Another set of infinitely many exceptional (X_ℓ) Laguerre polynomials

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Abstract

We present a new set of infinitely many shape invariant potentials and the corresponding exceptional (X_ℓ) Laguerre polynomials. They are to supplement the recently derived two sets of infinitely many shape invariant thus exactly solvable potentials in one dimensional quantum mechanics and the corresponding X_ℓ Laguerre and Jacobi polynomials (Odake and Sasaki, Phys. Lett. **B679** (2009) 414-417). The new X_ℓ Laguerre polynomials and the potentials are obtained by a simple limiting procedure from the known X_ℓ Jacobi polynomials and the potentials, whereas the known X_ℓ Laguerre polynomials and the potentials are obtained in the same manner from the mirror image of the known X_ℓ Jacobi polynomials and the potentials.

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1. Introduction

For a long time it was believed, due to Bochner's theorem [1], that among countless orthogonal polynomials, only the *classical orthogonal polynomials*, the Hermite, Laguerre, Jacobi and Bessel polynomials, satisfy second order differential equations. In 2008 the notion of the *exceptional* (X_ℓ) orthogonal polynomials was introduced by Gomez-Ullate et al [2] in the framework of Sturm-Liouville theory. These orthogonal polynomials are exceptional in the sense that they start at degree ℓ ($\ell \geq 1$) instead of degree 0 constant term, thus avoiding restrictions of Bochner's theorem and they satisfy second order differential equations. They constructed the lowest examples, the X_1 Laguerre and X_1 Jacobi polynomials explicitly. Reformulation [3] within the framework of one dimensional quantum mechanics and shape invariant potentials [4] followed. Two sets of infinitely many shape invariant potentials, the deformed radial oscillator potentials and the deformed trigonometric/hyperbolic Darboux-Pöschl-Teller (DPT) potentials, and the corresponding X_ℓ Laguerre and Jacobi polynomials ($\ell = 1, 2, \ldots, \infty$) were presented by the present authors [5] in June 2009. The $\ell = 1$ examples are the same as those given by Gomez-Ullate et al. [2] and Quesne [3]. Shape invariance of the ℓ-th members of the exactly solvable potentials are attributed to new polynomial identities of degree 3ℓ involving cubic products of the Laguerre or Jacobi polynomials and are proved elementarily in [6].

In this Letter, we present a new set of infinitely many shape invariant potentials, deformed radial oscillator potentials and the corresponding X_ℓ Laguerre polynomials. They are obtained from the known [5] deformed trigonometric DPT potential and the known X_ℓ Jacobi polynomials in a certain limit. On the other hand the deformed radial oscillator potential in [5] and the corresponding X_ℓ Laguerre polynomials are shown to be derived in the same way from the 'mirror image' of those deformed trigonometric DPT potential and the corresponding X_ℓ Jacobi polynomials given in [5]. The first ($\ell = 1$) members of the two exceptional Laguerre polynomials are identical and the X_1 Jacobi polynomials and their mirror images are also identical. This is one of the reasons why the new X_ℓ polynomials were not discovered earlier. The second ($\ell = 2$) member of the new exceptional Laguerre polynomials and its shape invariant potential are the same as those found by Quesne [7]. They were called type II exceptional Laguerre polynomials and were discussed in some detail in [8]. In this connection so-called type III Laguerre and Jacobi solutions were discussed by Quesne [7]. The nature of the two type III solutions in [7] will be explained in the final section.

This Letter is organised as follows. In section two we briefly recapitulate the scheme of deformation of shape invariant potentials in terms of a polynomial eigenfunction of degree ℓ. Various results of [5] are listed for comparison with the results to be derived. In section three the new set of infinitely many deformed oscillator potentials and the corresponding X_ℓ Laguerre polynomials are derived by a simple limiting procedure from the known deformed trigonometric DPT potentials and the corresponding X_ℓ Jacobi polynomials in [5]. Various limiting formulas are also displayed. The final section is for a summary and comments.

