, the original Darboux transformation, which is associated with Sturm-Louiville equation, has been generalized to many other differential and difference equations. It turns out that this approach often leads to nice representations in terms of determinants for solutions of nonlinear systems and thus an ideal method to construct $N-$soliton solutions (see [7][8][9][10]). In particular, Darboux transformations for certain multi-component integrable equations have been studied in [13][14].

The paper is organized as follows. In next section, we construct an elementary Darboux transformation for the general system (3)-(4), which naturally induces a Darboux transformation for the conjugate system. Then, we combine two Darboux transformations together and find a two-fold Darboux transformation, which turns out to be the proper one for the reduction we are interested in. The reduction problem will be tackled in the section 3 and an elegant Darboux transformation will be given there for our two component DNLS equation. In section 4, we iterate our Darboux transformation and give N-soliton solutions of two component DNLS equation in terms of determinants. Final section includes some discussion.

2 Darboux transformation in general

We now consider the general linear system (3)-(4) and manage to find a Darboux transformation for it. Our strategy is to find a proper Darboux transformation such that it can be easily reduced to the two-component DNLS case. To this aim, we start with an elementary Darboux transformation

$$\hat{\Phi} = T_1 \Phi$$

with Darboux matrix $T_1 = \zeta T_{11} + T_{10}$. After some calculations and analysis, we find that T_1 has to take the following explicit form

$$T_1 \quad = \quad \begin{pmatrix} a\zeta & c_1 & c_2 \\ c_3 & b\zeta & c\zeta \\ c_4 & d\zeta & e\zeta \end{pmatrix}, \tag{5}$$

where a, b, c, d and e are the functions of (x, t), while c_1, c_2, c_3 and c_4 need to be constants. For convenience, we make the assumption

$$c_1 = c_3 = 1, c_2 = c_4 = 0. \tag{6}$$

Since $\mathrm{Tr}(U) = 0$, $\mathrm{Tr}(V) = -9i\zeta^4$, we may assume

$$\det(T_1) = \zeta(\frac{\zeta^2}{\zeta_1^2} - 1) \tag{7}$$

where ζ_1 is a complex constant. Thus, the Darboux matrix T_1 is singular at $\zeta = \zeta_1$. Next we associate the entries of T_1 with a special solution of our linear systems (3)-(4). To this end, taking $(\varphi_1, \varphi_2, \varphi_3)^T$ as a corresponding solution of the Lax pairs at $\zeta = \zeta_1$ and requiring

$$T_1|_{\zeta=\zeta_1} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0, \tag{8}$$

we obtain

$$a = -\frac{\varphi_2}{\zeta_1 \varphi_1},\ b = \frac{-\varphi_1 + \zeta_1 Q_2 \varphi_3}{\zeta_1 \varphi_2},\ c = -Q_2,\ d = -\frac{\varphi_3}{\varphi_2},\ e = 1. \tag{9}$$

where Q_2 is a potential: $Q_{2,x} = q_2$.

Now we have the following

Theorem 1 *Let $(\varphi_1, \varphi_2, \varphi_3)^T$ be a particular solution of* (3)-(4) *at $\zeta = \zeta_1$ and the matrix T_1 be given by (5) with entries defined by* (6) *and* (9). *Then T_1 is a Darboux matrix for the linear system* (3)*)-(*(4)*, namely $\hat{\Phi} = T_1\Phi$ is a new solution of* (3)*)-(*(4)*). The transformations between fields are given by*

$$\hat{q}_1 = \frac{(q_1\varphi_2 + q_2\varphi_3 - 3i\zeta_1\varphi_1)\varphi_2}{\varphi_1^2}, \qquad \hat{r}_1 = \frac{(r_1\varphi_1 + 3i\zeta_1\varphi_2)\varphi_1 + (r_2\varphi_2 - r_1\varphi_3)\zeta_1 Q_2\varphi_1}{\varphi_2^2},$$

$$\hat{r}_2 = \frac{(r_1\varphi_3 - r_2\varphi_2)\zeta_1\varphi_1}{\varphi_2^2}, \qquad \hat{q}_2 = \frac{(\varphi_2 q_1 + \varphi_3 q_2 - 3i\zeta_1\varphi_1)\zeta_1 Q_2\varphi_2 - q_2\varphi_1\varphi_2}{\zeta_1\varphi_1^2},$$

where hatted quantities are transformed variables.

