Pure Skyrme-Faddeev-Niemi hopfions

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Abstract

The pure Skyrme-Faddeev-Niemi model (i.e., without quadratic kinetic term) with a potential is considered on the spacetime $\mathbb{S}^3 \times \mathbb{R}$. For one-vacuum potentials two types of exact Hopf solitons are obtained. Depending on the value of the Hopf index, we find compact or non-compact hopfions. The compact hopfions saturate a Bogomolny bound and lead to a fractional energy-charge formula $E \sim |Q|^{1/2}$, whereas the non-compact solitons do not saturate the bound and give $E \sim |Q|$. In the case of potentials with two vacua compact shell-like hopfions are derived. Some remarks on the influence of the potential on topological solutions in the full Skyrme-Faddeev-Niemi model or in (3+1) Minkowski space are also made.

1 Introduction

The Skyrme-Faddeev-Niemi (SFN) model [1], [2] is a field theory with hopfions as solitonic excitations. The model is given by the following Lagrange density

$$L = -\alpha(\partial_\mu \vec{n})^2 + \beta[\partial_\mu \vec{n} \times \partial_\nu \vec{n}]^2 + \lambda V(\vec{n}), \tag{1}$$

where $\vec{n} = (n^1, n^2, n^3)$ is a unit iso-vector living in $(3+1)$ dimensional Minkowski space-time. Additionally, α, β, λ are positive constants. The second term, referred to

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as the Skyrme term (strictly speaking the Skyrme term restricted to $\mathbb{S}^2$) is obligatory in the case of 3 space dimensions to avoid the Derrick argument for the non-existence of static, finite energy solutions. The requirement of the finiteness of the energy for static configurations leads to an asymptotic condition $\vec{n} \to \vec{n}_0$, as $\vec{x} \to \infty$, where $\vec{n}_0$ is a constant vector. Thus, static configurations are maps $\mathbb{R}^3 \cup \{\infty\} \cong \mathbb{S}^3 \to \mathbb{S}^2$ and therefore can be classified by the pertinent topological charge, i.e., the Hopf index $Q \in \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. Moreover, as the pre-image of a fixed $\vec{n} \in \mathbb{S}^2$ is isomorphic to $\mathbb{S}^1$, the position of the core of a soliton (pre-image of the antipodal point $-\vec{n}_0$) forms a closed, in general knotted, loop.

The physical interest of the SFN model is related to the fact that it may be applied to several important physical systems. In the context of condensed matter physics, it has been used to describe possible knotted solitons for multi-component superconductors [3]. In field theory, its importance originates in the attempts to relate it to the low energy (non-perturbative), pure gluonic sector of QCD [1], [4]. In this picture, relevant particle excitations, i.e., glueballs are identified with knotted topological solitons. This idea is in agreement with the standard picture of mesons, where quarks are connected by a very thin tube of the gauge field. Now, because of the fact that glueballs do not consist of quarks, such a flux-tube cannot end on sources. In order to form a stable object, the ends must be joined, leading to loop-like configurations.

Although the SFN model (or some generalization thereof) might provide the chance for a very elegant description of the physics of glueballs, this proposal has its own problems. First of all, one has to include a symmetry breaking potential term [5], although the potential would not be required for stability reasons. This is necessary in order to avoid the existence of massless excitations, i.e., Goldstone bosons appearing as an effect of the spontaneous global symmetry breaking. Indeed, the Lagrangian without a potential possesses global $O(3)$ symmetry while the vacuum state is only $O(2)$ invariant. Thus, two generators are broken and two massless bosons emerge. This feature of the SFN model has been recently discussed and some modifications have been proposed [5], [6].

Secondly, due to a non-trivial topological as well as geometrical structure of solitons one is left with numerical solutions only. The issue of obtaining the global minimum (and local minima) in a fixed topological sector is a highly complicated, only partially solved problem (see e.g. [7] for the case without potential). The interaction between hopfions is, of course, even more difficult.

In spite of the huge difficulties, some analytical results have been obtained. One has to underline, however, that they have been found entirely for the potential-less case. Let us mention the famous Vakulenko-Kapitansky energy-charge formula, $E \geq c_1 |Q|^{3/4}$ [8], [9]. Similar upper bounds $E \leq c_2 |Q|^{3/4}$ have also been reported [9]. Among analytical approaches which have been applied to the SFN model, one should mention the generalized integrability [10] and the first integration method [11], which were especially helpful in constructing vortex [12] and non-topological solutions [13].