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2. Deformed Shape Invariant Potentials

Here we follow the notation of [5] and recapitulate its main results for comparison with the new results to be derived in the next section. Exactly solvable deformation of the radial oscillator potential and the trigonometric DPT potential is most easily achieved at the prepotential level ($\ell = 1, 2, \ldots$):

$$w_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} w_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + \log\frac{\xi_\ell(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\eta(x);\boldsymbol{\lambda})}, \qquad (1)$$

in which w_0 is the undeformed prepotential and ξ_ℓ is related to the ℓ-th eigenpolynomial of the undeformed system, to be explained shortly. For the explanation of the sinusoidal coordinate $\eta(x)$ and the sets of parameters $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$, see [5]. These deformed prepotentials satisfy the shape invariance condition,

$$\begin{aligned}&\big(\partial_x w_\ell(x;\boldsymbol{\lambda})\big)^2 - \partial_x^2 w_\ell(x;\boldsymbol{\lambda})\\ &= \big(\partial_x w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)^2 + \partial_x^2 w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}). \qquad (2)\end{aligned}$$

The Hamiltonian and the other quantities are defined as:

$$\mathcal{H}_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda}) = p^2 + U_\ell(x;\boldsymbol{\lambda}),\ \ p=-i\partial_x, \qquad (3)$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \partial_x - \partial_x w_\ell(x;\boldsymbol{\lambda}),\ \ \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger = -\partial_x - \partial_x w_\ell(x;\boldsymbol{\lambda}), \qquad (4)$$

$$U_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \big(\partial_x w_\ell(x;\boldsymbol{\lambda})\big)^2 + \partial_x^2 w_\ell(x;\boldsymbol{\lambda}), \qquad (5)$$

$$\mathcal{H}_\ell(x;\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\phi_{\ell,n}(x;\boldsymbol{\lambda}), \qquad (6)$$

$$\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \psi_\ell(x;\boldsymbol{\lambda})P_{\ell,n}\big(\eta(x);\boldsymbol{\lambda}\big), \qquad (7)$$

$$\psi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{e^{w_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}}{\xi_\ell(\eta(x);\boldsymbol{\lambda})}. \qquad (8)$$

The polynomial eigenfunctions, *i.e.*, the *exceptional orthogonal polynomials* $P_{\ell,n}(\eta;\boldsymbol{\lambda})$ form the complete basis of the Hilbert space and satisfy the orthogonality:

$$\int_{x_1}^{x_2}\psi_\ell(x;\boldsymbol{\lambda})^2 P_{\ell,n}\big(\eta(x);\boldsymbol{\lambda}\big)P_{\ell,m}\big(\eta(x);\boldsymbol{\lambda}\big)dx = h_{\ell,n}(\boldsymbol{\lambda})\delta_{nm}. \qquad (9)$$

Here we list the explicit forms of various quantities. We attach superscripts L and J for the quantities related to the radial oscillator potential and the trigonometric DPT potential, respectively. We also attach superscripts 1 and 2 to distinguish those derived in the previous paper [5] and those new quantities to be introduced in the next section, respectively.

radial oscillator undeformed ($\ell = 0$) case:

$$\boldsymbol{\lambda}^{\text{L}} \stackrel{\text{def}}{=} g,\quad \boldsymbol{\delta}^{\text{L}} = 1,\quad g>0, \qquad (10)$$

$$\mathcal{E}_n^{\text{L}}(\boldsymbol{\lambda}) = 4n,\quad \eta^{\text{L}}(x) \stackrel{\text{def}}{=} x^2,\quad 0<x<\infty, \qquad (11)$$

$$\phi_0^{\text{L}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} e^{-\frac{x^2}{2}}x^g \Leftrightarrow w_0^{\text{L}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\frac{x^2}{2} + g\log x, \qquad (12)$$

$$P_n^{\text{L}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} L_n^{(g-\frac12)}(x), \qquad (13)$$

$$h_n^{\text{L}}(\boldsymbol{\lambda}) = \frac{1}{2\,n!}\,\Gamma(n+g+\tfrac12). \qquad (14)$$

ℓ-th deformed **radial oscillator**:

$$\xi_\ell^{\text{L1}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} L_\ell^{(g+\ell-\frac32)}(-x), \qquad (15)$$

$$\begin{aligned}P_{\ell,n}^{\text{L1}}(x;\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \xi_\ell^{\text{L1}}(x;g+1)P_n^{\text{L}}(x;g+\ell)\\ &\quad - \xi_{\ell-1}^{\text{L1}}(x;g+2)P_{n-1}^{\text{L}}(x;g+\ell), \qquad (16)\end{aligned}$$

$$h_{\ell,n}^{\text{L1}}(\boldsymbol{\lambda}) = \frac{n+g+2\ell-\frac12}{n+g+\ell-\frac12}\,h_n^{\text{L}}(g+\ell). \qquad (17)$$

trigonometric DPT undeformed ($\ell = 0$) case:

$$\boldsymbol{\lambda}^{\text{J}} \stackrel{\text{def}}{=} (g,h),\quad \boldsymbol{\delta}^{\text{J}} = (1,1),\quad g,h>0, \qquad (18)$$

$$\mathcal{E}_n^{\text{J}}(\boldsymbol{\lambda}) = 4n(n+g+h),\ \ \eta^{\text{J}}(x) \stackrel{\text{def}}{=} \cos 2x,\ \ 0<x<\frac{\pi}{2}, \qquad (19)$$

$$\begin{aligned}&\phi_0^{\text{J}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (\sin x)^g(\cos x)^h\\ &\qquad \Leftrightarrow w_0^{\text{J}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} g\log\sin x + h\log\cos x, \qquad (20)\end{aligned}$$

$$P_n^{\text{J}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n^{(g-\frac12,\,h-\frac12)}(x), \qquad (21)$$

$$h_n^{\text{J}}(\boldsymbol{\lambda}) = \frac{\Gamma(n+g+\frac12)\Gamma(n+h+\frac12)}{2\,n!\,(2n+g+h)\Gamma(n+g+h)}. \qquad (22)$$

ℓ-th deformed **trigonometric DPT**:

$$\xi_\ell^{\text{J1}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_\ell^{(-g-\ell-\frac12,\,h+\ell-\frac32)}(x),\quad h>g>0, \qquad (23)$$

$$\begin{aligned}P_{\ell,n}^{\text{J1}}(x;\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} a_{\ell,n}^{\text{J1}}(x;\boldsymbol{\lambda})P_n^{\text{J}}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\\ &\quad + b_{\ell,n}^{\text{J1}}(x;\boldsymbol{\lambda})P_{n-1}^{\text{J}}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \qquad (24)\end{aligned}$$

$$\begin{aligned}a_{\ell,n}^{\text{J1}}(x;\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \xi_\ell^{\text{J1}}(x;g+1,h+1)\\ &\quad + \frac{2n(-g+h+\ell-1)\,\xi_{\ell-1}^{\text{J1}}(x;g,h+2)}{(-g+h+2\ell-2)(g+h+2n+2\ell-1)}\\ &\quad - \frac{n(2h+4\ell-3)\,\xi_{\ell-2}^{\text{J1}}(x;g+1,h+3)}{(2g+2n+1)(-g+h+2\ell-2)}, \qquad (25)\end{aligned}$$

$$\begin{aligned}b_{\ell,n}^{\text{J1}}(x;\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \frac{(-g+h+\ell-1)(2g+2n+2\ell-1)}{(2g+2n+1)(g+h+2n+2\ell-1)}\\ &\qquad \times \xi_{\ell-1}^{\text{J1}}(x;g,h+2), \qquad (26)\end{aligned}$$

$$\begin{aligned}h_{\ell,n}^{\text{J1}}(\boldsymbol{\lambda}) &= \frac{(n+g+\ell+\frac12)(n+h+2\ell-\frac12)}{(n+g+\frac12)(n+h+\ell-\frac12)}\\ &\qquad \times h_n^{\text{J}}(g+\ell,h+\ell). \qquad (27)\end{aligned}$$

For the proof of shape invariance (2), see a recent paper [12].

3. Another set of deformed oscillator potentials & X_ℓ Laguerre polynomials

The *second set* of deformed radial oscillator and the corresponding X_ℓ Laguerre polynomials are the following.

2-nd ℓ-th deformed **radial oscillator**:

$$\xi^{\mathrm{L2}}_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} L_\ell^{(-g-\ell-\frac12)}(x), \qquad (28)$$

$$\begin{aligned} &P^{\mathrm{L2}}_{\ell,n}(x;\boldsymbol{\lambda})\\ &\stackrel{\text{def}}{=} \Bigl(\xi^{\mathrm{L2}}_\ell(x;g+1)-\frac{2n\,\xi^{\mathrm{L2}}_{\ell-2}(x;g+1)}{2g+2n+1}\Bigr)P^{\mathrm{L}}_n(x;g+\ell)\\ &\quad+\frac{2g+2n+2\ell-1}{2g+2n+1}\,\xi^{\mathrm{L2}}_{\ell-1}(x;g)P^{\mathrm{L}}_{n-1}(x;g+\ell), \end{aligned} \qquad (29)$$

$$h^{\mathrm{L2}}_{\ell,n}(\boldsymbol{\lambda})=\frac{n+g+\ell+\frac12}{n+g+\frac12}\,h^{\mathrm{L}}_n(g+\ell). \qquad (30)$$

