arXiv:0911.1499v1 [q-bio.NC] 8 Nov 2009

DYNAMICAL ASPECTS OF MEAN FIELD PLANE ROTATORS AND THE KURAMOTO MODEL

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Abstract. The Kuramoto model has been introduced in order to describe synchronization phenomena observed in groups of cells, individuals, circuits, etc... We look at the Kuramoto model with white noise forces: in mathematical terms it is a set of N oscillators, each driven by an independent Brownian motion with a constant drift, that is each oscillator has its own frequency, which, in general, changes from one oscillator to another (these frequencies are usually taken to be random and they may be viewed as a quenched disorder). The interactions between oscillators are of long range type (mean field). We review some results on the Kuramoto model from a statistical mechanics standpoint: we give in particular necessary and sufficient conditions for reversibility and we point out a formal analogy, in the $N \to \infty$ limit, with local mean field models with conservative dynamics (an analogy that is exploited to identify in particular a Lyapunov functional in the reversible set-up). We then focus on the reversible Kuramoto model with sinusoidal interactions in the $N \to \infty$ limit and analyze the stability of the non-trivial stationary profiles arising when the interaction parameter K is larger than its critical value K_c. We provide an analysis of the linear operator describing the time evolution in a neighborhood of the synchronized profile: we exhibit a Hilbert space in which this operator has a self-adjoint extension and we establish, as our main result, a spectral gap inequality for every $K > K_c$.

2010 *Mathematics Subject Classification: 82C20, 35P15, 60K35*

Keywords: Synchronization, Kuramoto model, (ir)reversibility, mean field Spin XY model, spectral gap inequality

1. Introduction

The Kuramoto model (*e.g.* [1, 21]) is defined by the set of coupled stochastic differential equations:

$$\mathrm{d}\varphi_j^\xi(t) \,=\, \xi_j \,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N} \sin\left(\varphi_j^\xi(t) - \varphi_i^\xi(t)\right)\,\mathrm{d}t + \sigma\,\mathrm{d}w_j(t)\,, \tag{1.1}$$

for $j = 1, 2, \ldots, N$, where

(1) $\{w_j(\cdot)\}_{j=1,\ldots,N}$ is a family of independent and identically distributed standard Brownian motions. We refer to this source of randomness as *thermal noise*.
(2) $\xi = \{\xi_j\}_{j=1,\ldots,N}$ is a family of independent identically distributed random variables. This is another source of noise, and we refer to it as *disorder*.
(3) K is a real parameter and $\sigma \geq 0$.

We stress from now that we consider the stochastic evolution (1.1) once a realization of the disorder variables ξ is chosen, so the disorder is of *quenched* type. Moreover the law of $\{w_j(\cdot)\}_{j=1,\ldots,N}$ and of the initial condition (specified below) does not depend on the values of the disorder variables.

Date: November 10, 2009.

Remark 1.1. It will be at times interesting to discuss the role of the sine drift in the model. We will therefore refer to a h-model when $K\sin(\cdot)$ is replaced by a smooth, *i.e.* C^∞, 2π-periodic function $h(\cdot)$.

The variables $\varphi_j^\xi(t)$ are actually angles, so we focus on $\varphi_j^\xi(t)\mathrm{mod}(2\pi)$, which is an element of $\mathbb{S} := \mathbb{R}/(2\pi\mathbb{Z})$. The existence and uniqueness of a unique (strong) solution to the system (1.1), when the initial condition $\{\varphi_j^\xi(0)\}_{j=1,\dots,N} \in \mathbb{R}^N$ and $\{w_j(\cdot)\}_{j=1,\dots,N}$ are independent and $\{\varphi_j^\xi(0)\}_{j=1,\dots,N}$ are square integrable random variables, is a standard result. In our case we may therefore choose $\{\varphi_j^\xi(0)\}_{j=1,\dots,N}$ arbitrarily distributed provided that it is concentrated on $[0, 2\pi)^N$.

The main result of this work is on the model in which there is no disorder, that is the case in which the law of ξ_1 is degenerate, so that $\xi_j = \xi$ for every j, with ξ is a real constant. In this case, with the change of variable $\varphi_j(t) := \varphi_j^\xi(t) - \xi t$ we have

$$\mathrm{d}\varphi_j(t) \;=\; -\frac{K}{N}\sum_{i=1}^{N} \sin\left(\varphi_j(t) - \varphi_i(t)\right)\,\mathrm{d}t + \sigma\,\mathrm{d}w_j(t)\,. \tag{1.2}$$

Disregarding the disorder is actually a major simplification first of all because, if $\sigma > 0$, the system (1.2) is *reversible* with respect to the (Gibbs) probability measure

$$\mu_{N,K}(\,\mathrm{d}\varphi) \;:=\; \frac{1}{Z_{N,K}}\exp\left(-\frac{2K}{\sigma^2}H_N(\varphi)\right)\lambda_N(\,\mathrm{d}\varphi)\,, \tag{1.3}$$

where $\varphi \in \mathbb{S}^N$, λ_N is the uniform probability measure on $\mathbb{S}^N$ (that is the N-fold product of Lebesgue measures normalized by $(2\pi)^N$),

$$H_N(\varphi) \;:=\; -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos\left(\varphi_j - \varphi_i\right)\,, \tag{1.4}$$

and $Z_{N,K} := \int_{\mathbb{S}^N}\exp(-2K\sigma^{-2}H_N(\varphi))\lambda_N(\,\mathrm{d}\varphi)$ is the *partition* function. In fact, the generator $L_{K,N}$ of the dynamics (1.2) acts on twice differentiable functions $F : \mathbb{S}^N \to \mathbb{R}$ as

$$L_{K,N}F(\varphi) \;=\; \frac{\sigma^2}{2}\sum_{i=1}^{N}\frac{\partial^2 F(\varphi)}{\partial\varphi_i^2} - K\sum_{i=1}^{N}\frac{\partial H_N(\varphi)}{\partial\varphi_i}\frac{\partial F(\varphi)}{\partial\varphi_i}\,, \tag{1.5}$$

and one directly verifies the symmetry $\int F\,L_{K,N}G\,\mathrm{d}\mu_{K,N} = \int G\,L_{K,N}F\,\mathrm{d}\mu_{K,N}$ for $F, G \in C^2$, which implies that $\mu_{K,N}$ is invariant for the dynamics [20, 13].

The measure $\mu_{K,N}$ is the Gibbs measure of a classical statistical mechanics model: the mean field spin XY model with single spin state space $\mathbb{S}$, *i.e.* mean field plane rotators [18].

Remark 1.2. It is important to notice that the system (1.1) is not reversible unless $\xi_i = 0$ for every i. Even the case $\xi_i = \mathit{const.} \neq 0$ for every i is not reversible, but, as we have argued, it maps to a reversible system. Notice in fact that, unless $\xi \equiv 0$, the transformation $\varphi_j^\xi \mapsto \varphi_j^\xi - \xi t$ maps to a system with time dependent interactions. This strongly hints to the absence of reversibility and it is indeed the case, but proving such a statement is more delicate: we address this point in Section 4 below. The aspect that we want to stress here is the *disorder induced non-equilibrium character* of the full Kuramoto model.

1.1. **Empirical measure and the large N limit.** Since we focus on the $\sigma > 0$ case, there is no loss in generality in choosing $\sigma = 1$ and we will do so from now on. We introduce the empirical measure

$$\nu_{N,t}(\mathrm{d}\theta) := \frac{1}{N}\sum_{j=1}^{N} \delta_{\varphi_j(t)}(\mathrm{d}\theta), \tag{1.6}$$

and observe that, by Itô's rule, for every $F \in C^2(\mathbb{S})$ and $t > 0$

$$\int_{\mathbb{S}} F(\theta)\nu_{N,t}(\mathrm{d}\theta) - \int_{\mathbb{S}} F(\theta)\nu_{N,0}(\mathrm{d}\theta) = \\ -K\int_0^t \int_{\mathbb{S}^2} F'(\theta)\sin(\theta-\theta')\nu_{N,s}(\mathrm{d}\theta)\nu_{N,s}(\mathrm{d}\theta')\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_{\mathbb{S}} F''(\theta)\nu_{N,s}(\mathrm{d}\theta)\,\mathrm{d}s + M_{N,F}(t)\,, \tag{1.7}$$

where $M_{N,F}(\cdot)$ is a continuous martingale with quadratic variation at time t, $\langle M_{N,F}\rangle(t)$, equal to $N^{-1}\int_0^t\int_{\mathbb{S}}(F'(\theta))^2\nu_{N,s}(\mathrm{d}\theta)\,\mathrm{d}s$. Therefore, by Doob's inequality, for every $T > 0$ we have that $\mathbf{E}[\sup_{t\in[0,T]}(M_{N,F}(t))^2]$ is bounded by $\langle M_{N,F}\rangle(T) \le T\|F'\|_\infty^2/N$: this guarantees that the thermal noise disappears as $N \to \infty$, so that the limit of the empirical measure, if it exists, is not random. To make this precise we introduce the space $C^0([0,T];\mathcal{M}_1(\mathbb{S}))$, where $\mathcal{M}_1(\mathbb{S})$ are the probability measures on $\mathbb{S}$ equipped with the topology of the weak convergence, and observe that if a subsequence of $\{\nu_{N,\cdot}\}_{N\in\mathbb{N}}$ (of elements of $C^0([0,T];\mathcal{M}_1(\mathbb{S})))$ converges to a limit $\nu_\cdot$, we have that for $t \in (0,T]$ and every $F \in C^2(\mathbb{S})$

$$\int_{\mathbb{S}} F(\theta)\nu_t(\mathrm{d}\theta) - \int_{\mathbb{S}} F(\theta)\nu_0(\mathrm{d}\theta) = \\ \frac{1}{2}\int_0^t \int_{\mathbb{S}} F''(\theta)\nu_s(\mathrm{d}\theta)\,\mathrm{d}s - K\int_0^t \int_{\mathbb{S}^2} F'(\theta)\sin(\theta-\theta')\nu_s(\mathrm{d}\theta)\nu_s(\mathrm{d}\theta')\,\mathrm{d}s\,. \tag{1.8}$$