Proof: What we need to do is to check that the following equations

$$T_{1x} + T_1U = \hat{U}T_1, \qquad T_{1t} + T_1V = \hat{V}T_1$$

hold. Where

$$\hat{U} = \zeta^2 U_2 + \zeta\hat{U}_1, \hat{V} = \zeta^4 V_4 + \zeta^3\hat{V}_3 + \zeta^2\hat{V}_2 + \zeta\hat{V}_1$$

and $\hat{U}_1$, $\hat{V}_3$, $\hat{V}_2$ and $\hat{V}_1$ are U_1, V_3, V_2, V_1 with the corresponding entries r_1, r_2, q_1 and q_2 are replaced respectively by $\hat{r}_1$, $\hat{r}_2$,$\hat{q}_1$ and $\hat{q}_2$. Checking can be done by direct calculations.

Remark 1 *It is interesting to note that under this Darboux transformation, we also have an alternative representation for $\hat{r}_2$: $d_x = \hat{r}_2$.*

To proceed, we notice that the two component DNLS equation also has the following Lax pairs

$$-\Psi_x = \Psi U, \tag{10}$$

$$-\Psi_t = \Psi V, \tag{11}$$

where $\Psi = \begin{pmatrix}\phi_1, & \phi_2, & \phi_3\end{pmatrix}$ and U and V are as above. This linear problem actually is the conjugate problem of (3)-(4). A simple but useful observation is

Lemma 1 *If the matrix T is a Darboux matrix of the original linear system* (3)-(4)*, then T^{-1} is a Darboux matrix of the conjugate linear system* (10)-(11).

Proof: Direct calculation.

Now we consider the conjugate linear system and its Darboux transformation. The analysis goes as in the case of the original linear system: Taking (χ_1, χ_2, χ_3) as a special solution of the system (10)-(11) at $\zeta = \zeta_2$, and constructing the following matrix

$$T_2 = \begin{pmatrix} \hat{a}\zeta & 1 & 0 \\ 1 & \hat{b}\zeta & \hat{c}\zeta \\ 0 & \hat{d}\zeta & \zeta \end{pmatrix}, \tag{12}$$

where

$$\hat{a} = -\frac{\chi_2}{\zeta_2\chi_1},\ \hat{b} = -\frac{\chi_1 + \zeta_2 R_2\chi_3}{\zeta_2\chi_2},\ \hat{c} = -\frac{\chi_3}{\chi_2},\ \hat{d} = R_2 \tag{13}$$

and

$$R_{2,x} = r_2,$$

we have

Theorem 2 *The matrix* T_2 *defined by* (12) *is an elementary Darboux matrix of the conjugate linear system* (10)-(11) *and the transformations between the field variables are given by*

$$\hat{q}_1 = \frac{q_1\chi_1^2 + 3i\zeta_2\chi_1\chi_2 + \zeta_2\chi_1 R_2(q_1\chi_3 - \chi_2 q_2)}{\chi_2^2}, \qquad \hat{r}_1 = \frac{(r_1\chi_2 + r_2\chi_3)\chi_2 - 3i\zeta_2\chi_1\chi_2}{\chi_1^2},$$
$$\hat{r}_2 = \frac{(3iR_2\zeta_2^2 - r_2)\chi_1\chi_2 - \zeta_2\chi_2 R_2(r_1\chi_2 + r_2\chi_3)}{\zeta_2\chi_1^2}, \quad \hat{q}_2 = \frac{(q_1\chi_3 - q_2\chi_2)\zeta_2\chi_1}{\chi_2^2}.$$

Proof: Direct calculation.

Similar to the **Remark 1**, we have

Remark 2 *An alternative formula for* $\hat{d}$ *is* $\hat{d}_x = r_2$*. Thus,* $\hat{d} = d$*.*

