Quantum Phase Transition in a Pseudo-hermitian Dicke model

Tetsuo Deguchi[1]* and Pijush K. Ghosh[2]†
[1] Department of Physics,
Graduate School of Humanities and Sciences,
Ochanomizu University, 2-1-1 Ohtsuka, Bunkyo-ku,
Tokyo 112-8610, Japan
[2] Department of Physics, Siksha-Bhavana,
Visva-Bharati University,
Santiniketan, PIN 731 235, India.

We show that a Dicke-type non-hermitian Hamiltonian admits entirely real spectra by mapping it to the "Dressed Dicke Model"(DDM) through a similarity transformation. We find a positive-definite metric in the Hilbert space of the non-hermitian Hamiltonian so that the time-evolution is unitary and allows a consistent quantum description. We then show that this non-hermitian Hamiltonian describing non-dissipative quantum processes undergoes Quantum Phase Transition(QPT). The exactly solvable limit of the non-hermitian Hamiltonian has also been discussed.

Although the choice of a proper set of hermitian operators is sufficient to ensure the reality of the entire spectra and unitary time-evolution for a quantum system, it is neither necessary nor dictated by any fundamental principle. It is known since the pioneering work of Bender and Boettcher[1] that $\mathcal{PT}$-symmetric non-hermitian operators with an appropriate inner-product in the Hilbert space give consistent description of non-dissipative quantum processes. The Hamiltonian that is non-hermitian with respect to the conventional inner-product in the Hilbert space becomes hermitian with respect to the new inner product and results of a hermitian theory follow naturally. The same problem can be studied using pseudo-hermitian operator[2, 3], i.e. an operator that is related to its adjoint through a similarity transformation. Both the approaches involving pseudo-hermiticity and $\mathcal{PT}$-invariance are complementary to each other and open up several new directions in the study of non-hermitian operators [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. It may be mentioned here that operators which are non-hermitian with respect to the conventional inner product in the Hilbert space are generally used to simulate dissipative processes. In this letter, we are concerned about a subclass of such non-hermitian operators which are also pseudo-hermitian and may be used consistently to describe non-dissipative processes with a modified inner product in the Hilbert space.

The study on QPT[15] has received considerable attention in recent times and reveals many aspects that are qualitatively different from that of phase transition at finite temperature. The investigations so far are mainly restricted to hermitian Hamiltonian, since an entirely real spectra with a well-defined ground-state is not guaranteed a priory for a non-hermitian Hamiltonian. The dynamics of QPT in a closed system is governed by non-dissipative terms, since the system at zero temperature is already in thermal equilibrium. Unlike the phase-transitions at finite temperature, the time-evolution from one phase to the other is expected to be unitary for a system undergoing QPT. It is thus nontrivial for a non-hermitian Hamiltonian to describe a QPT in a closed system, since the time-evolution is not necessarily unitary. It is natural to ask at this juncture whether or not one could discuss about QPT in a closed system within the framework of $\mathcal{PT}$-symmetric non-hermitian Hamiltonian. If the answer is in the affirmative, it may open up new directions in the study of several interlinked areas of physics like level statistics, quantum entanglement, quantum chaos etc. within the framework of pseudo-hermitian and/or $\mathcal{PT}$-symmetric non-hermitian Hamiltonian. The enlarged parameter-space of a non-hermitian Hamiltonian compared to its hermitian counterpart may prove to be an added advantage.

One of the main results of this letter is that a pseudo-hermitian deformation of the DDM indeed undergoes QPT. We consider the non-hermitian Dicke-type Hamiltonian[16],

$$\begin{aligned} H &= \omega a^\dagger a + \theta_1 e^{i\xi_1} a^2 + \theta_2 e^{-i\xi_1} {a^\dagger}^2 + \alpha e^{i\xi_2} J_- a^\dagger \\ &+ \beta e^{-i\xi_2} J_+ a + \gamma e^{i\xi_3} J_- a + \delta e^{-i\xi_3} J_+ a^\dagger + \omega_0 J_z, \end{aligned} \tag{1}$$

where $\omega, \omega_0, \theta_1, \ \theta_2, \ \alpha, \beta, \gamma, \delta, \xi_1, \xi_2, \xi_3$ are real parameters; a, $a^\dagger$ are the standard bosonic annihilation-creation operators and $J_z, \ J_\pm := J_x \pm i J_y$ are the generators of the $SU(2)$ algebra,

$$\begin{aligned} &\left[a, a^\dagger\right] = 1, \\ &[J_+, J_-] = 2J_z, \quad [J_z, J_\pm] = \pm J_\pm. \end{aligned} \tag{2}$$

