

# A generalized Grothendieck inequality and entanglement in XOR games

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## Abstract

Suppose Alice and Bob make local two-outcome measurements on a shared entangled state. For any  $d$ , we show that there are correlations that can only be reproduced if the local dimension is at least  $d$ . This resolves a conjecture of Brunner et al. [*Phys. Rev. Lett.* 100, 210503 (2008)] and establishes that the amount of entanglement required to maximally violate a Bell inequality must depend on the number of measurement settings, not just the number of measurement outcomes. We prove this result by establishing the first lower bounds on a new generalization of Grothendieck's constant.

## 1 Introduction

Grothendieck's inequality first arose in the study of norms on tensor products of Banach spaces [11]. It has since found many applications in mathematics and computer science, including approximation algorithms [3, 6] and communication complexity [15, 14]. In quantum information, it quantifies the difference between the classical and quantum values of certain simple Bell inequalities, as established by Tsirelson [22]. Tsirelson's work has been the starting point for considerable recent research into quantum nonlocality [5, 1, 7, 19].

We start by stating the inequality in its strongest form, in terms of *the real Grothendieck constant*  $K_G$ .

**Definition 1.** The *real Grothendieck constant of order  $n$* , is the smallest real number  $K_G(n)$  such that: For all positive integers  $r$  and for all real  $r \times r$  matrices  $M = (M_{ij})$ , the inequality

$$\max_{\substack{a_1, \dots, a_r \\ b_1, \dots, b_r}} \sum_{i,j} M_{ij} a_i \cdot b_j \leq K_G(n) \max_{\substack{\alpha_1, \dots, \alpha_r \\ \beta_1, \dots, \beta_r}} \sum_{i,j} M_{ij} \alpha_i \beta_j \quad (1)$$

holds, where the maximum on the left-hand side is taken over all sequences  $a_1, \dots, a_r, b_1, \dots, b_r$  of  $n$ -dimensional real unit vectors,  $a_i \cdot b_j$  denotes the Euclidean inner product of  $a_i$  and  $b_j$ , and the maximum on the right-hand side is taken over all for all sequences  $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r$  of real numbers in the set  $\{-1, +1\}$ .

The *real Grothendieck constant*, denoted  $K_G$ , is defined as  $\lim_{n \rightarrow \infty} K_G(n)$ .

The tightest version of the inequality known is due to Krivine [13], who proved that  $K_G \leq \pi / (2 \ln(1 + \sqrt{2})) \approx 1.78$ . Davie [8] and, independently, Reeds [18] are responsible for the best lower bounds: they showed that  $K_G \gtrsim 1.68$ . The exact value of  $K_G$  is unknown.

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In this paper, we give a new generalization of Grothendieck’s inequality. We replace the maximization over scalars on the right-hand side of Eq. (1) with a maximization over real unit vectors of dimension  $m < n$ . More formally:

**Definition 2.** Let  $m$  and  $n$  be positive integers with  $m < n$ . Let  $K_G(n \mapsto m)$  be the smallest real number such that: For all positive integers  $r$  and for all real  $r \times r$  matrices  $M = (M_{ij})$ , the inequality

$$\max_{\substack{a_1, \dots, a_r \\ b_1, \dots, b_r}} \sum_{i,j} M_{ij} a_i \cdot b_j \leq K_G(n \mapsto m) \max_{\substack{a'_1, \dots, a'_r \\ b'_1, \dots, b'_r}} \sum_{i,j} M_{ij} a'_i \cdot b'_j \quad (2)$$

holds, where the maximum on the left-hand side is taken over all sequences  $a_1, \dots, a_r, b_1, \dots, b_r$  of  $n$ -dimensional real unit vectors, and the maximum on the right-hand side is taken over all sequences  $a'_1, \dots, a'_r, b'_1, \dots, b'_r$  of  $m$ -dimensional real unit vectors. This generalizes Definition 1 in the sense that  $K_G(n) = K_G(n \mapsto 1)$ .

Building on the techniques Grothendieck used to prove the original lower bound on  $K_G$  [11], we prove the following lower bound on  $K_G(n \mapsto m)$ .

**Theorem 3.** For all  $m < n$ ,

$$K_G(n \mapsto m) \geq \frac{m}{n} \left( \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m+1}{2})} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \right)^2 \quad (3)$$

$$= 1 + \frac{1}{2m} - \frac{1}{2n} - O\left(\frac{1}{m^2}\right). \quad (4)$$

We do not need an upper bound on  $K_G(n \mapsto m)$  for our quantum application, so we don’t prove one. Note however that  $K_G(n \mapsto m) \leq K_G(n)$ , which does establish a trivial upper bound. A better upper bound could be obtained by using the techniques in [13].

**Application to nonlocal XOR games.** As a corollary of Theorem 3, we show that there are nonlocal quantum correlations that require entangled states with local support on a Hilbert space of dimension at least  $d$  (we allow arbitrary shared randomness). This resolves a question of Brunner et al. [5], proving that what they term *dimension witnesses* exist with binary outcomes.

Brunner et al. pointed out that the same result would follow if one could prove that the Grothendieck constants  $K_G(n)$  are strictly increasing in  $n$ . This is plausible but we do not know how to prove it. Our new proof sidesteps this issue.