Finally we may have a combined Darboux transformation in the following manner: we take a particular solution $\Phi_1 \equiv (\varphi_1, \varphi_2, \varphi_3)^T$ of (3)-(4) at $\zeta = \zeta_1$ and a particular solution (χ_1, χ_2, χ_3) of (10)-(11) at $\zeta = \zeta_2$. Then, with $(\varphi_1, \varphi_2, \varphi_3)^T$ we may use **Theorem 1** and have a Darboux transformation whose Darboux matrix is T_1. At this stage, (χ_1, χ_2, χ_3) is converted into a new solution $\Psi_1 \equiv (\hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_3) = (\chi_1, \chi_2, \chi_3)T_1^{-1}|_{\zeta=\zeta_2}$ for the conjugate linear system. This solution, with the help of **Theorem 2**, enables us to construct a Darboux matrix T_2 and take a Darboux transformation for the conjugate linear system, which in turn induces a transformation for the original linear system. Schematically it looks as

$$\Phi \xrightarrow[\text{seed:}\Phi_1]{T_1} \hat{\Phi} \xrightarrow[\text{seed:}\Psi_1]{T_2^{-1}} \Phi[1]$$

It is now easy to find the explicit formulae. Indeed, the three components of Ψ_1 reads as

$$\begin{aligned}
\hat{\chi}_1 &= \frac{\zeta_1\zeta_2\chi_1\varphi_1 + \zeta_1^2\chi_2\varphi_2 + \zeta_1^2\chi_3\varphi_3}{(\zeta_1^2 - \zeta_2^2)\varphi_2}, \\
\hat{\chi}_2 &= \frac{\zeta_1^2\chi_1\varphi_1 + \zeta_1\zeta_2\chi_2\varphi_2 + \zeta_1\zeta_2\chi_3\varphi_3}{(\zeta_1^2 - \zeta_2^2)\varphi_1}, \\
\hat{\chi}_3 &= \frac{\zeta_1(\zeta_1\chi_1\varphi_1 + \zeta_2\chi_2\varphi_2 + \zeta_2\chi_3\varphi_3)}{(\zeta_1^2 - \zeta_2^2)\varphi_1}Q_2 + \frac{\chi_3}{\zeta_2}.
\end{aligned}$$

Using this seed solution, we find that the functions appeared in T_2 in the present case read as

$$
\begin{aligned}
\hat{a} &= -\frac{\varphi_2(\zeta_1\chi_1\varphi_1+\zeta_2\chi_2\varphi_2+\zeta_2\chi_3\varphi_3)}{\zeta_2\varphi_1(\zeta_2\chi_1\varphi_1+\zeta_1\chi_2\varphi_2+\zeta_1\chi_3\varphi_3)},\\
\hat{b} &= \frac{\varphi_3}{\varphi_2}Q_2-\frac{\varphi_1(\zeta_1\zeta_2\chi_1\varphi_1+\zeta_1^2\chi_2\varphi_2+\zeta_2^2\chi_3\varphi_3)}{\zeta_1\zeta_2\varphi_2(\zeta_1\chi_1\varphi_1+\zeta_2\chi_2\varphi_2+\zeta_2\chi_3\varphi_3)},\\
\hat{c} &= -Q_2+\frac{(\zeta_2^2-\zeta_1^2)\chi_3\varphi_1}{\zeta_1\zeta_2(\zeta_1\chi_1\varphi_1+\zeta_2\chi\varphi_2+\zeta_2\chi_3\varphi_3)},\\
\hat{d} &= -\frac{\varphi_3}{\varphi_2}.
\end{aligned}
$$

The Darboux matrix we are seeking, $T=T_2^{-1}T_1$, which after removing an overall factor $\frac{\zeta_2^2}{(\zeta^2-\zeta_2^2)}$, is

$$
T=\begin{pmatrix} a\zeta^2-1 & c_1\zeta & c_2\zeta \\ c_3\zeta & b\zeta^2-1 & c\zeta^2 \\ c_4\zeta & d\zeta^2 & e\zeta^2-1 \end{pmatrix}, \tag{14}
$$

where

$$
a=\frac{D_2}{\zeta_1\zeta_2D_1},\quad b=\frac{D_3}{\zeta_1\zeta_2^2D_2},\quad e=\frac{D_4}{\zeta_1\zeta_2^2D_2}, \tag{15}
$$

$$
c=\frac{(\zeta_2^2-\zeta_1^2)\chi_3\varphi_2}{\zeta_1\zeta_2^2D_2},\qquad d=\frac{(\zeta_2^2-\zeta_1^2)\chi_2\varphi_3}{\zeta_1\zeta_2^2D_2}, \tag{16}
$$

$$
c_1=\frac{(\zeta_2^2-\zeta_1^2)\chi_2\varphi_1}{\zeta_1\zeta_2D_1},\quad c_2=\frac{(\zeta_2^2-\zeta_1^2)\chi_3\varphi_1}{\zeta_1\zeta_2D_1}, \tag{17}
$$