The Hamiltonian H commutes with the parity operator Π,

$$\Pi = e^{i\pi \hat{N}}, \quad \hat{N} = a^\dagger a + J_z + j, \tag{3}$$

where j is the total spin-angular momentum. The eigenstates of H have definite parity depending on whether the eigenvalues of the operator $\hat{N}$ are odd or even. In general, the Hamiltonian H is non-hermitian. The Hermitian Hamiltonian is obtained in the limit,

$$\alpha = \beta, \gamma = \delta, \theta_1 = \theta_2, \tag{4}$$

and is known as the DDM in the literature[17, 18]. The standard Dicke model is obtained by a further choice of $\theta_1 = \theta_2 = \xi_1 = \xi_2 = \xi_3 = 0$ and $\alpha = \beta = \gamma = \delta$. The Dicke Hamiltonian has been studied extensively from the viewpoint of QPT[19, 20, 21], level-statistics[21], quantum entanglement[22, 23] and exact solvability[24]. Certain spintronics based models[24, 25] with Dresselhaus and Rashba-type spin-orbit interactions can be mapped to the Dicke model, implying its relevance in the study of two dimensional semi-conductor physics. The Tavis-Cummings model[26] is obtained in the limit $\theta_1 = \theta_2 = \gamma = \delta = 0$ and it reduces to the Jaynes-Cummings model[27] if the fundamental representation of the $SU(2)$ is used. Non-hermitian versions of both the Tavis-Cummings and the Jaynes-Cummings models have been studied previously[12]. The Hamiltonian with $\omega_0 = \alpha = \beta = \gamma = \delta = 0$ is known as the 'Swanson model'[11] in the context of $\mathcal{PT}$-symmetric quantum mechanics and has been studied in some detail[11, 13]. In this letter, we study the Hamiltonian H with its full generality and show the existence of QPT for certain special choices of the parameters.

The Hamiltonian H can be mapped to a hermitian Hamiltonian $\mathcal{H}$ through a similarity transformation when the following relations are satisfied,

$$\begin{aligned} &\alpha\ \delta - \beta\ \gamma = 0, \quad \theta_1 = \theta_2 = 0 \\ &\alpha\ \delta\ \theta_1 - \beta\ \gamma\ \theta_2 = 0, \quad \theta_1 \neq 0 \neq \theta_2. \end{aligned} \tag{5}$$

To see this, define an operator ρ and its inverse as,

$$\begin{aligned} &\rho = e^{\hat{O}}, \quad \rho^{-1} = e^{-\hat{O}}, \\ &\hat{O} = \frac{1}{4} ln \left(\frac{\theta_1}{\theta_2}\right) a^\dagger a + \frac{1}{4} ln \left(\frac{\alpha\gamma}{\beta\delta}\right) (J_z + j). \end{aligned} \tag{6}$$

The operator ρ is positive-definite and well-defined provided the following relations are satisfied,

$$\frac{\theta_1}{\theta_2} > 0, \quad \frac{\alpha}{\beta} > 0, \quad \frac{\gamma}{\delta} > 0. \tag{7}$$

The conditions $\frac{\theta_1}{\theta_2} > 0$ and $\frac{\alpha\gamma}{\beta\delta} > 0$ are sufficient to ensure that ρ has the desired property. The much more stringent condition(7) is used to make the transformed Hamiltonian $\mathcal{H}$ hermitian. The operator $\hat{O}$ can be constructed for several special cases as follows:

$$\begin{aligned} &\hat{O} = \frac{1}{4} ln \left(\frac{\theta_1}{\theta_2}\right) a^\dagger a, \quad \frac{\theta_1}{\theta_2} > 0, \quad \alpha = \beta = \gamma = \delta = 0; \\ &\hat{O} = \frac{1}{4} ln \left(\frac{\alpha\gamma}{\beta\delta}\right) (J_z + j), \\ &\theta_1 = \theta_2 = 0, \quad \frac{\alpha}{\beta} > 0, \quad \frac{\gamma}{\delta} > 0; \\ &\hat{O} = \frac{1}{4} ln \left(\frac{\theta_1}{\theta_2}\right) a^\dagger a + \frac{1}{4} ln \left(\frac{\alpha}{\beta}\right) (J_z + j), \\ &\frac{\theta_1}{\theta_2} > 0, \quad \frac{\alpha}{\beta} > 0, \quad \gamma = \delta = 0; \\ &\hat{O} = \frac{1}{4} ln \left(\frac{\theta_1}{\theta_2}\right) a^\dagger a + \frac{1}{4} ln \left(\frac{\gamma}{\delta}\right) (J_z + j), \\ &\frac{\theta_1}{\theta_2} > 0, \quad \frac{\gamma}{\delta} > 0, \quad \alpha = \beta = 0. \end{aligned} \tag{8}$$

We will be working within the range of the parameters defined by Eq. (7) unless mentioned otherwise. Using the Baker-Campbell-Hausdorff formula,

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots, \tag{9}$$

we find,

$$\begin{aligned} \mathcal{H} &= \rho H \rho^{-1} \\ &= \omega a^\dagger a + \sqrt{\theta_1\theta_2}\left(e^{i\xi_1}\ a^2 + e^{-i\xi_1}\ {a^\dagger}^2\right) + \omega_0 J_z \\ &+ \sqrt{\alpha\beta}\left(e^{i\xi_2}\ J_- a^\dagger + e^{-i\xi_2}\ J_+ a\right) \\ &+ \sqrt{\gamma\delta}\left(e^{i\xi_3}\ J_- a + e^{-i\xi_3}\ J_+ a^\dagger\right), \end{aligned} \tag{10}$$

when the condition (5) is satisfied. Note that $\mathcal{H}$ is hermitian, since $\theta_1\theta_2$, $\alpha\beta$ and $\gamma\delta$ are positive-definite due to the condition (7). The Hamiltonian H is quasi-hermitian, i.e., related to the hermitian Hamiltonian $\mathcal{H}$ through a similarity transformation. The pseudo-hermiticity of H, i.e. $H^\dagger = \eta_+ H \eta_+^{-1}$, follows automatically where the metric η_+ is given by $\eta_+ := \rho^2$. The Hamiltonian H that is non-hermitian under the Dirac-hermiticity condition becomes hermitian with respect to the modified inner-product defined in the Hilbert space as, $\langle\langle u, v\rangle\rangle_{\eta_+} := \langle u, \eta_+ v\rangle$. In particular,

$$\langle u|Hv\rangle \neq \langle Hu|v\rangle, \quad \langle\langle u|Hv\rangle\rangle_{\eta_+} = \langle\langle Hu|v\rangle\rangle_{\eta_+}. \tag{11}$$

Thus, with the modified inner-product, the results of a hermitian Hamiltonian follow automatically.

A comment is in order at this point. The atomic inversion and the mean photon number are determined by the expectation values of the operators J_z and $a^\dagger a$, respectively. Both the operators J_z and $a^\dagger a$ commute with η_+ and hence, are hermitian with respect to the modified inner-product. However, operators like $J_x,\ J_y,\ a + a^\dagger$ and $i(a^\dagger - a)$, which are hermitian with respect to the Dirac-hermiticity condition, are no longer hermitian with respect to the modified inner product. It may be noted here that corresponding to each operator $\mathcal{A}$ that is hermitian with respect to the Dirac-hermiticity condition, the operator $\hat{\mathcal{A}} := \rho^{-1}\mathcal{A}\rho$ is hermitian with respect to the modified inner product[2]. Consequently, the operator $\hat{\mathcal{A}}$ is a physical observable in the Hilbert space of H that is endowed with the metric η_+. Following this prescription, a set of $SU(2)$ generators those are hermitian with respect to the modified inner-product can be constructed as follows:

$$\begin{aligned} \hat{J}_x &:= J_x cosh\Gamma - iJ_y sinh\Gamma \\ \hat{J}_y &:= J_y cosh\Gamma + iJ_x sinh\Gamma \\ \hat{J}_z &:= J_z, \quad \Gamma \equiv \frac{1}{4} ln\left(\frac{\alpha\gamma}{\beta\delta}\right). \end{aligned} \tag{12}$$