This is a weak form of the equation

$$\partial_t q_t(\theta) = \frac{1}{2}\frac{\partial^2 q_t(\theta)}{\partial\theta^2} + K\frac{\partial}{\partial\theta}\left[\left(\int_{\mathbb{S}} \sin(\theta-\theta')q_t(\theta')\,\mathrm{d}\theta'\right) q_t(\theta)\right], \tag{1.9}$$

when $\nu_t(\mathrm{d}\theta) = q_t(\theta)\,\mathrm{d}\theta$. So that if we assume that $\nu_{N,0}$ converges to a non random limit and if there is a unique solution to (1.8), then the evolution is non random and determined by (1.8).

More precisely, we have the following:

Proposition 1.3. *If there exists $\nu_0 \in \mathcal{M}_1(\mathbb{S})$ such that for every $\varepsilon > 0$ and every $F \in C^0(\mathbb{S};\mathbb{R})$ we have*

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\int_{\mathbb{S}} F(\theta)\nu_{N,0}(\mathrm{d}\theta) - \int_{\mathbb{S}} F(\theta)\nu_0(\mathrm{d}\theta)\right| > \varepsilon\right) = 0\,, \tag{1.10}$$

then for every $t > 0$ we have that for every ε and F

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\int_{\mathbb{S}} F(\theta)\nu_{N,t}(\mathrm{d}\theta) - \int_{\mathbb{S}} F(\theta)\nu_t(\mathrm{d}\theta)\right| > \varepsilon\right) = 0\,, \tag{1.11}$$

where $\nu_\cdot$ is the unique solution of (1.8). Moreover, for every $t > 0$ the measure ν_t is absolutely continuous with respect to the Lebesgue measure with (strictly) positive density $q_t(\cdot)$ and the function $(t,\theta) \mapsto q_t(\theta)$, from $(0,\infty)\times\mathbb{S}$ to $(0,\infty)$, is smooth and solves (1.9).

Proposition 1.3 is a particular (and particularly easy) case of far more general results (see for example [8, 15]). The derivation goes along proving tightness of $\{\nu_{N,\cdot}\}_{N\in\mathbb{N}}$ and then proving uniqueness for the limiting equation (1.8). In our case such an equation is particularly nice and the evolution is smoothing, so that even if the initial datum is not a function (*i.e* ν_0 is not absolutely continuous with respect to the Lebesgue measure) or it is not smooth, $\nu_t(\mathrm{d}\theta) = q_t(\theta)\,\mathrm{d}\theta$ and $q_t(\cdot) \in C^\infty(\mathbb{S})$ for every $t > 0$. These analytic aspects are taken up with more details in Section 3 (Proposition 3.1)

Remark 1.4. It is however important to recall here that Proposition 1.3 can be generalized to cover the disordered case (1.1). We refer to [7, 10] for precise statements, but, roughly, if the law of the random variable ξ_1 is denoted by $\mu(\mathrm{d}\xi)$ (let us for example assume that ξ_1 is bounded), the empirical average at time $t > 0$ converges as $N \to \infty$ to a measure with density $\int_{\mathbb{R}} q_t(\theta;\xi)\mu(\mathrm{d}\xi)$, where $\{q_t(\theta;\xi)\}_{t\geq 0,\theta\in\mathbb{S},\xi\in\mathbb{R}}$ is the unique solution to

$$\begin{aligned} \partial_t q_t(\theta;\xi) &= \frac{1}{2}\frac{\partial^2 q_t(\theta;\xi)}{\partial\theta^2} + \frac{\partial}{\partial\theta}\left[\left(\int_{\mathbb{R}}\left(\int_{\mathbb{S}} K\sin(\theta-\theta')q_t(\theta';\xi')\,\mathrm{d}\theta'\right)\mu(\mathrm{d}\xi') + \xi\right)q_t(\theta;\xi)\right], \\ q_0(\theta;\xi) &= \frac{\mathrm{d}\nu_0(\mathrm{d}\theta)}{\mathrm{d}\theta}, \end{aligned} \tag{1.12}$$

for every ξ in the support of μ. We have of course assumed, for simplicity, that ν_0 is absolutely continuous with respect to the Lebesgue measure.

Remark 1.5. The Kuramoto limit evolution (1.9) comes up also as *mesoscopic scaling limit* for the density of Kač models with conservative dynamics [5, 9], when the interaction potential is chosen equal to $\cos(\cdot)$. This is just a formal analogy, but it is going to be crucial for the sequel.

1.2. **Stationary profiles.** By the regularizing character of the evolution (Proposition 3.1), the stationary solutions to (1.8) coincide with the stationary solutions to (1.9) (we are of course interested only in non-negative solutions of total mass equal to one: Proposition 3.1 guarantees also the positivity of stationary solutions). Let us notice moreover that if $\hat{q}(\cdot)$ is a stationary solution, then $\hat{q}(\cdot + \theta_0)$ is a stationary solution too, for arbitrary choice of θ_0. This is due to the invariance of (1.2) under rotations (that can of course be read also out of (1.4)). Note that $\hat{q}(\cdot) = 1/(2\pi)$ is a solution to (1.2), regardless of the value of K but there may be more solutions: in fact every stationary solution can be written as $\hat{q}(\cdot + \theta_0)$ for some $\theta_0 \in [0, 2\pi)$ and

$$\hat{q}(\theta) := \frac{\exp(2Kr\cos(\theta)}{\int_{\mathbb{S}}\exp(2Kr\cos(\theta')\,\mathrm{d}\theta'}, \tag{1.13}$$

with r a non-negative solution to

$$r := \Psi(2Kr), \quad \text{with} \quad \Psi(x) := \frac{\int_{\mathbb{S}}\cos(\theta)\exp(x\cos(\theta))\,\mathrm{d}\theta}{\int_{\mathbb{S}}\exp(x\cos(\theta))\,\mathrm{d}\theta}. \tag{1.14}$$

In general, there is more than one solution to (1.14): in fact, there can be at most two, more precisely there is only the trivial solution $r = 0$ for $K \leq 1$ and there is also a second solution $r > 0$ if $K > 1$. This is because $\Psi'(0) = 1$ and because $\Psi(\cdot) : [0,\infty) \to [0,1)$ is strictly concave [16]. In terms of stationary solutions, this means that for $K \leq 1$ only the flat (*incoherent*) profile $1/(2\pi)$ is stationary, while for $K > 1$ also $\{\hat{q}(\cdot + \theta_0)\}_{\theta_0\in\mathbb{S}}$

is a family of stationary solutions (they are the solutions that exhibit the *coherence* or *synchronization* of the system).

The result we just stated, that is (1.13)-(1.14), is a classical one in the sense that it is of course closely linked to the solution of the mean field planar rotator model [18] (result completed by the concavity result proven in [16]). It is however worthwhile recalling the proof: every stationary solution $\hat{q}$ of (1.9) satisfies

$$\frac{1}{2}\hat{q}'(\theta) + K\left(\int_{\mathbb{S}} \sin(\theta-\theta')\hat{q}(\theta')\,\mathrm{d}\theta'\right)\hat{q}(\theta) \,=\, C\,, \tag{1.15}$$

for some constant C. Since we know that $\hat{q}(\cdot) > 0$, then (1.15) yields

$$\frac{1}{2}\left(\log \hat{q}(\theta)\right)' - K\left(\int_{\mathbb{S}} \cos(\theta-\theta')\hat{q}(\theta')\,\mathrm{d}\theta'\right)' \,=\, \frac{C}{\hat{q}(\theta)}\,, \tag{1.16}$$

which implies $C = 0$. At this point, by playing on the rotation symmetry, we may assume that $\int_{\mathbb{S}} \hat{q}(\theta)\sin(\theta)\,\mathrm{d}\theta = 0$, so that any stationary non-negative solution $\hat{q}(\cdot)$ with prescribed first Fourier cosine coefficient $\int_{\mathbb{S}} \hat{q}(\theta)\cos(\theta)\,\mathrm{d}\theta$ equal to r satisfies

$$\frac{1}{2}\hat{q}(\theta)' - Kr\cos(\theta)\hat{q}(\theta) \,=\, 0. \tag{1.17}$$

A solution to (1.17) is proportional to $\exp(2Kr\cos(\theta)$. By normalizing ($\hat{q}(\cdot)$ is a probability density) and recalling the constraint on the first Fourier cosine coefficient we get to (1.13)–(1.14).