$$
c_3=\frac{(\zeta_2^2-\zeta_1^2)\chi_1\varphi_2}{\zeta_1\zeta_2D_2},\quad c_4=\frac{(\zeta_2^2-\zeta_1^2)\chi_1\varphi_3}{\zeta_1\zeta_2D_2}, \tag{18}
$$

and

$$
\begin{aligned}
D_1 &= \zeta_1\chi_1\varphi_1+\zeta_2\chi_2\varphi_2+\zeta_2\chi_3\varphi_3,\\
D_2 &= \zeta_2\chi_1\varphi_1+\zeta_1\chi_2\varphi_2+\zeta_1\chi_3\varphi_3,\\
D_3 &= \zeta_1\zeta_2\chi_1\varphi_1+\zeta_2^2\chi_2\varphi_2+\zeta_1^2\chi_3\varphi_3,\\
D_4 &= \zeta_1\zeta_2\chi_1\varphi_1+\zeta_1^2\chi_2\varphi_2+\zeta_2^2\chi_3\varphi_3.
\end{aligned}
$$

The transformations between field variables can be reformed neatly

$$
q_1[1]=q_1-c_{1,x},\qquad q_2[1]=q_2-c_{2,x}, \tag{19}
$$

$$
r_1[1]=r_1-c_{3,x},\qquad r_2[1]=r_2-c_{4,x}, \tag{20}
$$

and c_i's are given by (17)-(18).

3 Reduction

In last section, we constructed a combined or two-fold Darboux transformation for our linear system (3)-(4). The relevant Darboux matrix and field variable transformations are given by (14) and (19)-(20) respectively. What we are interested in is to present a Darboux transformation for the two component DNLS equation and thus we have to do reduction. Next we will show that our Darboux transformation can be reduced easily to the interested case.

The constraints between field variables are

$$r_1 = -q_1^*, \;\; r_2 = -q_2^*$$

which should be kept invariant under Darboux transformation. Now we notice that, for the solution $(\varphi_1, \varphi_2, \varphi_3)^T$ of the linear system (3)-(3) at $\zeta = \zeta_1$, $(\varphi_1^*, \varphi_2^*, \varphi_3^*)$ is the solution of conjugate linear system equation (10)-(11) at $\zeta = \zeta_1^*$. Therefore, we use it as our seed for the second step Darboux transformation. Namely,

$$\Psi_1 = (\varphi_1^*, \varphi_2^*, \varphi_3^*), \;\; \zeta_2 = \zeta_1^*.$$

With these considerations, it is easy to verify that

$$c_1^* = -c_3, \;\; c_2^* = c_4$$

therefore

$$r_1[1] = -q_1[1]^*, \;\; r_2[1] = -q_2[1]^*$$

The final transformation is neatly written as

$$q_1[1] = q_1 - \frac{\zeta_1^{*2} - \zeta_1^2}{|\zeta_1^2|}\left(\frac{\varphi_1\varphi_2^*}{|\varphi_1^2|\zeta_1 + \zeta_1^*(|\varphi_2^2| + |\varphi_3^2|)}\right)_x, \tag{21}$$

$$q_2[1] = q_2 - \frac{\zeta_1^{*2} - \zeta_1^2}{|\zeta_1^2|}\left(\frac{\varphi_1\varphi_3^*}{|\varphi_1^2|\zeta_1 + \zeta_1^*(|\varphi_2^2| + |\varphi_3^2|)}\right)_x. \tag{22}$$

If we start with the vacuum solution $q_1 = q_2 = 0$, then the linear system (3)-(4) has a solution

$$\varphi_1 = e^{-2i\zeta_1^2 x - 9i\zeta_1^4 t}, \quad \varphi_2 = e^{i\zeta_1^2 x} = \varphi_3$$

which leads to

$$q_1[1] = q_2[1] = \frac{6\zeta_1^2 \mathrm{Im}(\zeta_1^2) e^{-iR}\left[\sinh(I)(\zeta_1^{*2} - 2|\zeta_1^2|) + \cosh(I)(\zeta_1^{*2} + 2|\zeta_1^2|)\right]}{\zeta_1^*\left[\sinh(I)(\zeta_1^2 + 2|\zeta_1^2|) + \cosh(I)(\zeta_1^2 - 2|\zeta_1^2|)\right]^2}$$

where $R = 3\mathrm{Re}(\zeta_1^2)x + 9\mathrm{Re}(\zeta_1^4)t$, $I = 3\mathrm{Im}(\zeta_1^2)x + 9\mathrm{Im}(\zeta_1^4)t$. It is nothing but a solution of the DNLS equation. To find more interesting ones we need to iterate our Darboux transformation and we will do so in next section.