Similarly, annihilation operator $\hat{a}$ and its adjoint $\hat{a}^\dagger$ can be obtained as,

$$\hat{a} := \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{4}} a, \quad \hat{a}^\dagger := \left(\frac{\theta_1}{\theta_2}\right)^{-\frac{1}{4}} a^\dagger. \tag{13}$$

The non-hermitian Hamiltonian H can be re-written in terms of these operators as,

$$\begin{aligned} H &= \omega \hat{a}^\dagger \hat{a} + \sqrt{\theta_1\theta_2}\left(e^{i\xi_1}\ \hat{a}^2 + e^{-i\xi_1}\ (\hat{a}^\dagger)^2\right) + \omega_0 \hat{J}_z \\ &+ \sqrt{\alpha\beta}\left(e^{i\xi_2}\ \hat{J}_- \hat{a}^\dagger + e^{-i\xi_2}\ \hat{J}_+ \hat{a}\right) \\ &+ \sqrt{\gamma\delta}\left(e^{i\xi_3}\ \hat{J}_- \hat{a} + e^{-i\xi_3}\ \hat{J}_+ \hat{a}^\dagger\right), \end{aligned} \tag{14}$$

where $\hat{J}_\pm := \hat{J}_x \pm i\hat{J}_y$.

The hermitian Hamiltonian $\mathcal{H}$ has the form of the DDM and has been extensively studied in the literature[17, 18]. In general, the Hamiltonian $\mathcal{H}$ is not exactly solvable. Using the Bogoliubov transformation,

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} cosh\theta & e^{i\phi} sinh\theta \\ e^{-i\phi} sinh\theta & cosh\theta \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{15}$$

either the counter-rotating terms $J_- a, J_+ a^\dagger$ or the double frequency terms $a^2, {a^\dagger}^2$ in the Hamiltonian $\mathcal{H}$ can be eliminated with all other terms appearing with renormalized coupling constants. Both the counter-rotating and the double frequency terms can be eliminated simultaneously for fixed values of θ and ϕ, if a constraint involving the parameters $\alpha, \beta, \gamma, \delta, \theta_1$ and θ_2 is also satisfied. Let us choose ϕ and θ as,

$$\begin{aligned} &\phi = -\xi_1, \quad \theta = tanh^{-1}\left(\frac{\Delta}{2\sqrt{\theta_1\theta_2}}\right); \\ &\Delta \equiv \omega - \left(\omega^2 - 4\theta_1\theta_2\right)^{\frac{1}{2}}, \quad \omega^2 > 4\theta_1\theta_2, \end{aligned} \tag{16}$$

so that the double-frequency terms are eliminated from $\mathcal{H}$. It may be mentioned here that the choice of ϕ and θ as,

$$\begin{aligned} &\phi = -\xi_1, \quad \tilde{\theta} = tanh^{-1}\left(\frac{\tilde{\Delta}}{2\sqrt{\theta_1\theta_2}}\right); \\ &\tilde{\Delta} \equiv \omega + \left(\omega^2 - 4\theta_1\theta_2\right)^{\frac{1}{2}}, \quad \omega^2 > 4\theta_1\theta_2, \end{aligned} \tag{17}$$

also removes the double-frequency terms. However, this solution leads to unphysical situations and is discarded henceforth. Using the condition (5) and demanding the removal of counter-rotating terms, the values of γ, δ and ξ_3 are determined as,

$$\gamma = \frac{\alpha\Delta}{2\theta_2}, \quad \delta = \frac{\beta\Delta}{2\theta_1}, \quad \xi_3 = \xi_1 + \xi_2. \tag{18}$$

The Hamiltonian $\mathcal{H}$ can be expressed in terms of the new canonical operators $b, b^\dagger$ as,

$$\begin{aligned} &\mathcal{H} = \omega_0 J_z + \Omega b^\dagger b + \Omega_0 + \Omega_1\left(e^{-i\xi_2}\ J_+ b + e^{i\xi_2}\ J_- b^\dagger\right), \\ &\Omega_0 = -\frac{\Delta}{2}, \quad \Omega = \frac{\left(\omega^2 - 4\theta_1\theta_2\right)\Delta}{4\theta_1\theta_2 - \omega\Delta}, \\ &\Omega_1 = \sqrt{\frac{\alpha\beta}{2\theta_1\theta_2}}\left(4\theta_1\theta_2 - \omega\Delta\right)^{\frac{1}{2}}, \end{aligned} \tag{19}$$

which has the form of the Tavis-Cummings model or the DDM in the rotating-wave approximation with the modified coupling constants. All these coupling constants Ω_0, Ω and Ω_1 are real and Ω is also positive-definite for $\omega^2 > 4\theta_1\theta_2$.