**Related work.** Definition 2 is but the latest in a long history of generalizations of Grothendieck’s inequality. Previously, Grothendieck’s inequality has been generalized as follows:

- Replacing the real scalars, vectors and matrices with complex ones results in our defining the *complex Grothendieck constant*.
- Restricting to matrices  $M$  with positive entries results in a tighter inequality [20].
- Rather than proving inequalities that hold for all matrices, we can prove inequalities that only hold for all matrices  $M$  of some fixed size, say  $r \times s$ . This refinement has been studied by Fishburn and Reeds [9], and results in the definition of a constant which they denote  $K_G(r, s)$ , not to be confused with our  $K_G(n \mapsto m)$ .
- Observe that Eq. (1) has a bipartite structure, in the following sense: on the left-hand side, the sum is of inner products  $a_i \cdot b_j$  of a vector from  $a_1, \dots, a_r$  with a vector from  $b_1, \dots, b_r$ ; there are no inner products  $a_i \cdot a_j$  or  $b_i \cdot b_j$ . A similar observation applies to the right-hand side. So if we consider a graph with vertices labelled by the vectors  $a_i$  and  $b_j$ , and draw an edge between vertex  $a_i$  and  $b_j$  whenever  $M_{ij} \neq 0$ , then the resulting “interaction graph” is bipartite. Alon et al. have generalized Grothendieck’s inequality to general graphs that are not necessarily bipartite [2].

There is some earlier work on lower bounding the amount of entanglement required to reproduce certain correlations. Wehner, Christandl and Doherty show how to obtain lower bounds using information-theoretic arguments [23].

The Hidden Matching quantum communication complexity problem (HM( $n$ )) [4] can be formulated as a nonlocal correlation, where a maximally entangled state of dimension  $n$  is used to reproduce the correlations perfectly. On the other hand, using the classical bounded error one-way communication complexity lower bound for HM( $n$ ), it follows that one needs  $\omega(\sqrt{n})$  bits of one-way communication to approximately reproduce these correlations classically. This in turn yields a lower bound on the dimension of the entangled state of  $\sqrt{n}/\log n$  for any quantum strategy that approximates these correlations. This follows because any smaller dimensional state can be used to establish a classical one-way protocol that approximates these correlations and uses less than  $\omega(\sqrt{n})$  bits of communication, by simply communicating a classical description of an approximation of the state that Bob has after Alice did her measurement.

**Outline.** The paper is structured as follows. We define notation in Section 2. In Section 3, we rework the definition of  $K_G(n \mapsto m)$  in order to work in the limit  $r \rightarrow \infty$ , which makes things simpler. Then, in Section 4, we prove our main result, Theorem 3. In Section 5, we describe the consequences for quantum nonlocality. Readers wishing to skip the details of the proof can read Section 5 immediately after Section 3.

## 2 Notation

We write  $[n]$  for the set  $\{1, \dots, n\}$ . The unit sphere in  $\mathbb{R}^n$  is denoted  $S_{n-1}$ . We write  $da$  for the Haar measure on  $S_{n-1}$ , normalized such that  $\int da = 1$ . The Dirac delta function on  $S_{n-1}$  is defined by  $\delta(a - b) = 0$  if  $a \neq b$  and  $\int da \delta(a - b) = 1$ . The norm  $\|a\|$  of a vector  $a$  is always the Euclidean norm. In the Introduction and Appendix, subscripts label vectors; in the remainder of the paper, subscripts on a vector denote its components. Variables in lowercase roman type will typically be vectors on the unit sphere; variables in lowercase greek type will typically be scalars.

## 3 An equivalent definition of $K_G(n \mapsto m)$

To establish a lower bound on  $K_G(n \mapsto m)$  per Eq. (2), we need to exhibit an  $r \times r$  matrix  $M$  and then calculate (or at least bound) both sides of Eq. (2). We will work in the limit  $r \rightarrow \infty$  and so we start by giving an alternative definition of  $K_G(n \mapsto m)$  that facilitates this.

**Lemma 4.** *The constant  $K_G(n \mapsto m)$  is given by*

$$K_G(n \mapsto m) = \sup_{M: S_{n-1} \times S_{n-1} \rightarrow [-1, 1]} \left( \frac{1}{D(M)} \int dadb M(a, b) a \cdot b \right). \quad (5)$$

where the supremum is over measurable functions  $M : S_{n-1} \times S_{n-1} \rightarrow [-1, 1]$  and the denominator

$$D(M) = \max_{A, B: S_{n-1} \rightarrow S_{m-1}} \int dadb M(a, b) A(a) \cdot B(b), \quad (6)$$

with the maximum over functions  $A, B : S_{n-1} \rightarrow S_{m-1}$ .

We informally describe why Lemma 4 is true. Fix an  $r \times r$  matrix  $M$ , and let  $a_1^*, \dots, a_r^*, b_1^*, \dots, b_r^*$  be the  $n$ -dimensional unit vectors that maximize

$$\max_{\substack{a_1, \dots, a_r \\ b_1, \dots, b_r}} \sum_{i, j} M_{ij} a_i \cdot b_j, \quad (7)$$

the left-hand side of Eq. (2). Here the vectors  $a_i^*$  and  $b_j^*$  are labelled by indices  $i$  and  $j$ , but these are just dummy indices and we could have written the sum with whatever indices we liked. The idea behind Lemma 4 is to *use the vectors themselves as labels*, which works as long as the vectors are all distinct. Thus we replace the matrix  $M_{ij}$  in Eq. 7 with an infinite matrix  $M(a, b)$  with rows labelled by unit vectors  $a$  and columns labelled by unit vectors  $b$ . The sum over  $i, j$  is replaced by integrals over  $a$  and  $b$ . The matrix element  $M(a_i^*, b_j^*)$  is  $M_{ij}$ ; all other entries of the matrix are zero.