4 Iterations: N-fold Darboux matrix

The appealing feature of a Darboux transformation is that it often leads to determinant representation for $N-$solitons. To this aim, one has to do iteration. In this section, we consider the iteration problem for our Darboux transformation.

First, let us rewrite our Darboux matrix T given by (14) with the reductions in mind. Introduce a new matrix

$$N(\zeta) = \mathrm{diag}(\frac{\zeta\varphi_1}{\zeta_1 D}, \frac{\zeta\varphi_2}{\zeta_1 D^*}, \frac{\zeta\varphi_3}{\zeta_1 D^*})$$

where $D = D_1|_{\chi_j=\varphi_j^*,\zeta_2=\zeta_1^*}$.Then, the Darboux matrix T takes the following form

$$T = \frac{\zeta^2 - \zeta_1^{*2}}{\zeta_1^{*2}} + \frac{\zeta_1^{*2} - \zeta_1^2}{\zeta_1^{*2}} N(\zeta) \begin{pmatrix} \zeta\varphi_1^* & \zeta_1^*\varphi_2^* & \zeta_1^*\varphi_3^* \\ \zeta_1^*\varphi_1^* & \zeta\varphi_2^* & \zeta\varphi_3^* \\ \zeta_1^*\varphi_1^* & \zeta\varphi_2^* & \zeta\varphi_3^* \end{pmatrix}.$$

Now, on the one hand we have already known

$$T|_{\zeta=\zeta_1} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0. \tag{23}$$

i.e. our seed $\Phi_1 = (\varphi_1, \varphi_2, \varphi_3)^T$ lies in the kernal of the matrix $T|_{\zeta=\zeta_1}$. On the other hand, let us suppose

$$T|_{\zeta=\zeta_1^*} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \frac{\zeta_1^{*2} - \zeta_1^2}{\zeta_1^*} N(\zeta_1^*) \begin{pmatrix} \varphi_1^*\psi_1 + \varphi_2^*\psi_2 + \varphi_1^*\psi_3 \\ \varphi_1^*\psi_1 + \varphi_2^*\psi_2 + \varphi_1^*\psi_3 \\ \varphi_1^*\psi_1 + \varphi_2^*\psi_2 + \varphi_1^*\psi_3 \end{pmatrix} = 0 \tag{24}$$

for certain vector function $\Psi_1 = (\psi_1, \psi_2, \psi_3)^T$, then for $\zeta_1 \neq \zeta_1^*$ one has to impose $\varphi_1^*\psi_1 + \varphi_2^*\psi_2 + \varphi_1^*\psi_3 = 0$, or

$$\Phi_1^\dagger \Psi_1 = 0,$$

obviously

$$\Psi_1^1 = (-\varphi_2^*, \varphi_1^*, 0)^T, \Psi_1^2 = (-\varphi_3^*, 0, \varphi_1^*)^T$$

meet the requirment

$$T|_{\zeta=\zeta_1^*} \Psi_1^k = 0. \tag{25}$$

We observe that the conditions (23) and (25) can in turn be used to determine the nine quantities appeared in T *uniquely.*

Now we are ready to do iterations. Assume that we are given N distinct complex numbers $\zeta_1, \zeta_2, ..., \zeta_N$ such that $\zeta_k^{*2} \neq \zeta_k^2$ $(k = 1, 2, ..., N)$. We further assume that the vector

$$\Phi_k = (\varphi_1^{(k)}, \varphi_2^{(k)}, \varphi_3^{(k)})^T$$

is a solution of linear equation at $\zeta = \zeta_k$, i.e.

$$[\partial_x - U(\zeta = \zeta_k)](\Phi_k) = 0, \qquad [\partial_t - V(\zeta = \zeta_k)](\Phi_k) = 0,$$

and

$$\Psi_k^1 = (-\varphi_2^{(k)*}, \varphi_1^{(k)*}, 0)^T, \quad \Psi_k^2 = (-\varphi_3^{(k)*}, 0, \varphi_1^{(k)*})^T,$$

which satisfy the orthogonal conditions $\Phi_k^\dagger \Psi_k^l = 0$.