Remark 1.6. Remarkably, a generalization of (1.13)–(1.14) holds also in the disordered case [19]. The key to such a derivation, like in the step above, is in the identification of the order parameter r, that captures the degree of *coherence* (or *synchronization*) of the oscillators. In statistical mechanics terms this is nothing but the fact that the Hamiltonian $H_N(\varphi)$ may be rewritten as $(1/2N)\sum_{i,j} S_i \cdot S_j = (1/2N)(\sum_i S_i)^2$, with $S_i = (\Re\exp(\mathrm{i}\varphi_i), \Im\exp(\mathrm{i}\varphi_i))$. However, if one considers an h-model (*cf.* Remark 1.1), the Hamiltonian cannot be expressed any longer as a function of the total magnetization $\sum_i S_i$. For the identification of the order parameter in this more general context we refer to [6].

1.3. **The gradient flow viewpoint.** For our purposes the following fact is of crucial importance: (1.9) can be rewritten in the *gradient form*

$$\partial_t q_t(\theta) \,=\, \nabla\left[q_t(\theta)\nabla\left(\frac{\delta\mathcal{F}(q_t)}{\delta q_t(\theta)}\right)\right]\,, \tag{1.18}$$

where we use ∇ for ∂_θ for visual impact, $\delta\mathcal{G}(q)/\delta q(\theta)$ is the standard L^2 Fréchet derivative of the functional $\mathcal{G}$ and

$$\mathcal{F}(q) \,:=\, \frac{1}{2}\int_{\mathbb{S}} q(\theta)\log q(\theta)\,\mathrm{d}\theta - \frac{K}{2}\int_{\mathbb{S}^2} \cos(\theta-\theta')q(\theta)q(\theta')\,\mathrm{d}\theta\,\mathrm{d}\theta'. \tag{1.19}$$

Note that $\mathcal{F} : L^2(\mathbb{S}) \to \mathbb{R}$ is Fréchet differentiable at $q(\cdot)$ for example if $q(\cdot)$ is continuous and $q(\cdot) > 0$ and Proposition 3.1 guarantees that the evolution may be cast in the form (1.18) for $t > 0$. A direct consequence of (1.18) is that

$$\frac{\partial\mathcal{F}(q_t)}{\partial t} \,=\, -\int_{\mathbb{S}} q_t(\theta)\left(\nabla\frac{\delta\mathcal{F}(q_t)}{\delta q_t(\theta)}\right)^2 \mathrm{d}\theta \,\leq\, 0\,. \tag{1.20}$$

A first consequence of this observation is:

Proposition 1.7. *If there exists* $t_2 > t_1 \geq 0$ *such that* $q_{t_1}(\cdot) = q_{t_2}(\cdot)$*, then there exists a constant* C *and a value* r *satisfying* (1.14) *such that* $q_t(\cdot) = \hat{q}(\cdot + c)$ *for every* $t \geq t_1$*.*

Proof. By Proposition 3.1 (guaranteeing smoothness and positivity of the solution) and by (1.20) we see that $\mathcal{F}(q_t)$ is constant for $t \in [t_1, t_2]$, so that $\nabla \delta \mathcal{F}(q_t)/\delta(q_t(\theta)) = 0$ for every θ and $t \in [t_1, t_2]$. But this implies that

$$\frac{1}{2}\nabla \left(\log q_t(\theta)\right) + K \left(\int_{\mathbb{S}} \sin(\theta - \theta') q_t(\theta') \, \mathrm{d}\theta'\right) \; = \; 0 \,, \tag{1.21}$$

which is precisely (1.16) with $C = 0$. Therefore for every $t \in [t_1, t_2]$ there exists a constant $\gamma(t)$ such that $q_t(\theta) = \hat{q}(\theta + \gamma(t))$, with $\hat{q}(\cdot)$ as in (1.13)-(1.14). Since $\hat{q}(\cdot + \gamma(t_1))$ is a stationary solution, the claim follows. □

Two observations are in order:

(1) Proposition 1.7 generalizes to the non-disordered h-model, when the latter is reversible (see Section 4), in the sense that the hypotheses imply $\nabla(\delta\mathcal{F}/\delta q_t) = 0$ and this condition identifies all the stationary solutions.
(2) Proposition 1.7 shows that there is no non-trivial stationary solution to (1.1) when $\xi_j \equiv \xi$, ξ a non-zero constant. This is simply because, otherwise, we would have a solution to (1.2) of the form $q(\cdot - t\xi)$, with $q(\cdot)$ non-constant, which violates Proposition 1.7. This is of interest also because it is not clear that Proposition 1.7 generalizes to disordered models. Clarifying the link between non-reversibility and coexistence of stationary and *rotating* solutions appears also to be an intriguing question.

1.4. **On synchronization stability.** The main result that we present addresses the important issue of the stability of the non-trivial stationary profiles $\hat{q}(\cdot)$, more precisely of the stability of the invariant *manifold* $\{\hat{q}(\cdot + \theta_0)\}_{\theta_0 \in \mathbb{S}}$. In the literature we find a full analysis of incoherence stability [22] (also in presence of disorder) as well as an analysis of synchronized profiles as bifurcation from the incoherent $1/2\pi$ profile (we refer to [1] and the several references therein). Our aim is to have a detailed non-perturbative analysis of the linearized evolution operator in the non disordered case, for every $K > K_c = 1$.

To address such an issue we observe that the linearized evolution $u_t(\cdot)$ around $\hat{q}(\cdot)$ obeys the equation $\partial_t u_t(\theta) = L_{\hat{q}} u_t(\theta)$ with $L_{\hat{q}}$ a linear operator with domain $D(L_{\hat{q}}) := \{C^2(\mathbb{S};\mathbb{R}) : \int_{\mathbb{S}} u(\theta)\,\mathrm{d}\theta = 0\}$ defined as

$$L_{\hat{q}} u(\theta) \; = \; \frac{1}{2}\Delta u(\theta) + K\nabla \left[\hat{q}(\theta) \int_{\mathbb{S}} \sin\left(\theta - \theta'\right) u\left(\theta'\right) \, \mathrm{d}\theta' + u(\theta) \int_{\mathbb{S}} \sin\left(\theta - \theta'\right) \hat{q}\left(\theta'\right) \, \mathrm{d}\theta'\right] . \tag{1.22}$$

It is easy to verify that $L_{\hat{q}}\hat{q}' = 0$, and this corresponds to the rotation invariance of the problem. However, what we are going to prove is that the remaining part of the spectrum is also real and it lies on the negative semi-axis. In order to make precise statements about $L_{\hat{q}}$ we introduce the Hilbert space $H_{-1,1/\hat{q}}$ of distributions u such that $u = \mathcal{U}'$, with $\mathcal{U} \in L^2(\mathbb{S};\mathbb{R})$. Of course the derivative is taken in the sense of distributions and $\mathcal{U}$ is determined, given u, only up to a constant: we remove this uncertainty by stipulating that $\int_{\mathbb{S}} (\mathcal{U}(\theta)/\hat{q}(\theta))\,\mathrm{d}\theta = 0$. The norm of $u \in H_{-1,1/\hat{q}}$ is defined by

$$\|u\|^2_{-1,1/\hat{q}} \; := \; \int_{\mathbb{S}} \frac{\mathcal{U}(\theta)^2}{\hat{q}(\theta)} \, \mathrm{d}\theta, \tag{1.23}$$

and the scalar product of u and $v \in H_{-1,1/\hat{q}}$ is going to be denoted by $\langle\langle u, v \rangle\rangle$: it is of course equal to $\int_{\mathbb{S}} (\mathcal{U}(\theta)\mathcal{V}(\theta)/\hat{q}(\theta))\, \mathrm{d}\theta$, with definition of $\mathcal{V}$ in analogy with $\mathcal{U}$. We will come back in the next section with more on $H_{-1,1/\hat{q}}$, but what one can verify directly is that $D(L_{\hat{q}})$ and $L_{\hat{q}} D(L_{\hat{q}})$ are subsets of $H_{-1,1/\hat{q}}$ and that $L_{\hat{q}}$ is symmetric as an operator on $H_{-1,1/\hat{q}}$, that is

$$\langle\langle u, L_{\hat{q}} v \rangle\rangle \;=\; \langle\langle v, L_{\hat{q}} u \rangle\rangle, \tag{1.24}$$

for every $u, v \in D(L_{\hat{q}})$ (for an explicit expression see (2.14)). We will actually prove (Proposition 2.6) that $L_{\hat{q}}$ is essentially self-adjoint. Moreover:

Theorem 1.8. *The spectrum of (the self-adjoint extension of) $L_{\hat{q}}$ is pure point and it lies in $(-\infty, 0]$. The value 0 is in the spectrum, with one-dimensional eigenspace (spanned, as we have seen, by $\hat{q}'$) and the distance between zero and the rest of the spectrum is of at least*

$$\lambda(K) \;:=\; \frac{\left(1 - K(1-r^2)\right)\left(1 - (I_0(2Kr))^{-2}\right)}{2Kr^2 \exp(8Kr) + \exp(4Kr)\left(1 - (I_0(2Kr))^{-2}\right)} \;>\; 0. \tag{1.25}$$

We stress that Theorem 1.8 holds as soon as there is a non-trivial solution r to (1.14), that is for every $K > 1$. We have:

$$\lambda(K) \overset{K \searrow 1}{\sim} \frac{K-1}{2} \quad \text{ and } \quad \lambda(K) \overset{K \to \infty}{\sim} \frac{\exp(-8K+2)}{4K}. \tag{1.26}$$

Numerically increases till $K = 1.033\ldots$, where it reaches the value $0.0028\ldots$, and then it decreases.