It remains to understand what happens if two or more vectors are the same, say  $a_1^* = a_2^*$ . In this case, we can replace the  $r \times r$  matrix  $M$  with an  $(r - 1) \times r$  matrix  $M'$  obtained from  $M$  by replacing the first two rows with their sum. We claim that the bound on  $K_G(n \mapsto m)$  established by  $M'$  is at least as good as the bound established by  $M$ . To see this, observe that replacing  $M$  with  $M'$  doesn't change the value of the left-hand side of Eq. 2. Replacing  $M$  with  $M'$  on the right-hand side of Eq. 2 is equivalent to performing the maximization

$$\max_{\substack{a'_1, \dots, a'_r \\ b'_1, \dots, b'_r}} \sum_{i,j} M_{ij} a'_i \cdot b'_j, \quad (8)$$

with the *additional* constraint that  $a'_1 = a'_2$ , which cannot increase the maximum. Thus the bound on  $K_G(n \mapsto m)$  obtained using  $M'$  is at least as good as that obtained using  $M$ . Thus it is okay to assume that all the vectors are distinct.

We give a formal proof of Lemma 4 in Appendix A.

## 4 Lower bound on $K_G(n \mapsto m)$

We prove Theorem 3 by considering a specific example due to Grothendieck himself [11]: for  $a, b \in S_{n-1}$ , take  $M(a, b) = a \cdot b$ .

We start by calculating the denominator  $D(M)$ . To do this, we need to work out which embeddings  $A, B : S_{n-1} \rightarrow S_{m-1}$  achieves the maximum in Eq. 6. It turns out that this is achieved when  $A$  and  $B$  are equal. Informally, we should try to preserve as much of the structure of  $S_{n-1}$  as possible, and it is natural to conjecture that the best embedding is a projection onto an  $m$ -dimensional subspace. This is indeed the case. We prove this in the following Lemma.

**Lemma 5.** *For the function  $M(a, b) = a \cdot b$ , the optimal embedding  $A : S_{n-1} \rightarrow S_{m-1}$  is a projection. In particular, the denominator  $D(M)$  is given by*

$$D = \frac{1}{m} \left( \int da \left( \sum_{i=1}^m a_i^2 \right)^{1/2} \right)^2, \quad (9)$$

where  $a_1, \dots, a_n$  are the components of  $a$ .

**Proof:** We prove this result in two steps. First, we show that the maximum is achieved by a *weighted* projection. Second, we show that the best projection is one with uniform weights.

We need to calculate

$$D(M) = \max_{A, B : S_{n-1} \rightarrow S_{m-1}} \int dadb M(a, b) A(a) \cdot B(b), \quad (10)$$

with the maximum over functions  $A, B : S_{n-1} \rightarrow S_{m-1}$ . For  $M(a, b) = a \cdot b$ , we can write

$$(a \cdot b)(A(a) \cdot B(b)) = (a \otimes A(a)) \cdot (b \otimes B(b)), \quad (11)$$

(this trick is motivated by a similar one used by Krivine in proving his upper bound on  $K_G$  [13]), which allows us to write  $D(M)$  as a maximization over the inner product of two vectors,

$$D = \max_{A, B: S_{n-1} \rightarrow S_{m-1}} \left( \int da a \otimes A(a) \right) \cdot \left( \int db b \otimes B(b) \right) \quad (12)$$

$$= \max_{A: S_{n-1} \rightarrow S_{m-1}} \left\| \int da a \otimes A(a) \right\|^2, \quad (13)$$

where the second equality follows from the fact that the inner product is maximized when vectors are parallel. Let  $\int da a \otimes A(a) = \chi v$ , where  $v$  is an  $(n+m)$ -dimensional unit vector and  $\chi \geq 0$  is what we want to maximize. Applying the singular value decomposition—known in quantum information theory as the Schmidt decomposition (see for example [17])—we can write

$$v = \sum_{i=1}^m \sqrt{\gamma_i} x_i \otimes y_i, \quad (14)$$

where, for each  $i \in [m]$ ,  $\gamma_i \geq 0$ ,  $\sum_i \gamma_i = 1$ , and  $\{x_1, \dots, x_m\}$  and  $\{y_1, \dots, y_m\}$  are orthonormal sets in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively. Therefore, in order to maximize

$$\chi = v \cdot \int da a \otimes A(a) = \int da \sum_i \sqrt{\gamma_i} (a \cdot x_i) (A(a) \cdot y_i) = \int da A(a) \cdot \left( \sum_i \sqrt{\gamma_i} (a \cdot x_i) y_i \right), \quad (15)$$