The action of the operators $\mathcal{A}_\ell(g)$ and $\mathcal{A}_\ell(g)^\dagger$ on the eigenfunction $\phi_{\ell,n}(x;g)$ (or the action of the forward and backward shift operators on the eigenpolynomial $P_{\ell,n}(\eta(x);g)$) is the same as the L1 case, eq.(22) in [5]. It is easy to verify that $\xi^{\mathrm{L2}}_\ell(x;\boldsymbol{\lambda})$ is of the same sign for $0<x<\infty$ so that the deformation is non-singular. It is important to note that the first members of the deforming polynomials are essentially the same for the L1 and L2:

$$\xi^{\mathrm{L1}}_1(x;g)=-\xi^{\mathrm{L2}}_1(x;g). \qquad (31)$$

Since the normalisation of the ξ's are irrelevant to the deformation, the first deformed potential and the eigenpolynomials are identical:

$$\mathcal{H}^{\mathrm{L1}}_1(x;g)=\mathcal{H}^{\mathrm{L2}}_1(x;g),\quad P^{\mathrm{L1}}_{1,n}(x;g)=-P^{\mathrm{L2}}_{1,n}(x;g). \qquad (32)$$

This together with the same situation in the Jacobi case (49) are one of the reasons why the second sets are not recognised earlier. It is straightforward to verify that Quesne's type II potential eq.(2.17) lower sign of [7] is the same as the second deformed potential $U^{\mathrm{L2}}_2(x;\boldsymbol{\lambda})$, with the replacements $\omega\to 2$, $l\to g+1$. Correspondingly her type II X_2 Laguerre polynomials $\tilde{L}^{(\alpha)}_{2,\nu+2}(z)$ eq.(2.26) agree with the above $P^{\mathrm{L2}}_{\ell,n}(x;g)$, $\tilde{L}^{(g+\frac32)}_{2,n+2}(x)=2P^{\mathrm{L2}}_{2,n}(x;g)$. Her type I polynomials also coincide with ours, $\tilde{L}^{(g+\frac32)}_{1,n+2}(x)=2P^{\mathrm{L1}}_{2,n}(x;g)$.

The above second set is obtained by a simple limiting procedure from the deformed trigonometric DPT potential and the corresponding X_ℓ Jacobi polynomials (23)–(27) given in the preceding section.

In fact, the limit formulas of the base polynomials

$$\lim_{\beta\to\infty}P_n^{(\alpha,\,\pm\beta)}\bigl(1-2x\beta^{-1}\bigr)=L_n^{(\alpha)}(\pm x) \qquad (33)$$

are well known. The radial oscillator potential is known to be obtained from the trigonometric DPT potential in the limit of infinite coupling $h\to\infty$ with the rescaling of the coordinate:

$$x=\frac{x^{\mathrm{L}}}{\sqrt{h}},\quad 0<x<\frac{\pi}{2}\Leftrightarrow 0<x^{\mathrm{L}}<\frac{\pi}{2}\sqrt{h}\,. \qquad (34)$$

We then have

$$\eta^{\mathrm{J}}(x)=1-2\eta^{\mathrm{L}}(x^{\mathrm{L}})h^{-1}+O(h^{-2}), \qquad (35)$$

$$\lim_{h\to\infty}\bigl(w^{\mathrm{J}}_0(x;g,h)+\tfrac12 g\log h\bigr)=w^{\mathrm{L}}_0(x^{\mathrm{L}};g). \qquad (36)$$

Since a constant shift of the prepotential does not affect the Hamiltonian, these lead to the limit relations for the Hamiltonians and eigenfunctions

$$\lim_{h\to\infty}P^{\mathrm{J}}_n\bigl(\eta^{\mathrm{J}}(x);g,h\bigr)=P^{\mathrm{L}}_n\bigl(\eta^{\mathrm{L}}(x^{\mathrm{L}});g\bigr), \qquad (37)$$

$$\lim_{h\to\infty}h^{-1}\mathcal{H}^{\mathrm{J}}_0(x;g,h)=\mathcal{H}^{\mathrm{L}}_0(x^{\mathrm{L}};g), \qquad (38)$$

$$\lim_{h\to\infty}h^{-1}\mathcal{E}^{\mathrm{J}}_n(g,h)=\mathcal{E}^{\mathrm{L}}_n(g). \qquad (39)$$