The paper is organized as follows: in Section 2 we study $L_{\hat{q}}$ and prove a spectral gap inequality, the essential self-adjontness of the operator and the fact that the spectrum is pure point. The nonlinear evolution properties mentioned in this introduction are treated in Section 3 and the (ir)reversibility issues are considered in Section 4.

2. Synchronization stability

In this section we prove the main result (Theorem 1.8). We assume $K > 1$ and, for simplicity, we drop the hat from $\hat{q}(\cdot)$, so that a stationary solution is denoted by $q(\cdot)$.

2.1. **Some properties of the stationary profile.**

We first rewrite (1.13)-(1.14) by using the Bessel function notation:

$$q(\theta) \;:=\; \frac{1}{2\pi I_0(2Kr)} \exp\left(2Kr\cos(\theta)\right), \tag{2.1}$$

and $r \in (0,1)$ is the unique positive solution of

$$r \;:=\; \Psi(2Kr), \quad \text{with} \quad \Psi(x) \;:=\; \frac{I_1(x)}{I_0(x)}. \tag{2.2}$$

We have used the standard notation for the modified Bessel functions of order 0 and 1, explicitly

$$I_\nu(x) \;:=\; \frac{1}{2\pi} \int_0^{2\pi} (\cos(\theta))^\nu \exp\left(x\cos(\theta)\right)\, \mathrm{d}\theta, \qquad \text{for } \nu = 0, 1. \tag{2.3}$$

As already mentioned before, uniqueness of r is a non-trivial fact that follows from [16, Lemma 4], that establishes in particular the concavity of $\Psi(\cdot)$ on the positive semi-axis.

One can therefore define, via (2.2)), the function $[1,\infty) \ni K \mapsto r(K) \in [0,1)$ (one sets $r(1) := 0$ by continuity). We have that, for $K > 1$, $1-K(1-r^2) \in (0,1/2)$ or (equivalently)

$$\sqrt{1-\frac{1}{K}} < r(K) < \sqrt{1-\frac{1}{2K}}. \tag{2.4}$$

These bounds are easily checked for K close to 1 and K large, and and the numerical plots of the three functions appearing in (2.4) ca be found in Figure 1. We could not find quick proofs of (2.4): a proof of the upper bound is a byproduct of one of the arguments that we develop below (see the proof of Lemma 2.2), while we prove here the lower bound by using Bessel functions properties.

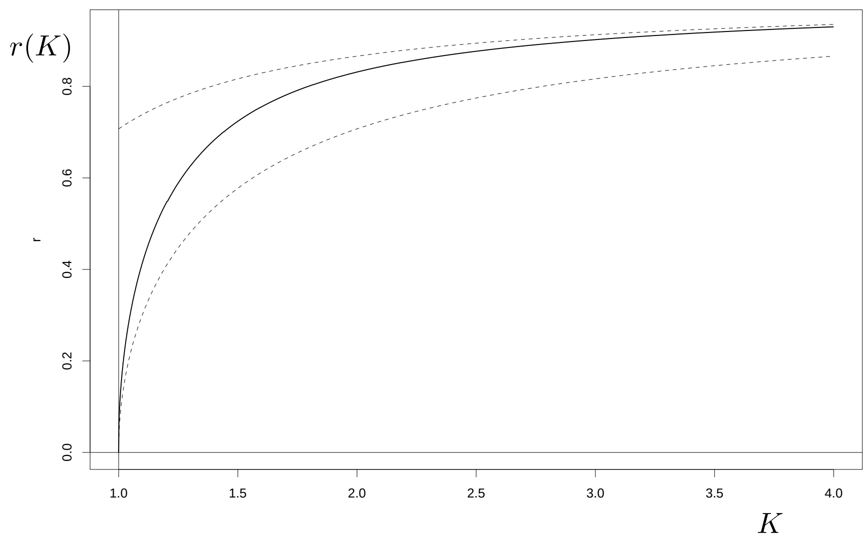

FIGURE 1. The function $K \mapsto r(K)$ and (dashed lines) the bounds of (2.4). The upper bound is proven, the lower bound is verified for K close to 1 and K large.

Proof of (2.4), *lower bound.* By the change of variables $(r,K) \mapsto (r, 2Kr) =: (r,y)$ we see that what we have to prove is equivalent to showing that

$$\Psi^2(y) + \frac{2}{y}\Psi(y) - 1 \overset{y>0}{>} 0\,, \tag{2.5}$$

Apply now the identity [23]

$$\frac{I_1(y)}{I_0(y)} = \frac{y}{2}\left(\frac{1}{1+\frac{y}{2}\frac{I_2(y)}{I_1(y)}}\right), \tag{2.6}$$

so that

$$1-\frac{2}{y}\Psi(y) = \frac{1}{1+\frac{2}{y}\frac{I_1(y)}{I_2(y)}}\ (>0), \tag{2.7}$$

and therefore (2.5) is equivalent to

$$\Psi^2(y)\left(1+\frac{2}{y}\frac{I_1(y)}{I_2(y)}\right) = \frac{I_1(y)^2}{I_0(y)I_2(y)} > 1\,, \tag{2.8}$$

where the intermediate step follows from the identity $yI_2(y) + 2I_1(y) - yI_0(y) = 0$ [23]. But this is equivalent to $I_1(y)/I_0(y) > I_2(y)/I_1(y)$ for $y > 0$, a fact that is proven in [11, (3.8)]. □

In the sequel we will also use the notations

$$J(\theta) := -K\sin(\theta) \quad \text{and} \quad \widetilde{J}(\theta) := K\cos(\theta)\,, \tag{2.9}$$

so that

$$\frac{q'}{2q} = J * q\,, \tag{2.10}$$

and with this change of notation (1.22) reads

$$L_q u = \frac{1}{2}u'' - \left(q\,(J*u) + u\,(J*q)\right)'\,. \tag{2.11}$$

2.2. **Rigged Hilbert spaces and $H_{-1,1/q}$.** We now introduce a *rigged Hilbert spaces* structure [4, pp. 81-82]. The *pivot* (Hilbert) space is $H := \{u \in L^2(\mathbb{S}) : \int u = 0\}$ (of course the scalar product is $(u,v) := \int uv$ and the norm is denoted by $\|\cdot\|_2$). The second Hilbert space we consider is $V := H_{1,q}$, closure of the set of periodic C^1 functions u such that $\int u = 0$ with respect to the norm

$$\|u\|_{1,q} := \sqrt{\int_{\mathbb{S}} (u')^2 q}, \tag{2.12}$$

so that $V \subset H$ and the canonical injection of V into H is continuous (by the Poincaré inequality). Note that V is dense in H. We consider then the dual space V' of V and the duality functional in V' defined by $\varphi_u(v) := (u,v)$ for every given $u \in H$ ($\varphi_u : V \to \mathbb{R}$). We can define $T : H \longrightarrow V'$ by setting $Tu := \varphi_u$. One can then show that $T(H)$ is dense in V' and T injects H into V' in a continuous way [4, p. 82]. This injection allows considering H as a subset of V', by identifying u and Tu. Moreover if $u \in H$ we have that $\|u\|_{V'} = \|Tu\|_{V'}$ can be made more explicit: given $u \in H$ we call $\mathcal{U}$ the primitive of H such that $\int U/q = 0$ and we observe that

$$\|u\|_{V'} = \sup_{v\in V} \frac{(u,v)}{\|v\|_V} = \sup_{v \in H_{1,q}} \frac{\int \mathcal{U} v'}{\sqrt{\int q (v')^2}} = \sqrt{\int \frac{\mathcal{U}^2}{q}}, \tag{2.13}$$

where the last step follows on one hand by the Cauchy-Schwarz inequality (establishing the upper bound) and by choosing $v' = \mathcal{U}/q$ in the supremum (establishing the lower bound).