we should choose  $A(a)$  to be

$$\frac{\sum_i \sqrt{\gamma_i} (a \cdot x_i) y_i}{\left\| \sum_i \sqrt{\gamma_i} (a \cdot x_i) y_i \right\|} = \frac{\sum_i \sqrt{\gamma_i} (a \cdot x_i) y_i}{(\sum_i \gamma_i (a \cdot x_i)^2)^{1/2}}, \quad (16)$$

a weighted projection onto some  $m$ -dimensional subspace, the particular choice of which does not matter. Substituting this into Eq. (15) and then Eq. (13) and choosing a basis for  $\mathbb{R}^n$  by extending  $x_1, \dots, x_m$  so that  $a_i = a \cdot x_i$  establishes that

$$D = \max_{A, B: S_{n-1} \rightarrow S_{m-1}} \int da db (a \cdot b) (A(a) \cdot B(b)) = (\chi(\gamma_1, \dots, \gamma_m))^2, \quad (17)$$

where

$$\chi(\gamma_1, \dots, \gamma_m) = \int da \left( \sum_{i=1}^m \gamma_i a_i^2 \right)^{1/2}. \quad (18)$$

It remains to show that weights  $\gamma_i$  can be taken to be equal. To prove this, suppose that  $\chi$  is maximized by  $(\gamma_1^*, \gamma_2^*, \dots, \gamma_m^*)$ . Then, by symmetry, the maximum is also achieved by  $(\gamma_2^*, \gamma_1^*, \dots, \gamma_m^*)$ , and indeed, by any other permutation  $\sigma$  of the  $\gamma_i^*$ . Hence

$$\chi(\gamma_1^*, \dots, \gamma_m^*) = \frac{1}{m!} \sum_{\sigma} \chi(\gamma_{\sigma(1)}^*, \dots, \gamma_{\sigma(m)}^*) \quad (19)$$

$$= \frac{1}{m!} \sum_{\sigma} \int da \left( \sum_{i=1}^m \gamma_{\sigma(i)}^* a_i^2 \right)^{1/2} \quad (20)$$

$$= \int da \frac{1}{m!} \sum_{\sigma} \left( \sum_{i=1}^m \gamma_{\sigma(i)}^* a_i^2 \right)^{1/2} \quad (21)$$

$$\leq \int da \left( \frac{1}{m!} \sum_{\sigma} \sum_{i=1}^m \gamma_{\sigma(i)}^* a_i^2 \right)^{1/2} \quad (22)$$

by Jensen's inequality and the concavity of  $(\cdot)^{1/2}$ . But the coefficient of  $a_i^2$  in this expression is just

$$\frac{1}{m!} \sum_{\sigma} \gamma_{\sigma(i)}^* = \frac{1}{m} \sum_i \gamma_i^* = \frac{1}{m} \times 1 = \frac{1}{m}. \quad (23)$$

Thus the maximum is achieved by uniform weights. ■

With Lemma 5 in hand, the proof of Theorem 3 is straightforward.

**Proof of Theorem 3:** Take  $M(a, b) = a \cdot b$  in Lemma 5. It follows from Lemma 5, that

$$K_G(n \mapsto m) \geq \frac{m}{n} \left( \frac{Y_n}{Y_m} \right)^2, \quad (24)$$

where

$$Y_k := \int_{a \in S_{n-1}} da \left( \sum_{i=1}^k a_i^2 \right)^{1/2},$$

and we evaluated the numerator in Eq. (17) by observing that it is the same as the denominator when  $m = n$ , and so we already calculated it as a special case of Lemma 5. We can evaluate  $Y_k$  using a trick similar to that used to calculate the surface area of the  $n$ -sphere. Define

$$C_k := \int_{a \in \mathbb{R}^n} da \left( \sum_{i=1}^k a_i^2 \right)^{1/2} e^{-\|a\|_2^2}.$$

Introducing spherical coordinates, and writing  $r = \|a\|_2$ , we have

$$C_k = Y_k \int_0^\infty dr r^{n-1} (r^2)^{1/2} e^{-r^2} = \frac{1}{2} Y_k \Gamma\left(\frac{n+1}{2}\right), \quad (25)$$

where  $\Gamma$  is the well-known gamma function.

On the other hand, we have

$$C_k = \int_{-\infty}^\infty da_1 \cdots da_k \left( \sum_{i=1}^k a_i^2 \right)^{1/2} e^{-(a_1^2 + \cdots + a_k^2)} \int_{-\infty}^\infty da_{k+1} \cdots da_n e^{-(a_{k+1}^2 + \cdots + a_n^2)}. \quad (26)$$

We can interpret  $\left( \sum_{i=1}^k a_i^2 \right)^{1/2}$  as the norm of a point in  $k$ -dimensional space, and write the first integral (over  $k$  variables) as

$$\Omega_k \int_0^\infty dr' r'^k e^{-r'^2} = \frac{2\pi^{k/2}}{\Gamma(\frac{k}{2})} \cdot \frac{\Gamma(\frac{k+1}{2})}{2},$$

where  $\Omega_k$  is the surface area of a unit sphere in  $k$  dimensions. The second integral of Eq. (26) is simply  $(\sqrt{\pi})^{n-k}$ . Comparing these two ways to evaluate  $C_k$ , we conclude that

$$Y_k = 2\pi^{n/2} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+1}{2})}$$

and

$$K_G(n \mapsto m) \geq \frac{m}{n} \left( \frac{Y_n}{Y_m} \right)^2 \quad (27)$$

$$= \frac{m}{n} \left( \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m+1}{2})} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \right)^2. \quad (28)$$