With these seed solutions, we define

$$T_k = \frac{\zeta^2 - \zeta_k^{*2}}{\zeta_k^{*2}} + \frac{\zeta_k^{*2} - \zeta_k^2}{\zeta_k^{*2}} N_k(\zeta) \begin{pmatrix} \zeta\varphi_1^{(k)}[k-1]^* & \zeta_k^*\varphi_2^{(k)}[k-1]^* & \zeta_k^*\varphi_3^{(k)}[k-1]^* \\ \zeta_i^*\varphi_1^{(k)}[k-1]^* & \zeta\varphi_2^{(k)}[k-1]^* & \zeta\varphi_3^{(k)}[k-1]^* \\ \zeta_i^*\varphi_1^{(k)}[k-1]^* & \zeta\varphi_2^{(k)}[k-1]^* & \zeta\varphi_3^{(k)}[k-1]^* \end{pmatrix},$$

where

$$\begin{aligned} D[k] &= \zeta_k \left|\varphi_1^{(k)}[k-1]\right|^2 + \zeta_k^*(\left|\varphi_2^{(k)}[k-1]\right|^2 + \left|\varphi_3^{(k)}[k-1]\right|^2), \\ N_k(\zeta) &= \mathrm{diag}\left(\frac{\zeta\varphi_1^{(k)}[k-1]^*}{\zeta_i D[k]}, \frac{\zeta\varphi_2^{(k)}[k-1]^*}{\zeta_i D[k]^*}, \frac{\zeta\varphi_3^{(k)}[k-1]^*}{\zeta_i D[k]^*}\right), \end{aligned}$$

and our notation is the following

$$\Phi_j[k] = \begin{pmatrix} \varphi_1^{(j)}[k] \\ \varphi_2^{(j)}[k] \\ \varphi_3^{(j)}[k] \end{pmatrix} = T_k T_{k-1} ... T_1|_{\zeta=\zeta_j} \begin{pmatrix} \varphi_1^{(j)} \\ \varphi_2^{(j)} \\ \varphi_3^{(j)} \end{pmatrix},$$

and $\Phi_j[0] = \Phi_j$.

The N-times iterated Darboux matrix is given by

$$T = T_N T_{N-1} \cdots T_1.$$

It is easy to see that, similar to the equation (23), the following relations hold

$$T_k T_{k-1} \cdots T_1|_{\zeta=\zeta_k} \Phi_k = T_k|_{\zeta=\zeta_k} \Phi_k[k-1] = 0 \qquad (k = 1, 2, ..., N).$$

Furthermore, we recursively define

$$\Psi_k^l[0] = \Psi_k^l, \quad \Psi_k^l[j-1] = T_{j-1}T_{j-2} \cdots T_1|_{\zeta=\zeta_k^*} \Psi_k^l,$$

then we have

Proposition 1 $\Phi_k^\dagger[k-1]\Psi_k^l[k-1] = 0$.

Proof: We know $\Phi_k^\dagger[0]\Psi_k^l[0] = 0$. Let us suppose $\Phi_k^\dagger[m]\Psi_k^l[m] = 0$ $(0 \le m < k-1)$. Then thanks to $\Phi_k[m+1] = T_{m+1}|_{\zeta=\zeta_k}\Phi_k[m]$ and $\Psi_k^l[m+1] = T_{m+1}|_{\zeta=\zeta_k^*}\Psi_k^l[m]$, we have

$$\Psi_k^\dagger[m+1]\Phi_k^l[m+1] = \Psi_k^\dagger[m]T_{m+1}^\dagger|_{\zeta=\zeta_k}T_{m+1}|_{\zeta=\zeta_k^*}\Phi_k^l[m] = 0,$$

because of $T_{m+1}^\dagger|_{\zeta=\zeta_k}T_{m+1}|_{\zeta=\zeta_k^*} = \frac{1}{|\zeta_{m+1}^4|}(\zeta_k^{*2} - \zeta_{m+1}^{*2})(\zeta_k^{*2} - \zeta_{m+1}^2)$. Therefore, the lemma follows from the mathematical induction.