As already mentioned in the introduction, the scalar product in $V' = H_{-1,1/q}$ is denoted by $\langle\langle \cdot,\cdot \rangle\rangle$. We observe also that these steps allow the precise identification of the functions in $H_{-1,1/q}$: $u \in H_{-1,1/q}$ if and only if $u = U'$ (in the sense of distributions), with $U \in L^2(\mathbb{S})$.

At this point it is crucial to observe that $D(L_q)$ (recall that $D(L_q)$ is the subset of periodic C^2 functions u such that $\int u = 0$) is dense in $H_{-1,1/q}$ and that for $u, v \in D(L_q) \subset H_{-1,1/q}$, we have (use (2.10))

$$\langle\langle v, L_q u\rangle\rangle = \langle\langle L_q v, u\rangle\rangle = -\frac{1}{2}\int_{\mathbb{S}} \frac{uv}{q} + \left(v, \widetilde{J} * u\right). \tag{2.14}$$

In words: L_q is a symmetric operator on $H_{-1,1/q}$.

2.3. **Estimates on the Dirichlet form.** It is going to be useful to introduce also the Hilbert space $L^2_{1/q}$, which coincides with $L^2(\mathbb{S})$ as a set of functions, but we equip it with the scalar product

$$\langle\!\langle u, u \rangle\!\rangle \;:=\; (u, u/q). \tag{2.15}$$

We therefore introduce, for $u \in D(L_q)$, the Dirichlet form $\mathcal{D}(u) := -\langle\!\langle u, L_q u \rangle\!\rangle$. By (2.14) we have

$$\mathcal{D}(u) \;=\; \frac{1}{2}\langle\!\langle u, u \rangle\!\rangle - \left(u, \widetilde{J} * u\right). \tag{2.16}$$

Our aim is to bound from below $\mathcal{D}(u)$ and we start with two technical lemmas. The first one yields the spectral decomposition of $(u, \widetilde{J} * u)$, viewed as a quadratic form on $L^2_{1/q}$.

Lemma 2.1. *We have the orthogonal decomposition*

$$L^2_{1/q} \;=\; V_0 \oplus^{\perp} V_{1/2} \oplus^{\perp} V_{K-1/2}, \tag{2.17}$$

where

$$V_0 \;:=\; \left\{ \theta \mapsto a_0 + \sum_{j \ge 2} (a_j \cos(j\theta) + b_j \sin(j\theta)) : \sum_j a_j^2 + b_j^2 < \infty \right\}, \tag{2.18}$$

and both $V_{1/2}$ and $V_{K-1/2}$ are one dimensional subspaces generated respectively by $\theta \mapsto \sin(\theta)q(\theta)$ $(= -q'(\theta)/2Kr)$ and by $\theta \mapsto \cos(\theta)q(\theta)$. Moreover, when $u \in V_\lambda$ we have

$$\widetilde{J} * u \;=\; \frac{\lambda}{q} u. \tag{2.19}$$

Proof. The $L^2_{1/q}$-orthogonality statements $V_0 \perp V_{1/2}$ and $V_0 \perp V_{K-1/2}$ follow directly from the orthogonality in L^2 of the family $\{\cos(j\theta), \sin(j\theta)\}_{j=0,1,\ldots}$. Instead $V_{1/2} \perp V_{K-1/2}$ because $\int_{-\pi}^{\pi} q(\theta)\cos(\theta)\sin(\theta)\,\mathrm{d}\theta = 0$.

The validity of (2.19) follows by direct computation: for $u \in V_0$

$$\widetilde{J} * u(\theta) \;=\; K\cos(\theta) \int_0^{2\pi} \cos(\theta')u(\theta')\,\mathrm{d}\theta' + K\sin(\theta) \int_0^{2\pi} \sin(\theta'))u(\theta')\,\mathrm{d}\theta', \tag{2.20}$$

which is equal to zero because u does not contain the first harmonics. The other two cases follow by using the same trigonometric identity and the following two (clearly equivalent) identities:

$$\int_0^{2\pi} q(\theta)\sin^2(\theta)\,\mathrm{d}\theta \;=\; \frac{1}{2K}, \qquad \int_0^{2\pi} q(\theta)\cos^2(\theta)\,\mathrm{d}\theta \;=\; \left(1 - \frac{1}{2K}\right). \tag{2.21}$$

□

The second lemma is more technical and its interest will become clear in the proof of Proposition 2.3.

Lemma 2.2. *We have*

$$\min_{u \in V_0 \colon \int u = 0} \langle\!\langle 1+u, 1+u \rangle\!\rangle \;=\; \langle\!\langle 1+\hat{u}, 1+\hat{u} \rangle\!\rangle \;=\; (2\pi)^2 \frac{2K-1}{2K(1-r^2)-1}, \tag{2.22}$$

where

$$\hat{u}(\theta) \,=\, -1 - \frac{2\pi\left(1-\frac{1}{2K}\right)}{\left(r^2-\left(1-\frac{1}{2K}\right)\right)}\, q(\theta) \,+\, \frac{2\pi r}{\left(r^2-\left(1-\frac{1}{2K}\right)\right)}\, q(\theta)\cos(\theta), \tag{2.23}$$

Proof. We have to minimize a quadratic functional under the linear constraints $(1,u)=0$ and $u\in V_0$. This corresponds to the three constraints:

$$\int_0^{2\pi} u(\theta)\,\mathrm{d}\theta \,=\, 0, \qquad \int_0^{2\pi} \cos(\theta)u(\theta)\,\mathrm{d}\theta \,=\, 0 \quad \text{and} \quad \int_0^{2\pi} \sin(\theta)u(\theta)\,\mathrm{d}\theta \,=\, 0. \tag{2.24}$$

The extrema (minima, by convexity) of such a problem can be found by the Lagrange multipliers method and they are of the form

$$\hat{u}(\theta) \,=\, -1 + \lambda q(\theta) + \mu q(\theta)\cos(\theta) + \eta q(\theta)\sin(\theta), \tag{2.25}$$

with λ, μ and η three real numbers. The constraints (2.24), via (2.2) and (2.21), yield the linear system $\eta=0$, $\lambda+\mu r=2\pi$ and $2K\lambda r+\mu(2K-1)=0$, which has a solution if and only if $2K(1-r^2)-1\neq 0$. Since the minimum exists for every K and since $2K(1-r^2)-1\to 1$ for $K\searrow 1$ we see that $2K(1-r^2)-1>0$ for every K (this is the upper bound in (2.4)). The proof is completed by making λ and μ explicit and using that the expression in (2.22) is equal to $\lambda^2+2\lambda\mu r+\mu^2(1-(1/2K))$. □

The following is one of our main statements:

Proposition 2.3. *There exists $c_K\in(0,1/2)$ such that, if $u\in L^2_{1/q}$ is such that $(1,u)=\int_0^{2\pi} u=0$, then*

$$\mathcal{D}(u) \,\geq\, c_K \left\langle u-u_{1/2}, u-u_{1/2}\right\rangle, \tag{2.26}$$

where $u_{1/2}:=q'\langle u,q'\rangle/\langle q',q'\rangle$. In particular, $\mathcal{D}(u)\geq 0$.

Proof. Lemma 2.1 shows that, if we write $u=v_0+v_{1/2}+\widetilde{v}$ (according to the decomposition (2.17): of course $v_{1/2}=u_{1/2}$) we have

$$\mathcal{D}(u) \,=\, -(K-1)\,\langle\,\widetilde{v},\widetilde{v}\,\rangle + \frac{1}{2}\langle v_0,v_0\rangle\,. \tag{2.27}$$

We write $\widetilde{v}(\theta)=\widetilde{c}q(\theta)\cos(\theta)$ and $v_0=a_0+\sum_{j\geq 2}(a_j\cos(j\theta)+b_j\sin(j\theta))$ and, with this notations, one directly sees, using the definitions (2.1) and (2.2), that the constraint $(1,u)=0$ is equivalent to

$$r\widetilde{c} \,=\, -2\pi a_0. \tag{2.28}$$

Let us observe that we can assume $\widetilde{c}\neq 0$: if $\widetilde{c}=0$ then $v-v_{1/2}=v_0$ and, in view of (2.27), (2.26) holds. From now on we perform estimates for arbitrary, but fixed, values of $\widetilde{c}\neq 0$ (hence a_0 is fixed too).