For all integers  $1 \leq m < n$ , this bound is nontrivial, i.e., is strictly greater than 1. This is because the function

$$f(n) = \frac{1}{\sqrt{n}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \quad (29)$$

is strictly increasing for  $n = 1, 2, \dots$  (see Appendix B for a proof). Asymptotically, we have

$$K_G(n \mapsto m) \geq 1 + \frac{1}{2m} - \frac{1}{2n} - O\left(\frac{1}{m^2}\right), \quad (30)$$

where the approximation follows from the asymptotic series (see answer to Exercise 9.60 in [10])

$$\frac{\Gamma(k + \frac{1}{2})}{\Gamma(k)} = \sqrt{k} \left(1 - \frac{1}{8k} + \frac{1}{128k^2} + \dots\right). \quad (31)$$

■

## 5 Quantum nonlocality

Here we describe the application to quantum nonlocality. Suppose that two parties, Alice and Bob, each have a  $d$ -dimensional quantum system, described by Hilbert spaces  $\mathcal{H}_A \cong \mathbb{C}^d$  and  $\mathcal{H}_B \cong \mathbb{C}^d$ , respectively. Alice and Bob each make a two-outcome measurement on their own system, resulting in outcomes  $\alpha, \beta \in \{\pm 1\}$ , respectively. Suppose the set of Alice's possible measurements is  $M_A$ , and the set of Bob's possible measurements is  $M_B$ . An observable is a Hermitian operator with eigenvalues in  $\{\pm 1\}$ . Alice's  $a$ th possible measurement is specified by an observable  $A_a$  on  $\mathcal{H}_A$ ; Bob's  $b$ th measurement by an observable  $B_b$  on  $\mathcal{H}_B$  (and all observables specify valid measurements). If the joint system of Alice and Bob is in pure state  $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ , then the *joint correlation*—the expectation of the product of Alice and Bob's outcomes, given that Alice performs measurement  $a$  and Bob measurement  $b$ —is

$$E[\alpha\beta|ab] = \langle \psi | A_a \otimes B_b | \psi \rangle. \quad (32)$$

In the computer science literature, such correlations are studied in the context of *XOR nonlocal games* [7].

We say that a set of joint correlations,  $\{E[\alpha\beta|ab] : a \in [a_{\max}], b \in [b_{\max}]\}$ , is *pure- $d$ -quantum-realizable* if there is a state  $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$  and for all  $a \in [a_{\max}]$ , there are observables  $A_a$  on  $\mathbb{C}^d$  and for all  $b \in [b_{\max}]$ , there are observables  $B_b$  on  $\mathbb{C}^d$  such that  $E[\alpha\beta|ab] = \langle \psi | A_a \otimes B_b | \psi \rangle$ . A set of joint correlations is  *$d$ -quantum-realizable* if it is a probabilistic mixture of pure- $d$ -quantum-realizable correlations (this definition accounts for allowing Alice and Bob to share an arbitrary large amounts of shared randomness, use POVMs, and share a mixed state). A set of joint correlations is *finitely quantum-realizable* if there is some  $d$  such that the correlations are  $d$ -quantum-realizable.

We prove the following theorem.

**Theorem 6.** *For any  $d$ , there are correlations that are finitely quantum-realizable, but which are not  $d$ -quantum-realizable.*

We now describe the correlations that we use to prove Theorem 6. Fix some integer  $n$ . Alice and Bob's possible measurements are parametrized by unit vectors in  $\mathbb{R}^n$ ,  $a$  and  $b$ , respectively. (Note that each party here has an infinite number of possible measurements; we'll reprove the theorem with finite sets of measurements in the next subsection.) The joint correlations are given by

$$E[\alpha\beta|ab] = a \cdot b, \quad (33)$$

where  $a \cdot b$  is just the Euclidian inner product of  $a$  and  $b$ . For all  $n$ , these correlations are finitely quantum-realizable, as the following result shows.

**Lemma 7** (Tsirelson [22]). *Let  $|\psi\rangle$  be a maximally entangled state on  $\mathbb{C}^d \otimes \mathbb{C}^d$  where  $d = 2^{\lfloor n/2 \rfloor}$ . Then there are two mappings from unit vectors in  $\mathbb{R}^n$  to observables on  $\mathbb{C}^d$ , one taking  $a$  to  $A_a$ , the other taking  $b$  to  $B_b$ , such that*

$$\langle \psi | A_a \otimes B_b | \psi \rangle = a \cdot b,$$

for all unit vectors  $a, b$ .

To show they are not  $d$ -quantum-realizable, we will use the following characterization.

**Lemma 8** ([22, 1]). *Suppose Alice and Bob measure observables  $A_a$  and  $B_b$  on a pure quantum state  $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ . Then we can associate a real unit vector  $A(a) \in \mathbb{R}^{2d^2}$  with  $A_a$  (independent of  $B_b$ ), and a real unit vector  $B(b) \in \mathbb{R}^{2d^2}$  with  $B_b$  (independent of  $A_a$ ) such that*

$$E[\alpha\beta|ab] = \langle \psi | A_a \otimes B_b | \psi \rangle = A(a) \cdot B(b). \quad (34)$$

**Proof of Theorem 6:** Let  $n = 2d^2 + 1$ , and consider the joint correlations described in Eq. 33. By Lemma 7, these correlations are finitely quantum-realizable. To show they are not  $d$ -quantum-realizable, we will show that they lie outside the convex hull of the set of pure- $d$ -quantum-realizable correlations.