Based on **Proposition 1**, we obtain

$$T_k T_{k-1} ... T_1|_{\zeta=\zeta_k^*}\Psi_k^l \quad = \quad T_k|_{\zeta=\zeta_k^*}\Psi_k^l[k-1] = 0, \qquad (l = 1, 2).$$

Therefore, we have

$$T|_{\zeta=\zeta_k}\Phi_k = 0, \quad T|_{\zeta=\zeta_k^*}\Psi_k^1 = 0, \quad T|_{\zeta=\zeta_k^*}\Psi_k^2 = 0, \tag{26}$$

for $k = 1, 2, ..., N$. We also notice that our iterated Darboux matrix T is taking of the form

$$T = \sum_{k=0}^{2N} \zeta^k T_k = \sum_{n=1}^{N} \begin{pmatrix} a_{2n}\zeta^{2n} & c_1^{(2n-1)}\zeta^{2n-1} & c_2^{(2n-1)}\zeta^{2n-1} \\ c_3^{(2n-1)}\zeta^{2n-1} & b_{2n}\zeta^{2n} & c_{2n}\zeta^{2n} \\ c_4^{(2n-1)}\zeta^{2n-1} & d_{2n}\zeta^{2n} & e_{2n}\zeta^{2n} \end{pmatrix} + (-1)^N.$$

Above coefficients can be determinated in T by solving the linear algebraic systems (26). The solution formulae are obtained from

$$q_1[N] = q_1 + \left(\frac{H_2}{H_1}\right)_x, \qquad q_2[N] = q_2 + \left(\frac{H_3}{H_1}\right)_x,$$

where

$$H_1 = \begin{vmatrix} \zeta_1^{2N}\varphi_1^{(1)} & \zeta_1^{2N-1}\varphi_2^{(1)} & \zeta_1^{2N-1}\varphi_3^{(1)} & ... & \zeta_1^2\varphi_1^{(1)} & \zeta_1\varphi_2^{(1)} & \zeta_1\varphi_3^{(1)} \\ -\zeta_1^{2N*}\varphi_2^{(1)*} & \zeta_1^{2N-1*}\varphi_1^{(1)*} & 0 & ... & -\zeta_1^{2*}\varphi_2^{(1)*} & \zeta_1^*\varphi_1^{(1)*} & 0 \\ -\zeta_1^{2N*}\varphi_3^{(1)*} & 0 & \zeta_1^{2N-1*}\varphi_1^{(1)*} & ... & -\zeta_1^{2*}\varphi_3^{(1)*} & 0 & \zeta_1^*\varphi_1^{(1)*} \\ ... & ... & ... & ... & ... & ... & ... \\ \zeta_N^{2N}\varphi_1^{(N)} & \zeta_N^{2N-1}\varphi_2^{(N)} & \zeta_N^{2N-1}\varphi_3^{(N)} & ... & \zeta_N^2\varphi_1^{(N)} & \zeta_N\varphi_2^{(N)} & \zeta_N\varphi_3^{(N)} \\ -\zeta_N^{2N*}\varphi_2^{(N)*} & \zeta_n^{2N-1*}\varphi_1^{(N)*} & 0 & ... & -\zeta_N^{2*}\varphi_2^{(N)*} & \zeta_N^*\varphi_1^{(N)*} & 0 \\ -\zeta_N^{2N*}\varphi_3^{(N)*} & 0 & \zeta_n^{2N-1*}\varphi_1^{(N)*} & ... & -\zeta_N^{2*}\varphi_3^{(N)*} & 0 & \zeta_N^*\varphi_1^{(N)*} \end{vmatrix},$$

and H_2 and H_3 are H_1 with the $3N-1$th column and the $3N$th column replaced by L_1 respectively. Where

$$L_1 = \begin{pmatrix} -\varphi_1^{(1)}, & \varphi_2^{(1)*}, & \varphi_3^{(1)*}, & ..., & -\varphi_1^{(N)}, & \varphi_2^{(N)*}, & \varphi_3^{(N)*}, \end{pmatrix}^T,$$

To demonstrate the usefulness our solution formulae, we calculate solutions for the two component DNLS equation. Selecting

$$\zeta_1 = 1 + \frac{1}{3}i, \zeta_2 = 1 + \frac{2}{3}i, \Phi_1 = (e^{-2i\zeta_1^2 x - 9i\zeta_1^4 t}, 0, e^{i\zeta_1^2 x})^T, \Phi_2 = (e^{-2i\zeta_2^2 x - 9i\zeta_2^4 t}, e^{i\zeta_2^2 x}, e^{i\zeta_2^2 x})^T,$$

and substituting them into (4) we could have the solutions. Figure 1 and Figure 2 show these solutions by plotting $|q_1^2|$ and $|q_2^2|$. It is pointed out that while the second figure exhibits standard two-soliton scattering, the first one demonstrates a fission process.