Note then that $\langle\widetilde{v},\widetilde{v}\rangle=\widetilde{c}^2(1-(1/(2K)))$, by (2.21), and we can therefore write

$$\mathcal{D}(u) \,=\, -(K-1)\left(1-\frac{1}{2K}\right)\widetilde{c}^2 \,+\, \frac{a_0^2}{2}\left\langle 1+\frac{u_2}{a_0}, 1+\frac{u_2}{a_0}\right\rangle, \tag{2.29}$$

where of course we have set $u_2:=v_0-a_0$. By Lemma 2.2 we therefore obtain that

$$\min_{\substack{u:(1,u)=0\\ \text{given } \widetilde{c}}} \mathcal{D}(u) \,=\, \mathcal{D}(\widetilde{v}+a_0(1+\hat{u}))\,, \tag{2.30}$$

where we recall that $\widetilde{c}$ and a_0 are related via (2.28). Note that, by (2.4), $c_1(K) > 0$ for $K > 1$. For sake of compactness let us introduce $\hat{v}_0 := a_0(1+\hat{u})(\in V_0)$, so that $u = (v_0 - \hat{v}_0) + \hat{v}_0 + v_{1/2} + \widetilde{v}$ and, looking back at (2.27), we see that

$$\begin{aligned}\mathcal{D}(u) &= -(K-1)\langle \widetilde{v}, \widetilde{v}\rangle + \frac{1}{2}\langle \hat{v}_0, \hat{v}_0\rangle + \langle \hat{v}_0, v_0 - \hat{v}_0\rangle + \frac{1}{2}\langle v_0 - \hat{v}_0, v_0 - \hat{v}_0\rangle \\ &= \mathcal{D}(\widetilde{v} + \hat{v}_0) + \langle \hat{v}_0, v_0 - \hat{v}_0\rangle + \frac{1}{2}\langle v_0 - \hat{v}_0, v_0 - \hat{v}_0\rangle , \\ &= \min_{\substack{u:(1,u)=0 \\ \text{given } \widetilde{c}}} \mathcal{D}(u) + \langle \hat{v}_0, v_0 - \hat{v}_0\rangle + \frac{1}{2}\langle v_0 - \hat{v}_0, v_0 - \hat{v}_0\rangle ,\end{aligned} \tag{2.31}$$

which implies in particular

$$\langle \hat{v}_0, v_0 - \hat{v}_0\rangle + \frac{1}{2}\langle v_0 - \hat{v}_0, v_0 - \hat{v}_0\rangle \geq 0 , \tag{2.32}$$

Since a lengthy computation yields

$$\frac{\mathcal{D}(\widetilde{v} + \hat{v}_0)}{\langle \widetilde{v}, \widetilde{v}\rangle + \langle \hat{v}_0, \hat{v}_0\rangle} = 1 - K(1-r^2) =: c_K \in (0, 1/2), \tag{2.33}$$

where $c_K \in (0,1/2)$ is just a restatement of (2.4), we get

$$\begin{aligned}\mathcal{D}(u) &= c_K\left(\langle \widetilde{v}, \widetilde{v}\rangle + \langle \hat{v}_0, \hat{v}_0\rangle\right) + \langle \hat{v}_0, v_0 - \hat{v}_0\rangle + \frac{1}{2}\langle v_0 - \hat{v}_0, v_0 - \hat{v}_0\rangle \\ &\geq c_K\left(\langle \widetilde{v}, \widetilde{v}\rangle + \langle \hat{v}_0, \hat{v}_0\rangle\right) + 2c_K\langle \hat{v}_0, v_0 - \hat{v}_0\rangle + c_K\langle v_0 - \hat{v}_0, v_0 - \hat{v}_0\rangle \\ &= c_K\left(\langle \widetilde{v}, \widetilde{v}\rangle + \langle v_0, v_0\rangle\right) ,\end{aligned} \tag{2.34}$$

and the proof is complete. □

For the next result we point out that, by the definition of $H^{-1}_{1/q}$ in terms of rigged Hilbert spaces, we know that there exists $c > 0$ such that $\langle\langle u, u\rangle\rangle \leq c(u,u)$ for every $u \in L^2$ with $(1,u) = 0$. Therefore

$$\langle\langle u, u\rangle\rangle \leq c_P^2 \langle u, u\rangle , \tag{2.35}$$

with $c_P^2 = c \max q$. This can be proven directly and c_P can be made explicit, in fact $\int \mathcal{U}/q = 0$ tells us that $\mathcal{U}(\theta_0) = 0$ for some θ_0, so that $\langle\langle u, u\rangle\rangle \leq (\min q)^{-1}\int_{\theta_0}^{\theta_0+2\pi}\mathcal{U}^2$ (we are looking at $\mathcal{U}$ as a periodic function with domain $\mathbb{R}$) and the Poincaré inequality tells us that $\int_{\theta_0}^{\theta_0+2\pi}\mathcal{U}^2$ is smaller than $\int_{\theta_0}^{\theta_0+2\pi}(\mathcal{U}')^2 = (u,u)$. Therefore we can choose

$$c_P^2 = \frac{\max q}{\min q} = \exp(4Kr). \tag{2.36}$$

.

Lemma 2.4. *For $u \in L^2(\mathbb{S})$ such that $(1,u) = 0$ we have*

$$\|u - u_{1/2}\|_{L^2_{1/q}} \geq C\left\| u - \frac{\langle\langle u, q'\rangle\rangle}{\langle\langle q', q'\rangle\rangle} q'\right\|_{H_{-1,1/q}} , \tag{2.37}$$

where $C > 0$ is given by

$$C^2 := \frac{\left(1 - (I_0(2Kr))^{-2}\right)}{2Kr^2c_P^4 + c_P^2\left(1 - (I_0(2Kr))^{-2}\right)}. \tag{2.38}$$

Of course Proposition 2.3 and Lemma 2.4 yields the spectral gap inequality: for every u such that $\langle\!\langle u, q' \rangle\!\rangle = 0$ we have

$$\mathcal{D}(u) \ge c_K C^2 \langle\!\langle u, u \rangle\!\rangle. \tag{2.39}$$

Proof. Set $e = q' / \langle q', q' \rangle^{1/2}$, so that $u_{1/2} = \langle u, e \rangle e$. By (2.35) we see that (2.37) follows if one can show

$$\begin{aligned} \|u - u_{1/2}\|_{L^2_{1/q}} &\ge c_P C \left\| u - \frac{\langle\!\langle u, e \rangle\!\rangle}{\langle\!\langle e, e \rangle\!\rangle} e \right\|_{L^2_{1/q}} \\ &= c_P C \sqrt{\langle u - u_{1/2}, u - u_{1/2} \rangle + \left(\langle u, e \rangle - \frac{\langle\!\langle u, e \rangle\!\rangle}{\langle\!\langle e, e \rangle\!\rangle} \right)^2}, \end{aligned} \tag{2.40}$$

and this is equivalent (note that $c_P C \in (0,1)$) to

$$\left\langle u - u_{1/2}, u - u_{1/2} \right\rangle \ge C_0 \left(\langle u, e \rangle - \frac{\langle\!\langle u, e \rangle\!\rangle}{\langle\!\langle e, e \rangle\!\rangle} \right)^2 = C_0 \left(\frac{\langle\!\langle u - u_{1/2}, e \rangle\!\rangle}{\langle\!\langle e, e \rangle\!\rangle} \right)^2, \tag{2.41}$$

with $C_0 := (c_P C)^2/(1 - (c_P C)^2)$. By the Cauchy-Schwarz inequality and by (2.35) we have

$$\langle\!\langle u - u_{1/2}, e \rangle\!\rangle^2 \le \langle\!\langle u - u_{1/2}, u - u_{1/2} \rangle\!\rangle \langle\!\langle e, e \rangle\!\rangle \le c_P^2 \left\langle u - u_{1/2}, u - u_{1/2} \right\rangle \langle\!\langle e, e \rangle\!\rangle, \tag{2.42}$$

so that (2.41) holds if $C_0 c_P^2 / \langle\!\langle e, e \rangle\!\rangle \le 1$, which is equivalent to $c_P^2 C^2 \le \langle\!\langle e, e \rangle\!\rangle / (c_P^2 + \langle\!\langle e, e \rangle\!\rangle)$. This is a condition on C that we can verify explicitly by using $\langle\!\langle e, e \rangle\!\rangle = (1 - I_0^{-2}(2Kr))/(2Kr^2)$. □

2.4. **Self-adjointness and spectral properties of L_q.** Let us recall that H is the space L^2 with zero average constraint and that L_q is viewed as an operator on $H_{-1,1/q} \supset H$. The first result is a technical lemma:

Lemma 2.5. *Fix $K > 1$. There exists $c > 0$ such that for every $u \in H$, we have*

$$\langle\!\langle u, u \rangle\!\rangle \ge c \, \|u_{1/2}\|_2^2, \tag{2.43}$$

where $u_{1/2}$ is the orthogonal projection, in $L^2_{1/q}$, on $V_{1/2}$.