We do this in the standard way, using a Bell inequality. Consider the following linear function on the correlations.

$$B(E[\alpha\beta|ab]) = \int dadb(a \cdot b)E[\alpha\beta|ab], \quad (35)$$

where the integral is over all unit vectors  $a, b$ . Substituting for  $E[\alpha\beta|ab]$  using Eq. 33, we have

$$B(E[\alpha\beta|ab]) = \int dadb(a \cdot b)^2. \quad (36)$$

For any pure- $d$ -quantum-realizable correlations, by Lemma 8 there are vectors  $A(a)$  and  $B(b)$  in  $\mathbb{R}^{2d^2}$ , such that the resulting correlations are given by

$$E[\alpha\beta|ab]_d = A(a) \cdot B(b). \quad (37)$$

Evaluating  $B$  on these correlations, we must have

$$B(E[\alpha\beta|ab]_d) = \int dadb(a \cdot b)A(a) \cdot B(b) \quad (38)$$

$$\leq \max_{A,B} \int dadb(a \cdot b)A(a) \cdot B(b) \quad (39)$$

$$\leq \frac{1}{K_G^{\leq}(n \mapsto 2d^2)} \int dadb(a \cdot b)^2 \quad (40)$$

$$= \frac{1}{K_G^{\leq}(n \mapsto 2d^2)} B(E[\alpha\beta|ab]). \quad (41)$$

where  $K_G^{\leq}(n \mapsto 2d^2)$  is our lower bound on  $K_G(n \mapsto 2d^2)$ . Since  $n = 2d^2 + 1$  and  $K_G^{\leq}(n \mapsto n - 1) > 1$  by Theorem 3, we conclude that the correlations in Eq. 33 are not  $d$ -quantum-realizable.  $\blacksquare$

## 5.1 Reducing the number of questions

A possible objection to the example above is that the number of questions is taken to be infinite. Here we reduce to a finite number of questions by considering a discretization of the unit  $n$ -sphere by means of an  $\varepsilon$ -net.

**Definition 9** ( $\varepsilon$ -net). For fixed  $\varepsilon > 0$ , a set of vectors  $E_n^\varepsilon = \{w_1, w_2, \dots \in S_{n-1}\}$  is an  $\varepsilon$ -net for  $S_{n-1}$  if for all  $a \in S_{n-1}$ , there exists a vector  $u \in E$  that satisfies  $\|a - u\|_2 \leq \varepsilon$ .

**Lemma 10.** For  $0 < \varepsilon < 1$ , there is an  $\varepsilon$ -net for  $S_{n-1}$  with  $|E_n^\varepsilon| = (3/\varepsilon)^n$ .

**Proof:** We follow [12, Lemma II.4]. Let  $E_n^\varepsilon$  be a maximal set of vectors satisfying  $\|u - v\|_2 \geq \varepsilon$  for all  $u, v \in E_n^\varepsilon$ , where the existence of such a set is guaranteed by Zorn's lemma. Then  $E_n^\varepsilon$  is an  $\varepsilon$ -net for  $S_{n-1}$ . We



bound  $|E_n^\varepsilon|$  using a volume argument. The open balls of radius  $\varepsilon/2$  around each point  $u \in E_n^\varepsilon$  are pairwise disjoint and all contained in the ball of radius  $1 + \varepsilon/2$  about the origin. Hence

$$|E_n^\varepsilon| \leq \frac{(1 + \varepsilon/2)^n}{(\varepsilon/2)^n} = \left(\frac{2}{\varepsilon} + 1\right)^n \leq \left(\frac{3}{\varepsilon}\right)^n. \quad (42)$$

■

To convert the quantum correlations above (Eq. 33) into ones with only a finite number of settings, fix  $0 < \varepsilon < 1$  (to be chosen later) and let  $E_n^\varepsilon$  be an  $\varepsilon$ -net for  $S_{n-1}$ . We shall consider the following correlations. Alice's set of possible measurements is  $E_n^\varepsilon$ , and so is Bob's (note that we implicitly apply Lemma 7 here). If Alice performs a measurement  $u \in E_n^\varepsilon$  and Bob a measurement  $v \in E_n^\varepsilon$ , the joint correlation should satisfy

$$E[\alpha\beta|uv] = u \cdot v, \quad (43)$$

just as in our earlier example. These correlations, being a subset of those considered above, are finitely quantum-realizable.