The Hamiltonian $\mathcal{H}$ in Eq. (19) is exactly solvable[20]. It can be decomposed in terms of two mutually commuting operators K and L as follows,

$$\begin{aligned} &\mathcal{H} = \Omega K + \Omega_1 L + \Omega_0, \\ &K = b^\dagger b + J_z, \\ &L = e^{-i\xi_2}\ J_+ b + e^{i\xi_2}\ J_- b^\dagger + \frac{\omega_0 - \Omega}{\Omega_1} J_z. \end{aligned} \tag{20}$$

The operator K is diagonal for a fixed spin j with the eigenvalues of J_z as $(m-j)$, $m = 0, 1, \dots 2j$ and that of the bosonic number operator $b^\dagger b$ as n, $n = 0, 1, 2 \dots$. The operator L and hence, the operator $\mathcal{H}$ can be diagonalized in the basis spanned by the eigenstates of K. Let $|n, m; j\rangle_{\mathcal{H}}$ be a complete set of orthonormal eigenstates of $\mathcal{H}$ with the eigenvalues $E_{n,m;j}$. The orthonormality of $|n, m; j\rangle_{\mathcal{H}}$ is based on the standard inner-product in the Hilbert space. The eigenstates of H with the same eigenvalues $E_{n,m;j}$ are determined as,

$$|n, m; j\rangle_H = \rho^{-1}\ \ |n, m; j\rangle_{\mathcal{H}}, \tag{21}$$

which form a complete set of orthonormal eigenstates under the modified inner-product defined in the Hilbert space of H. Consequently, the non-hermitian Hamiltonian H is also exactly solvable and admits consistent quantum description.

The expectation value of an operator X in the Hilbert space of H is determined as,

$$\langle\langle X\rangle\rangle_{\eta_+} = \langle n, m; j|\rho X \rho^{-1}|n, m; j\rangle_{\mathcal{H}}. \tag{22}$$

Both J_z and $a^\dagger a$ are hermitian with respect to the Dirac-hermiticity condition as well as with respect to the modified inner product. In particular, both J_z and $a^\dagger a$ commute with ρ, leading to the results:

$$\begin{aligned} &\langle\langle J_z\rangle\rangle_{\eta_+} = \langle n, m; j|J_z|n, m; j\rangle_{\mathcal{H}}, \\ &\langle\langle a^\dagger a\rangle\rangle_{\eta_+} = \langle n, m; j|a^\dagger a|n, m; j\rangle_{\mathcal{H}}. \end{aligned} \tag{23}$$

Thus, both $\langle\langle J_z\rangle\rangle_{\eta_+}$ and $\langle\langle a^\dagger a\rangle\rangle_{\eta_+}$ are real. However, in general, $\langle\langle J_x\rangle\rangle_{\eta_+}$, $\langle\langle J_y\rangle\rangle_{\eta_+}$, $\langle\langle a + a^\dagger\rangle\rangle_{\eta_+}$ and $\langle\langle i(a^\dagger - a)\rangle\rangle_{\eta_+}$ are complex,

$$\begin{aligned} &\langle\langle J_x\rangle\rangle_{\eta_+} = cosh\Gamma\langle J_x\rangle_{\mathcal{H}} + isinh\Gamma\langle J_y\rangle_{\mathcal{H}} \\ &\langle\langle J_y\rangle\rangle_{\eta_+} = -isinh\Gamma\langle J_x\rangle_{\mathcal{H}} + cosh\Gamma\langle J_y\rangle_{\mathcal{H}} \end{aligned}$$