Similar limit formulas hold for the ℓ deformed systems:

$$\lim_{h\to\infty}\xi^{\mathrm{J1}}_\ell\bigl(\eta^{\mathrm{J}}(x);g,h\bigr)=\xi^{\mathrm{L2}}_\ell\bigl(\eta^{\mathrm{L}}(x^{\mathrm{L}});g\bigr), \qquad (40)$$

$$\lim_{h\to\infty}\bigl(w^{\mathrm{J1}}_\ell(x;g,h)+\tfrac12(g+\ell)\log h\bigr)=w^{\mathrm{L2}}_\ell(x^{\mathrm{L}};g), \qquad (41)$$

$$\lim_{h\to\infty}h^{-1}\mathcal{H}^{\mathrm{J1}}_\ell(x;g,h)=\mathcal{H}^{\mathrm{L2}}_\ell(x^{\mathrm{L}};g), \qquad (42)$$

$$\lim_{h\to\infty}P^{\mathrm{J1}}_{\ell,n}\bigl(\eta^{\mathrm{J}}(x);g,h\bigr)=P^{\mathrm{L2}}_{\ell,n}\bigl(\eta^{\mathrm{L}}(x^{\mathrm{L}});g\bigr). \qquad (43)$$

By changing the roles of g and h in (23)–(27), we obtain the *second set* of deformed trigonometric DPT.

2-nd ℓ-th deformed **trigonometric DPT**:

$$\xi^{\mathrm{J2}}_\ell(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}P_\ell^{(g+\ell-\frac32,\,-h-\ell-\frac12)}(x),\quad g>h>0, \qquad (44)$$

$$\begin{aligned} P^{\mathrm{J2}}_{\ell,n}(x;\boldsymbol{\lambda})&\stackrel{\text{def}}{=}a^{\mathrm{J2}}_{\ell,n}(x;\boldsymbol{\lambda})P^{\mathrm{J}}_n(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\\ &\quad+b^{\mathrm{J2}}_{\ell,n}(x;\boldsymbol{\lambda})P^{\mathrm{J}}_{n-1}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \end{aligned} \qquad (45)$$

$$\begin{aligned} a^{\mathrm{J2}}_{\ell,n}(x;\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\xi^{\mathrm{J2}}_\ell(x;g+1,h+1)\\ &\quad-\frac{2n(g-h+\ell-1)\,\xi^{\mathrm{J2}}_{\ell-1}(x;g+2,h)}{(g-h+2\ell-2)(g+h+2n+2\ell-1)}\\ &\quad-\frac{n(2g+4\ell-3)\,\xi^{\mathrm{J2}}_{\ell-2}(x;g+3,h+1)}{(2h+2n+1)(g-h+2\ell-2)}, \end{aligned} \qquad (46)$$

$$\begin{aligned} b^{\mathrm{J2}}_{\ell,n}(x;\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\frac{(g-h+\ell-1)(2h+2n+2\ell-1)}{(2h+2n+1)(g+h+2n+2\ell-1)}\\ &\qquad\times\xi^{\mathrm{J2}}_{\ell-1}(x;g+2,h), \end{aligned} \qquad (47)$$

$$\begin{aligned} h^{\mathrm{J2}}_{\ell,n}(\boldsymbol{\lambda})&=\frac{(n+h+\ell+\frac12)(n+g+2\ell-\frac12)}{(n+h+\frac12)(n+g+\ell-\frac12)}\\ &\qquad\times h^{\mathrm{J}}_n(g+\ell,h+\ell). \end{aligned} \qquad (48)$$

The action of the operators $\mathcal{A}_\ell(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ on the eigenfunction $\phi_{\ell,n}(x;\boldsymbol{\lambda})$ (or the action of the forward and backward shift operators on the eigenpolynomial $P_{\ell,n}(\eta(x);\boldsymbol{\lambda})$) is the same as the J1 case, eq.(37) in [5]. Again the deforming polynomial $\xi^{\mathrm{J2}}_\ell(x;\boldsymbol{\lambda})$ ($g>h>0$) is of the same sign in $-1<x<1$ and the deformation is non-singular. It is interesting to note that the first members of the deforming polynomials are essentially the same for the J1 and J2:

$$\xi^{\mathrm{J1}}_1(x;g,h)=-\xi^{\mathrm{J2}}_1(x;g,h). \qquad (49)$$

This means that there is no difference between the J1 and J2 deformations in the $\ell=1$ case.