The  $\varepsilon$ -net divides the unit sphere into  $|E_n^\varepsilon|$  regions. (For  $u \in E_n^\varepsilon$ , let  $R_u$  be the set of points on  $S_{n-1}$  that are closer to  $u$  than to any other point in  $E_n^\varepsilon$ , and assign points equidistant to two or more points in the net in some arbitrary way.) Consider the Bell inequality

$$B_{\text{finite}}(E[\alpha\beta|uv]) = \sum_{u \in E_n^\varepsilon} \int_{a \in R_u} da \sum_{v \in E_n^\varepsilon} \int_{b \in R_v} db (a \cdot b) E[\alpha\beta|uv]. \quad (44)$$

Evaluating this on the correlations  $E[\alpha\beta|uv] = u \cdot v$ , we obtain

$$B_{\text{finite}}(E[\alpha\beta|uv]) = \sum_{u \in E_n^\varepsilon} \int_{a \in R_u} da \sum_{v \in E_n^\varepsilon} \int_{b \in R_v} db (a \cdot b)(u \cdot v) \quad (45)$$

$$\geq -2\varepsilon + \int da \int db (a \cdot b)^2 \quad (46)$$

$$= B(E[\alpha\beta|ab]) - 2\varepsilon, \quad (47)$$

where we used

$$u \cdot v = a \cdot b + (u - a) \cdot b + u \cdot (v - b) \quad (48)$$

$$\geq a \cdot b - \|u - a\|_2 - \|v - b\|_2 \quad (49)$$

$$\geq a \cdot b - 2\varepsilon, \quad (50)$$

and related the value of the Bell inequality to the one earlier in this section with an infinite number of questions.

Now consider a pure  $d$ -dimensional quantum strategy. Let  $A(u)$  be the  $2d^2$ -dimensional real unit vector associated with Alice's measurement  $u$  and  $B(v)$  be the vector associated with Bob's measurement  $v$  by Lemma 8. This mapping induces a mapping for the correlations where we had an infinite number of questions. First map  $a$  to the closest point  $u$  in the  $\varepsilon$ -net, then to the vector  $A(u)$ . We now evaluate the Bell inequality with an infinite number of settings in terms of the one with a finite number:

$$B(E[\alpha\beta|ab]_d) \geq \sum_{u \in E_n^\varepsilon} \int_{a \in R_u} da \sum_{v \in E_n^\varepsilon} \int_{b \in R_v} db (a \cdot b) A(u) \cdot B(v) \quad (51)$$

$$= \sum_{u \in E_n^\varepsilon} \int_{a \in R_u} da \sum_{v \in E_n^\varepsilon} \int_{b \in R_v} db (a \cdot b) E[\alpha\beta|uv] \quad (52)$$

$$= B_{\text{finite}}(E[\alpha\beta|uv]_d). \quad (53)$$

Combining the above calculations with Eq. 41, we get

$$B_{\text{finite}}(E[\alpha\beta|uv]_d) \leq B(E[\alpha\beta|ab]_d) \tag{54}$$

$$\leq \frac{1}{K_G^{\leq}(n \mapsto 2d^2)} B(E[\alpha\beta|ab]) \tag{55}$$

$$\leq \frac{1}{K_G^{\leq}(n \mapsto 2d^2)} [B_{\text{finite}}(E[\alpha\beta|ab]) + 2\varepsilon] \tag{56}$$

Now choose  $n > 2d^2$  and  $\varepsilon$  such that  $B_{\text{finite}}(E[\alpha\beta|uv]_d)$  is strictly less than  $B_{\text{finite}}(E[\alpha\beta|uv])$ .

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## A Convenient form of $K_G(n \mapsto m)$

Here we prove Lemma 4. Observe that we can rewrite the conventional definition of  $K_G(n \mapsto m)$  as

$$K_G(n \mapsto m) = \lim_{r \rightarrow \infty} \sup_{M_{ij}} \left( \frac{\max_{(a_i, b_j)} \sum_{i,j} M_{ij} a_i \cdot b_j}{\max_{(a'_i, b'_j)} \sum_{i,j} M_{ij} a'_i \cdot b'_j} \right).$$

The following two propositions give the result.

**Proposition 11.** *For all positive integers  $r$ , and for all  $r \times r$  real-valued matrices  $M_{ij}$ , there exists a measurable function  $M' : S_{n-1} \times S_{n-1} \rightarrow [-1, 1]$ , such that*

$$\frac{\int da db M'(a, b) a \cdot b}{\max_{A, B: S_{n-1} \rightarrow S_{m-1}} \int da db M'(a, b) A(a) \cdot B(b)} \geq \frac{\max_{(a_i, b_j)} \sum_{i,j} M_{ij} a_i \cdot b_j}{\max_{(a'_i, b'_j)} \sum_{i,j} M_{ij} a'_i \cdot b'_j}. \quad (57)$$

**Proof:** Let  $f, g : [r] \rightarrow S_{n-1}$  and  $f', g' : [r] \rightarrow S_{m-1}$  be vector valued functions, and let  $f^*$  and  $g^*$  be such that they give a sequence  $(f^*(i), g^*(j))_{i,j=1}^r = (a_i^*, b_j^*)_{i=1}^r$  that maximizes  $\sum_{i,j} M_{ij} a_i \cdot b_j$ . Set

$$M'(a, b) = \sum_{i,j} M_{ij} \delta(a - a_i^*) \delta(b - b_j^*),$$

where  $\delta(\cdot)$  denotes the Dirac delta function. This causes the numerators of (57) to be equal. For the denominator of left-hand side of (57), we have

$$\max_{A,B:S_{n-1} \rightarrow S_{m-1}} \int dadb M'(a,b) A(a) \cdot B(b) = \max_{A',B':(f^*(i),g^*(j)) \rightarrow S_{m-1}} \sum_{i,j} M_{ij} A'(f^*(i)) \cdot B'(g^*(j)) \quad (58)$$

$$\leq \max_{f',g'} \sum_{i,j} M_{ij} f'(i) \cdot g'(j), \quad (59)$$

where the inequality follows because because the second maximization is over a subset of the set that the last equation is maximized over. This gives the result.  $\blacksquare$