$$\langle\langle a+a^{\dagger}\rangle\rangle_{\eta_+} = \left(\frac{\theta_1}{\theta_2}\right)^{-\frac{1}{4}} \langle a\rangle_{\mathcal{H}} + \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{4}} \langle a^{\dagger}\rangle_{\mathcal{H}}$$
$$\langle\langle a^{\dagger}-a\rangle\rangle_{\eta_+} = \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{4}} \langle a^{\dagger}\rangle_{\mathcal{H}} - \left(\frac{\theta_1}{\theta_2}\right)^{-\frac{1}{4}} \langle a\rangle_{\mathcal{H}}, \quad (24)$$

where $\langle J_x\rangle_{\mathcal{H}}$ and $\langle J_y\rangle_{\mathcal{H}}$ are real, while $\langle a\rangle_{\mathcal{H}}$ and $\langle a^{\dagger}\rangle_{\mathcal{H}}$ are complex. As discussed before, corresponding to each operator $\mathcal{A}$ in the Hilbert space of $\mathcal{H}$, the physical observable in the Hilbert space of H that is endowed with the metric η_+ is $\hat{\mathcal{A}} = \rho^{-1}\mathcal{A}\rho$. The expectation values of these capped operators are real, since $\langle\langle\hat{\mathcal{A}}\rangle\rangle_{\eta_+} = \langle n,m;j|\mathcal{A}|n,m;j\rangle_{\mathcal{H}}$. Thus, a complete and consistent description of the pseudo-hermitian H is allowed with the proper identification of the physical observables,

The Hamiltonian $\mathcal{H}$ in Eq. (19) is known to exhibit QPT[20, 21]. Although $\mathcal{H}$ is exactly solvable, it becomes tedious to calculate the eigenspectra for large j. The Holstein-Primakoff representation of the $SU(2)$ generators

$$J_- = \left(2j-\zeta^{\dagger}\zeta\right)^{\frac{1}{2}}\zeta, \;\; J_+ = \zeta^{\dagger}\left(2j-\zeta^{\dagger}\zeta\right)^{\frac{1}{2}}, \;\; J_z = \zeta^{\dagger}\zeta - j, \quad (25)$$

where $\zeta, \zeta^{\dagger}$ are the bosonic annihilation and creation operators satisfying $\left[\zeta,\zeta^{\dagger}\right]=1$, can be used to study the thermodynamic limit $j\to\infty$. It is important to note that among the $SU(2)$ generators, only the combination J_z+j appears in the expressions of the parity operator Π and the similarity operator ρ. Consequently, ρ and Π are well-defined in the thermodynamic limit $j\to\infty$. Following the standard method described in[20, 21], the normal phase of $\mathcal{H}$ can be found to be described in the range $\lambda_1 < \lambda_1^c \equiv \sqrt{\Omega\omega_0}$, while the 'super-radiant phase' is described in the range $\lambda_1 > \lambda_1^c$, where $\lambda_1 \equiv \sqrt{2j}\Omega_1$. The Hamiltonian H and $\mathcal{H}$ have the same eigenspectra, since they are related to each other through a similarity transformation. Moreover, note that the operators $\hat{O}$, ρ and ρ^{-1} are well-defined in the thermodynamic limit $j\to\infty$. Thus, the Hamiltonian H also undergoes QPT with the normal phase described in the range $\lambda_1 < \lambda_1^c$, while the 'super-radiant phase' is described in the range $\lambda_1 > \lambda_1^c$. The values of the mean photon number and the atomic inversion above the critical value λ_1^c can be determined as follows:

$$j^{-1}\langle\langle a^{\dagger}a\rangle\rangle_{\eta_+} = \frac{1}{2}\left(1-\frac{\omega_0^2\Omega^2}{\Omega_1^4}\right)\frac{\Omega_1^2}{\Omega^2}$$
$$j^{-1}\langle\langle J_z\rangle\rangle_{\eta_+} = -\frac{\omega_0\Omega}{\Omega_1^2}; \;\; \lambda_1 > \lambda_1^c. \quad (26)$$

This is one of the main results of this letter.