In fact this is not a new set but simply a 'mirror image' of the first set (23)–(27), under the transformation:

$$x \to y \stackrel{\text{def}}{=} \frac{\pi}{2} - x, \quad 0 < x < \frac{\pi}{2}, \quad 0 < y < \frac{\pi}{2}, \tag{50}$$

and renaming of the coupling constants $g \leftrightarrow h$. By the parity property of the Jacobi polynomial $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ and

$$\eta^{\mathrm{J}}(x) = -\eta^{\mathrm{J}}(y), \quad w_0^{\mathrm{J}}(x;g,h) = w_0^{\mathrm{J}}(y;h,g), \tag{51}$$

we obtain

$$\xi_\ell^{\mathrm{J2}}(x;g,h) = (-1)^\ell \xi_\ell^{\mathrm{J1}}(-x;h,g). \tag{52}$$

Hence the above assertion is demonstrated:

$$\mathcal{H}_\ell^{\mathrm{J2}}(x;g,h) = \mathcal{H}_\ell^{\mathrm{J1}}\big(y;h,g\big), \tag{53}$$

$$P_{\ell,n}^{\mathrm{J2}}(x;g,h) = (-1)^{\ell+n} P_{\ell,n}^{\mathrm{J1}}(-x;h,g). \tag{54}$$

Now it is easy to verify that the original (L1) set of the ℓ-th deformed oscillator and the X_ℓ Laguerre polynomials (15)–(17) are obtained from the second set of the trigonometric DPT and the X_ℓ Jacobi polynomials (44)–(48) in the same $h \to \infty$ limit with the corresponding rescaling of the coordinate (34):

$$\lim_{h\to\infty} \xi_\ell^{\mathrm{J2}}\big(\eta^{\mathrm{J}}(x);g,h\big) = \xi_\ell^{\mathrm{L1}}\big(\eta^{\mathrm{L}}(x^{\mathrm{L}});g\big), \tag{55}$$

$$\lim_{h\to\infty} \big(w_\ell^{\mathrm{J2}}(x;g,h) + \tfrac{1}{2}(g+\ell)\log h\big) = w_\ell^{\mathrm{L1}}(x^{\mathrm{L}};g), \tag{56}$$

$$\lim_{h\to\infty} h^{-1}\mathcal{H}_\ell^{\mathrm{J2}}(x;g,h) = \mathcal{H}_\ell^{\mathrm{L1}}(x^{\mathrm{L}};g), \tag{57}$$

$$\lim_{h\to\infty} P_{\ell,n}^{\mathrm{J2}}\big(\eta^{\mathrm{J}}(x);g,h\big) = P_{\ell,n}^{\mathrm{L1}}\big(\eta^{\mathrm{L}}(x^{\mathrm{L}});g\big). \tag{58}$$

4. Summary and Comments

We have presented a new set of infinitely many deformed radial oscillator potentials, which are shape invariant and thus exactly solvable. The corresponding X_ℓ^{L2} Laguerre polynomials are also given. They can be derived from the first set of trigonometric DPT and the corresponding X_ℓ Jacobi polynomials in a certain limit. The shape invariance relations for the first and second sets of deformations are attributed to the same cubic identities [6], both for the Laguerre and Jacobi cases. Various properties of the X_ℓ Laguerre and Jacobi polynomials of both kinds will be discussed in a forthcoming article [9]. In particular, we will present equivalent but much simpler forms of the X_ℓ Laguerre and Jacobi polynomials. It would be a good challenge to search matrix models associated with these exceptional orthogonal polynomials.

In Quesne's paper [7] the explicit forms are given of the candidates of the second type of the exceptional (X_2) Laguerre polynomials together with the corresponding deformed potential, which are shown to be the same as those given in this Letter. In the same paper [7], Quesne reported two non shape invariant but exactly solvable potentials called type III, each related to the radial oscillator and the DPT. Let us remark that they can be easily obtained by applying Adler's [10] modification of Crum's method [11] to the radial oscillator and DPT, with the specification to remove the first and second excited states of the undeformed system. The resulting eigenfunctions (the groundstate included) constitute the complete set of the Hilbert space, as proved in the paper [10]. Let us note that starting from an exactly solvable system, an infinite variety of exactly solvable potentials and the corresponding eigenfunctions can be constructed by Adler's method [10]. None of the derived systems, however, is shape invariant even if the starting system is.

Before closing this Letter, let us also mention that the deformation in terms of a degree ℓ eigenpolynomial, applied to the discrete quantum mechanical Hamiltonians for the Wilson and Askey-Wilson polynomials, produced two sets of infinitely many shape invariant systems together with exceptional (X_ℓ) Wilson and Askey-Wilson polynomials ($\ell = 1, 2, \ldots$) [12].

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