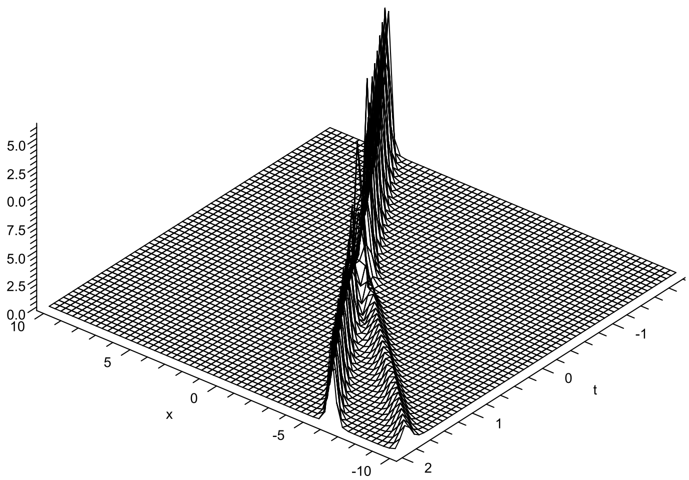

Figure 1: $|q_1^2|$

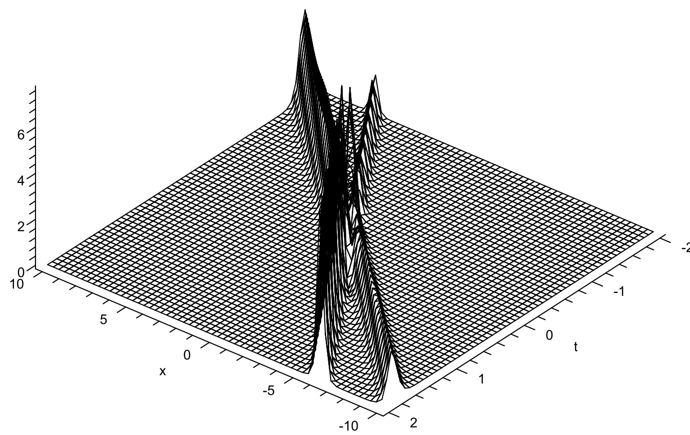

Figure 2: $|q_2^2|$

5 Conclusion

Above we found a Darboux transformation for the two component DNLS equation and obtained a closed formula for its solutions. We remark that our Darboux transformation can be easily generalized to multi component case. In fact, the Darboux matrix in this case is

$$T = \frac{\zeta^2 - \zeta_1^{*2}}{\zeta_1^{*2}} + \frac{\zeta_1^{*2} - \zeta_1^2}{\zeta_1^{*2}} N(\zeta) \begin{pmatrix} \zeta\varphi_1^* & \zeta_1^*\varphi_2^* & \zeta_1^*\varphi_3^* & \cdots & \zeta_1^*\varphi_n^* \\ \zeta_1^*\varphi_1^* & \zeta\varphi_2^* & \zeta\varphi_3^* & \cdots & \zeta\varphi_n^* \\ \zeta_1^*\varphi_1^* & \zeta\varphi_2^* & \zeta\varphi_3^* & \cdots & \zeta\varphi_n^* \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \zeta_1^*\varphi_1^* & \zeta\varphi_2^* & \zeta\varphi_3^* & \cdots & \zeta\varphi_n^* \end{pmatrix},$$

where

$$N(\zeta) = \mathrm{diag}\left(\frac{\zeta\varphi_1}{\zeta_1 D}, \frac{\zeta\varphi_2}{\zeta_1 D^*}, \cdots, \frac{\zeta\varphi_n}{\zeta_1 D^*}\right),$$

with

$$D = \zeta_1|\varphi_1^2| + \zeta_1^*|\varphi_2^2| + \ldots + \zeta_1^*|\varphi_n^2|.$$

and solution formulae may be derived.

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