Proof. Using the explicit expression for $u_{1/2}$ (Proposition 2.3), and

$$\left\langle u, q' \right\rangle = \int (\log q)' u = - \int (\log q)'' \mathcal{U}, \tag{2.44}$$

we see that

$$\begin{aligned} &\|u_{1/2}\|_2^2 \le (\max q) \left\langle u_{1/2}, u_{1/2} \right\rangle = \\ &\qquad \frac{\max q}{\langle q', q' \rangle} \left(\int (\log q)'' \mathcal{U} \right)^2 \le \frac{2\pi \max q^2 \max |(\log q)''|^2}{\langle q', q' \rangle} \int \mathcal{U}^2 / q =: \frac{1}{c} \langle\!\langle u, u \rangle\!\rangle, \end{aligned} \tag{2.45}$$

where the last step is the definition of c. □

Proposition 2.6. *L_q is essentially self-adjoint.*

Proof. We start by introducing for $u, v \in D(L_q)$

$$\mathcal{E}_1(v,u) := \langle\langle v, (1-L_q)u \rangle\rangle = -\int v(\theta)\left(\int_0^\theta \frac{\mathcal{U}}{q}\right) \mathrm{d}\theta + \frac{1}{2}\int \frac{vu}{q} - \int v\widetilde{J} * u. \tag{2.46}$$

The right-most expression shows that $\mathcal{E}_1(u,v)$ is well defined as long as $u, v \in H$ (this generalizes the definition of $\mathcal{E}_1(\cdot,\cdot)$ and we will take this definition from now on). Moreover $\mathcal{E}_1(\cdot,\cdot)$ is a continuous and coercive bilinear form on $H \times H$, that is there exists $c \in (0,1)$ such that

$$\mathcal{E}_1(u,v) \le \frac{1}{c}\|u\|_2\|v\|_2, \quad \text{and} \quad \mathcal{E}_1(u,u) \ge c\|u\|_2^2. \tag{2.47}$$

The second inequality follows from Proposition 2.3 ad Lemma 2.5. Now observe that for every $f \in H_{-1,1/q}$ the linear form $v \mapsto \langle\langle v, f\rangle\rangle$, from H to $\mathbb{R}$, is continuous (it is continuous also as a map from $H_{-1,1/q}$ to $\mathbb{R}$) and therefore, by the Lax-Milgram Theorem [4, Cor. V.8], we have that there exists a unique $u \in H$ such that

$$\mathcal{E}_1(v,u) = \langle\langle v, f\rangle\rangle, \quad \text{for every } v \in H. \tag{2.48}$$

Since we can write

$$\langle\langle v, f\rangle\rangle = -\int v(\theta)\left(\int_0^\theta \frac{\mathcal{F}}{q}\right) \mathrm{d}\theta, \tag{2.49}$$

from (2.46), (2.49) and (2.48) we see that

$$-\int_0^\theta \frac{\mathcal{U}}{q} + \frac{u(\theta)}{2q(\theta)} - \left(\widetilde{J} * u\right)(\theta) = -\int_0^\theta \frac{\mathcal{F}}{q}, \tag{2.50}$$

for (Lebesgue) almost every θ. Since $u \in H$, the primitive of $\mathcal{U}/q$ is C^1 and its (weak) second derivative is square integrable. If $f \in H (\subset H_{-1,1/q})$ then the same is true for the right-hand side in (2.50). Since $\widetilde{J} * u$ is C^∞, we see that u is C^1 (more precisely, has a C^1 version). So the left-most term in (2.50) is at least C^3. If now we assume that f is C^0, we can therefore conclude that $u \in C^2$.

To sum up: if f is periodic, C^0, with $\int f = 0$, then $u \in D(L_q)$ and $(1-L_q)u = f$, which follows by taking applying $\partial_\theta(q(\theta)\partial_\theta\,\cdot\,)$ to both terms in (2.50) (and by using (2.10)). Since such functions f are dense in $H_{-1,1/q}$, we see that the range of $1-L_q$ is dense, so that its kernel is $\{0\}$, and this implies that L_q is essentially self-adjoint ([4, Prop. VII.6]). ☐

Proposition 2.7. *The spectrum of L_q is pure point.*

Proof. We are going to prove this by showing that the resolvent of L_q is compact, namely that $(\lambda - L_q)^{-1}$ is compact for λ in the resolvent set. It suffices to prove such a result for one value of λ [12, p. 187] and we choose $\lambda = 1$, which is in the resolvent set thanks to Proposition 2.3 and Proposition 2.6. So let us consider $f := (1-L_q)^{-1}u$, $u \in H_{-1,1/q}$, so that f is in the domain of $1 - L_q$ and we have

$$\langle\langle f, (1-L_q)f\rangle\rangle = \langle\langle f, u\rangle\rangle. \tag{2.51}$$

But, by (2.47), $\langle\langle f, (1-L_q)f\rangle\rangle$ is bounded below by $c\|f\|_2^2$, so that

$$c\,\|f\|_2 \le \frac{\langle\langle f,u\rangle\rangle}{\|f\|_2} \le \frac{1}{C}\frac{\langle\langle f,u\rangle\rangle}{\sqrt{\langle\langle f,f\rangle\rangle}} \le \frac{1}{C}\sqrt{\langle\langle u,u\rangle\rangle}, \tag{2.52}$$

where we have used the continuous injection of H into $V' = H_{-1,1/q}$ (C is the constant arising when comparing the norms of these two spaces). Therefore $(1-L_q)^{-1}$ maps sequences that are bounded in $H_{-1,1/q}$ to sequences that are bounded in H. We are therefore left with showing that the embedding of H into $H_{-1,1/q}$ is compact. This just follows by the Cauchy-Schwarz inequality: for every $v \in H$ we have $|\mathcal{V}(\theta) - \mathcal{V}(\theta')| \le \|v\|_2 \sqrt{|\theta - \theta'|}$ and , since we are on a bounded interval with periodic boundary conditions and $\int \mathcal{V}/q = 0$, this yields that $\{\mathcal{V} : v \in H \text{ and } \|v\|_2 \le const.\}$ is a compact subset of C^0 (Ascoli-Arzelá Theorem), and hence of $L^2_{1/q}$. That is, a bounded subset of H is a relatively compact subset of $H_{-1,1/q}$ and this completes the proof. □

3. The nonlinear evolution

We need the following result on the nonlinear evolution:

Proposition 3.1. *For every $\nu_0 \in \mathcal{M}_1(\mathbb{S})$ there is a unique element $\nu_\cdot$ of $C^0([0,\infty); \mathcal{M}_1(\mathbb{S}))$ such that (1.8) is satisfied for every $F \in C^2(\mathbb{S})$ and every $t > 0$. Moreover for $t > 0$ the measure ν_t is absolutely continuous with respect to the Lebesgue measure and, if we denote by $q_t(\cdot)$ its density, the function $(t, \theta) \mapsto q_t(\theta)$, with domain of definition $(0,\infty) \times \mathbb{S}$, is C^∞, strictly positive and it solves (1.9).*

Proof. The uniqueness result is proven for example in [15] (with $\mathbb{S}$ replaced by $\mathbb{R}$) and in [8] (for a general, bounded or unbounded, domain subset of $\mathbb{R}^d$). The case of periodic boundary conditions is not treated explicitly but the argument of proof goes through with minor modifications. Existence is established by the tightness result on the particle system, which follows by methods that are by now standard: it suffices in fact to show that, for every (smoooth) F, $\{\int_{\mathbb{S}} F \,\mathrm{d}\nu_{N,t}\}_N$ is tight (see *e.g.* [13, 15, 8]) and this follows immediately from the fact that the drift in (1.2) is bounded.

For the regularity and positivity aspects we use the fact that, if for $t > 0$ there is a solution to

$$\partial_t u(t,\theta) \;=\; \frac{1}{2}\frac{\partial^2 u(t,\theta)}{\partial\theta^2} + K\frac{\partial}{\partial\theta}\left[(H(t,\theta))\, u(t,\theta)\right], \tag{3.1}$$

with

$$H(t,\theta) \;:=\; \int_{\mathbb{S}} \sin(\theta-\theta')\nu_t(\,\mathrm{d}\theta') \;=\; \sin(\theta)\int_{\mathbb{S}}\cos(\theta')\nu_t(\,\mathrm{d}\theta') - \cos(\theta)\int_{\mathbb{S}}\sin(\theta')\nu_t(\,\mathrm{d}\theta'), \tag{3.2}$$

(note that $H(\cdot,\cdot)$ is continuous in time and C^∞ in space) such that for every $F \in C^0(\mathbb{S})$

$$\lim_{t\searrow 0}\int_{\mathbb{S}} F(\theta)u(t,\theta)\,\mathrm{d}\theta \;=\; \int F\,\mathrm{d}\nu_0, \tag{3.3}$$

then one directly verifies that $\widetilde{\nu}_\cdot \in C^0([0,T]; \mathcal{M}_1(\mathbb{S}))$, defined by $\widetilde{\nu}_t(\,\mathrm{d}\theta) = u(t,\theta)\,\mathrm{d}\theta$ for $t > 0$ and $\widetilde{\nu}_0 = \nu_0$, is a solution to (1.8). Hence $\widetilde{\nu}_\cdot = \nu_\cdot$ by uniqueness.