**Proposition 12.** *For any measurable function  $M'(a,b)$  with  $a,b \in S_{n-1}$ , and any  $\varepsilon > 0$ , there exist an  $r$  and matrix  $M_{ij} \in \mathbb{R}^r \times \mathbb{R}^r$ , such that*

$$\frac{\max_{(a_i,b_j)} \sum_{i,j} M_{ij} a_i \cdot b_j}{\max_{(a'_i,b'_j)} \sum_{i,j} M_{ij} a'_i \cdot b'_j} \geq \frac{\int da db M'(a,b) a \cdot b}{\max_{A,B:\mathbb{R}^n \rightarrow \mathbb{R}^m} \int da db M'(a,b) A(a) \cdot B(b)} - \varepsilon. \quad (60)$$

**Proof:** First note that since  $|M'(\cdot, \cdot)|$  is a measurable function, the integral  $\int dadb |M'(a,b)|$  is bounded. Therefore, without loss of generality, we may assume that  $\int dadb |M'(a,b)| = 1$ .

Suppose we divide the unit  $n$ -sphere up into  $r$  disjoint regions  $R_1, \dots, R_r \subseteq S_{n-1}$  whose sizes decrease with increasing  $r$ , and set

$$M_{ij} = \int_{a \in R_i} \int_{b \in R_j} dadb M'(a,b).$$

Let  $(a_i^*, b_j^*)_{i,j=1}^r$  be a sequence that maximizes  $\sum_{i,j} M_{ij} a_i \cdot b_j$ , and define  $\delta := \max_{i,j} \{|a_i^* \cdot b_j^* - a \cdot b| \mid a \in R_i, b \in R_j\}$ . Then by the triangle inequality, we have

$$\left| \sum_{i,j} \int_{a \in R_i} da \int_{b \in R_j} db M(a,b) (a_i^* \cdot b_j^* - a \cdot b) \right| \leq \sum_{i,j} \int_{a \in R_i} da \int_{b \in R_j} db |M'(a,b)| |a_i^* \cdot b_j^* - a \cdot b| \leq \delta.$$

Hence, for the numerators we get:  $\max_{(a_i,b_j)} \sum_{i,j} M_{ij} a_i \cdot b_j \geq \int da db M'(a,b) a \cdot b - \delta$ . For the denominators, we have

$$\max_{A,B:\mathbb{R}^n \rightarrow \mathbb{R}^m} \int da db M'(a,b) A(a) \cdot B(b) \geq \max_{(a'_i,b'_j)} \sum_{i,j} M_{ij} a'_i \cdot b'_j,$$

since we can always pick  $A(a) = a'_i$  and  $B(b) = b'_j$  for all  $a \in R_i$  and  $b \in R_j$ . The result follows from the fact that we can let  $\delta$  become arbitrarily small by increasing  $r$ .  $\blacksquare$

## B Proof that the bound on $K_G(n \mapsto m)$ is nontrivial

Here we establish the following lemma.

**Lemma 13.** *The function*

$$f(n) = \frac{1}{\sqrt{n}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \quad (61)$$

*is strictly increasing on integers  $n = 1, 2, \dots$*

**Proof:** For  $n \leq 9$ , just evaluate  $f(n)$ . For  $n > 9$ , we use the following bound on  $\log \Gamma(x)$ , first proved by Robbins [21] for integer values of  $x$ , but which Matsunawa observed [16, Remark 4.1] is also valid for real values of  $x \geq 2$ :

$$\sqrt{2\pi x}^{x+1/2} e^{-x+1/(12x+1)} < \Gamma(x+1) < \sqrt{2\pi x}^{x+1/2} e^{-x+1/(12x)}. \quad (62)$$

Using this bound, we obtain

$$\begin{aligned} \log \frac{f(n+1)}{f(n)} &= -\frac{1}{2} \log\left(1 + \frac{1}{n}\right) + \log \frac{n}{2} + 2 \log \Gamma\left(\frac{n}{2}\right) - 2 \log \Gamma\left(\frac{n+1}{2}\right) \\ &\geq -\frac{1}{2} \log\left(1 + \frac{1}{n}\right) + \log\left(1 + \frac{1}{n/2-1}\right) - n \log\left(1 + \frac{1}{n-2}\right) + \frac{2}{6n-11} - \frac{2}{6n-6}. \end{aligned}$$

Now use

$$\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} \leq \log\left(1 + \frac{1}{n}\right) \leq \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}, \quad (63)$$

(which is valid for all  $n \geq 1$ ), and we obtain

$$\log \frac{f(m+10)}{f(m+9)} \geq \frac{14m^7 + 679m^6 + 13923m^5 + 155346m^4 + 1005620m^3 + 3684139m^2 + 6679947m + 3828140}{12(m+7)^4(m+8)(m+9)^3(6m+43)},$$

which is obviously positive when  $m \geq 0$ , i.e., when  $n \geq 9$ . Thus  $f(n)$  is strictly increasing.  $\blacksquare$