The hermitian Hamiltonian $\mathcal{H}$ in Eq. (14) with $\theta_1 = \theta_2 = \xi_1 = \xi_2 = \xi_3 = 0$ and $\sqrt{\gamma\delta} = \sqrt{\alpha\beta} \equiv \frac{\lambda_2}{\sqrt{2j}}$ reduces to the 'standard Dicke model' which is known to undergo QPT for $\lambda_2 > \lambda_2^c \equiv \frac{\sqrt{\omega\omega_0}}{2}$ [21]. The non-hermitian Hamiltonian H in (1) with $\theta_1=\theta_2=0$, $\xi_1=\xi_2=\xi_3=0$, $\gamma=\pm\alpha$, $\delta=\pm\beta$,

$$\tilde{H} = \omega\; a^{\dagger}a + +\omega_0\; J_z + \alpha\; J_- a^{\dagger} + \beta\; J_+ a \pm \alpha J_- a \pm \beta J_+ a^{\dagger}, \quad (27)$$

is equivalent to the 'standard Dicke Model' through the similarity transformation $H_{Dicke} = \rho\tilde{H}\rho^{-1}$ with the operator $\hat{O}$ given by,

$$\hat{O} = \frac{1}{2}ln(\frac{\alpha}{\beta})\,(J_z+j)\,, \quad \frac{\alpha}{\beta} > 0. \quad (28)$$

Thus, the non-hermitian Hamiltonian $\tilde{H}$ also undergoes QPT for $\lambda_2 > \lambda_2^c$. The values of the atomic inversion and the mean photon number above the critical value λ_2^c are identical to that of the standard Dicke model:

$$j^{-1}\langle\langle J_z\rangle\rangle_{\eta_+} = -\left(\frac{\lambda_2^c}{\lambda_2}\right)^2,$$
$$j^{-1}\langle\langle a^{\dagger}a\rangle\rangle_{\eta_+} = \frac{2\lambda_2^2}{\omega^2}\left[1-\left(\frac{\lambda_2^c}{\lambda_2}\right)^4\right], \;\; \lambda_2 > \lambda_2^c. \quad (29)$$

The results for finite j, as quoted in Ref. [21] for H_{Dicke}, are equally applicable for $\tilde{H}$, since $\langle\langle J_z\rangle\rangle_{\eta_+} = \langle J_z\rangle_{H_{Dicke}}$ and $\langle\langle a^{\dagger}a\rangle\rangle_{\eta_+} = \langle a^{\dagger}a\rangle_{H_{Dicke}}$.

The Hamiltonian $\mathcal{H}$ in Eq. (14) with its full generality also undergoes QPT for $|\mu|<1$,

$$\mu \equiv \frac{\omega_0\left(\omega+2\sqrt{\theta_1\theta_2}\right)}{\left(\lambda_3+\lambda_4\right)^2}, \;\; \lambda_3 \equiv \sqrt{\frac{\alpha\beta}{2j}}, \;\; \lambda_4 \equiv \sqrt{\frac{\gamma\delta}{2j}}. \quad (30)$$

Consequently, H with the parameters satisfying the relations in Eq. (5) also undergoes quantum phase transition for $|\mu|<1$. The values of mean photon number and the atomic inversion for $\mu<1$ can be determined as follows:

$$j^{-1}\langle\langle a^{\dagger}a\rangle\rangle_{\eta_+} = \frac{1}{2}(1-\mu^2)\left(\frac{\lambda_3+\lambda_4}{\omega+2\sqrt{\theta_1\theta_2}}\right)^2$$
$$j^{-1}\langle\langle J_z\rangle\rangle_{\eta_+} = -\mu; \;\; \mu<1. \quad (31)$$

The mean photon number vanishes identically and the atomic inversion is equal to -1 for $\mu>1$. The QPT in the Tavis-Cummings model and the Dicke model appear as special cases of the general result described by Eqs. (30) and (31).

We have shown that a non-hermitian version of the DDM undergoes QPT. This is the first time in the literature that QPT for pseudo-hermitian operators has been described and definitely broadens the scope of studying QPT in various other non-hermitian models. For the particular case of the pseudo-hermitian DDM, it is to be seen whether or not the QPT is related to a change in level-statistics and/or cross-over from entangled to disentangled states, as is the case for the standard Dicke Hamiltonian[21, 22]. Finally, as mentioned earlier, the DDM can be mapped to certain spintronics-based models[24, 25]. Our results on QPT can be directly extended to such models and may prove to be the testing ground of pseudo-hermitian quantum mechanics through appropriate quantum engineering of two-dimensional semiconductor devices.

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* Electronic address: deguchi@phys.ocha.ac.jp
† Electronic address: pijushkanti.ghosh@visva-bharati.ac.in

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