Equation (3.1) is a parabolic linear partial differential equation on which there is much literature. For our purpose the results by D. G. Aronson in [2] turn out to be particularly relevant. In particular, Aronson shows that the equation (3.1) (consider $\theta \in \mathbb{R}$ for the moment) admits a fundamental solution $\Gamma(t,\theta;s,\theta')$ $(t > s)$ so that the solution at time t can be written as $\int_{\mathbb{R}} \Gamma(t,\theta;0,\theta')u(0,\theta')\,\mathrm{d}\theta'$, at least when $u(0,\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R})$ (of course the solution has to be interpreted in a weak sense: see [2, pp. 608-609] for the very general set up in which such a result it is proven). A number of results on the fundamental solution are proven [2, Section 7] and notably that it is a continuous function in (t,θ) for $t > s$ and that, if we assume that $\sup_{t\in[s,s+T],\theta\in\mathbb{R}} H(t,\theta) =: M < \infty$, there exists $C = C(T,M) > 0$ such that

$$\frac{1}{C\sqrt{t-s}}\exp\left(-\frac{C(\theta-\theta')^2}{\sqrt{t-s}}\right) \;\le\; \Gamma(t,\theta;s,\theta') \;\le\; \frac{C}{\sqrt{t-s}}\exp\left(-\frac{(\theta-\theta')^2}{C\sqrt{t-s}}\right), \tag{3.4}$$

for every θ, θ' and every $t \in (s, s+T]$. Moreover ([2, Corollary 12.1]) the result applies also to initial data that are measures: namely, if μ is a measure on $\mathbb{R}$ such that for $t > 0$ the map $\theta \mapsto \int_{\mathbb{R}} \Gamma(t, \theta; 0, \theta')\mu(\mathrm{d}\theta)$ is a function in $L^2_{\mathrm{loc}}(\mathbb{R})$ then such a map defines a weak solution to (3.1) on $(0, T] \times \mathbb{R}$ (namely a weak solution to (3.1) on $[t_0, T] \times \mathbb{R}$ for every $t_0 \in (0, T)$).

Let us now specialize to our case: $H(t, \cdot)$ is smooth and 2π-periodic, so that $\Gamma(t, \theta; s, \theta') = \Gamma(t, \theta + 2\pi; s, \theta' + 2\pi)$. Let us apply the result we have just stated with μ defined by requiring that its restriction to $[2\pi j, 2\pi(j+1))$ coincides with the image of the measure ν_0 under the application $[0, 2\pi) \ni \theta \mapsto \theta + 2\pi j \in [2\pi j, 2\pi(j+1))$, for every j. Therefore for $t > 0$

$$\int_{\mathbb{R}} \Gamma(t, \theta; 0, x)\mu(\mathrm{d}x) \,=\, \sum_{j \in \mathbb{Z}} \int_{[0,2\pi)} \Gamma(t, \theta; 0, \theta' + 2\pi j)\nu(\mathrm{d}\theta') \,=:\, v(t, \theta), \tag{3.5}$$

and, by (3.4), $v(t, \cdot)$ is bounded as soon as $t > 0$. Therefore $v(\cdot, \cdot)$ is a weak solution and, in turns, $v(t, \theta)\,\mathrm{d}\theta$ coincides with $\nu_t(\mathrm{d}\theta)$, which therefore has a representation in terms of the fundamental solution Γ and this implies not only that the solution becomes a bounded function as soon as $t > 0$, but also (by the lower bound in (3.4)) that it is strictly positive.

At this point, since we know that the solution is bounded, the smoothness in both variables of the solution (for $t > 0$) may be derived by standard methods: this issue is taken up for example in [3] for a slightly different evolution equation, or, more generally, in [17]. □

4. On the irreversibility of the Kuramoto model

In this section we consider the Kuramoto h-model, *i.e.* (1.1) with *general* drift, as in Remark 1.1, and $\sigma = 1$.

Existence of a unique invariant probability measure (for each fixed N) is a well known fact, but one can actually prove that such an invariant measure has a positive C^∞ density $\rho : \mathbb{S}^N \to (0, \infty)$. These issues are treated in detail for example in [14, Ch.s 3 and 5], where one finds also an extensive treatment of the entropy productions for Markov processes (with references to the vast literature on the subject). The entropy production rate e_p for a stationary process X is defined as the limit as $T \to \infty$, when it exists, of the relative entropy of the law of $\{X_t\}_{t \in [0,T]}$ with respect to the law of $\{X_{T-t}\}_{t \in [0,T]}$, divided by T. For a large class of models e_p takes the form of the steady state average of time integral of the square of a suitable *flux*. This is true also in our case, namely $e_p = \frac{1}{2} \int_{\mathbb{S}^N} \sum_{j=1}^N J_j\left(\underline{\varphi}\right)^2 \rho(\underline{\varphi})\,\mathrm{d}\underline{\varphi}$, where

$$J_j\left(\underline{\varphi}\right) \,:=\, 2\xi_j - \frac{2}{N}\sum_{i=1}^N h(\varphi_j - \varphi_i) - \frac{\partial}{\partial \varphi_j} \log \rho\left(\underline{\varphi}\right). \tag{4.1}$$

The key point is that $e_p = 0$ if and only if the system is reversible [14, Th. 5.4.6] (of course reversibility calls for specifying an invariant probability with respect to which the system is reversible, but in our set-up there is only one invariant measure). Therefore our system is reversible if and only if $J_j(\cdot) \equiv 0$ for every j: let us spell it out

$$\frac{\partial}{\partial \varphi_j} \log \rho\left(\underline{\varphi}\right) \,=\, 2\xi_j - \frac{2}{N}\sum_{i=1}^N h(\varphi_j - \varphi_i) \quad \text{for every } j \text{ and } \underline{\varphi}. \tag{4.2}$$

This expression directly implies that $\int_{\mathbb{S}} h(\theta)\,\mathrm{d}\theta = 2\pi\xi_j$ for every j, that is ξ_j does not depend on j. Without loss of generality we may therefore assume $\xi_j = 0$ for every j (recall

Remark 1.2), which entails $\int_{\mathbb{S}} h(\theta)\, \mathrm{d}\theta = 0$ and therefore the primitive $\widetilde{h}$ of h is 2π-periodic (we make the arbitrary choice $\widetilde{h}(0) = 0$). By integrating (4.2) we obtain that

$$\log \rho(\varphi) \;=\; \frac{2}{N} \sum_i \widetilde{h}(\varphi_j - \varphi_i) + c_j\left(\underline{\varphi}\right), \tag{4.3}$$

where $c_j\left(\underline{\varphi}\right)$ does not depend on φ_j. For $j = 1$ and φ_i fixed for $i = 3, 4, \ldots$ we can rewrite (4.3) as

$$\log \rho(\varphi_1, \varphi_2) \;=\; \frac{2}{N}\widetilde{h}(\varphi_1 - \varphi_2) + g_1(\varphi_1) + g_2(\varphi_2), \tag{4.4}$$

where $g_1(\varphi_1) := (2/N) \sum_{i \geq 3} \widetilde{h}(\varphi_j - \varphi_i)$ and $g_2(\varphi_2) := c_1\left(\underline{\varphi}\right)$. We can of course repeat the same steps with $j = 2$ obtaining thus

$$\log \rho(\varphi_1, \varphi_2) \;=\; \frac{2}{N}\widetilde{h}(\varphi_2 - \varphi_1) + f_1(\varphi_1) + f_2(\varphi_2), \tag{4.5}$$

with f_1 and f_2 defined in analogy with g_1, g_2 (but we are simply interested in the fact that they are smooth functions from $\mathbb{S}$ to $\mathbb{R}$). From (4.4) and (4.5) we infer that

$$\widetilde{h}(\varphi_1 - \varphi_2) - \widetilde{h}(\varphi_2 - \varphi_1) \;=\; f(\varphi_1) + g(\varphi_2), \tag{4.6}$$

for suitable smooth functions f and g from $\mathbb{S}$ to $\mathbb{R}$. This tells us in particular that $f(c + \theta) + g(c)$ does not depend on c (it is equal to $\widetilde{h}(\theta) - \widetilde{h}(-\theta)$) and, therefore, that $f(x) + g(x)$ is a constant and $f(c + x) - f(x) = f(c) - f(0)$ for every c and x (that is f is constant, since it is continuous and periodic). We have therefore reached the conclusion that $\theta \mapsto \widetilde{h}(\theta) - \widetilde{h}(-\theta)$ is a constant, which has therefore to be zero.

We sum up the argument we have just developed in the following statement:

Proposition 4.1. *For every N the dynamics defined by* (1.1)*, generalized as in Remark 1.1, is reversible if and only if the following two conditions are satisfied:*

(1) $\xi_1 = \xi_j$ *for every* j*;*
(2) $h(\cdot) - \xi_1 : \mathbb{R} \mapsto \mathbb{R}$ *is an odd function.*

Acknowledgments

We are very grateful to Francesco Caravenna for suggesting Lemma 2.4. G.G. thanks also Marek Biskup, François Delarue and Lorenzo Zambotti for fruitful suggestions and discussions. This work has been supported by the ANR grant MANDy.

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