Another approach, which sheds some light on the properties of hopfions and allows for analytical calculations is the substitution of the flat Minkowski space-time by $\mathbb{S}^3 \times \mathbb{R}$ [14], [15], where an infinite set of static and time dependent solutions where found.

The main aim of the present paper is to analytically investigate the role of the potential term in a simplified version of the SFN model, and to study the resulting compact and non-compact soliton solutions. The influence of the potential term on qualitative and quantitative properties of topological solitons has been established in a version of the SFN model in (2+1) dimensions, i.e., in the baby Skyrme model [16], [17], [18]. Our strategy will be two-fold: we perform the $\alpha \to 0$ limit, that is, we neglect the quadratic part of the action, and we assume the space-time $\mathbb{S}^3 \times \mathbb{R}$. The first assumption is quite acceptable as the obtained model still allows to circumvent the Derrick arguments. In fact, as we comment in the summary, the solution of the model in the limit $\alpha \rightarrow 0$ probably can be viewed as a zero order approximation to a true soliton of the full theory. The limit $\alpha \rightarrow 0$ has been previously investigated in the context of the baby Skyrme model and the Skyrme-Faddeev-Niemi model (without potential term [19] - the so-called strong coupling limit). The second assumption takes us rather far from the standard SFN model but it is the price we have to pay if we want to perform all calculations in an analytical way while preserving the topological properties.

2 The pure Skyrme-Faddeev-Niemi model on $\mathbb{S}^3 \times \mathbb{R}$

2.1 Equations of motion

After the limit $\alpha \rightarrow 0$ we get the following pure SFN model

$$L = \beta[\partial_\mu \vec{n} \times \partial_\nu \vec{n}]^2 + \lambda V(\vec{n}), \tag{2}$$

where for the moment we choose for the potential

$$V = \frac{1}{2}(1 - n^3). \tag{3}$$

In 2+1 dimensional Minkowski space-time, i.e., in the baby Skyrme model, this potential is known as the old baby Skyrme potential. It should be stressed that the fact that the model is solvable does not depend on a particular form of the potential. However, specific quantitative as well as qualitative properties of the topological solutions are strongly connected with the form of the potential.
Coordinates on $\mathbb{S}^3 \times \mathbb{R}$ are chosen such that the metric is

$$ds^2 = dt^2 - R_0^2 \left(\frac{dz^2}{4z(1-z)} + (1-z)d\phi_1^2 + zd\phi_2^2 \right), \tag{4}$$

where $z \in [0, 1]$ and the angles $\phi_1, \phi_2 \in [0, 2\pi]$, R_0 denotes the radius of $\mathbb{S}^3$.
After the stereographic projection

$$\vec{n} = \frac{1}{1+|u|^2} \left(u + \bar{u}, -i(u - \bar{u}), 1 - |u|^2 \right). \tag{5}$$

we get

$$L = 8\beta \frac{(u_\mu \bar{u}^\mu)^2 - u_\mu^2 \bar{u}_\nu^2}{(1+|u|^2)^4} + \lambda \frac{|u|^2}{1+|u|^2} \tag{6}$$

where $u_\mu \equiv \partial_\mu u$, etc. The corresponding field equations read

$$\partial_\mu \left(\frac{\mathcal{K}^\mu}{(1+|u|^2)^2} \right) + \frac{2\bar{u}}{(1+|u|^2)^3} \mathcal{K}_\mu \partial^\mu u - \frac{\lambda}{4} \frac{\bar{u}}{(1+|u|^2)^2} = 0 \tag{7}$$

and its complex conjugate. Here

$$\mathcal{K}^\mu = 4\beta \frac{(u_\nu \bar{u}^\nu)\bar{u}^\mu - \bar{u}_\nu^2 u^\mu}{(1+|u|^2)^2}. \tag{8}$$

Thus,

$$\partial_\mu \mathcal{K}^\mu - \frac{\lambda}{4}\bar{u} = 0, \tag{9}$$

where we used the following identity

$$\mathcal{K}^\mu \bar{u}_\mu = 0. \tag{10}$$

In the subsequent analysis we assume the standard Ansatz

$$u = e^{i(m_1\phi_1 + m_2\phi_2)} f(z), \tag{11}$$

where $m_1, m_2 \in Z$. This ansatz exploits the base space symmetries of the theory, which for static configurations is equal to the isometry group SO(4) of the base space $\mathbb{S}^3$. This group has rank two, so it allows the separation of two angular coordinates $e^{im_l\phi_l}$, $l = 1, 2$, see e.g. [15] for details. We remark that, in addition, this theory has infinitely many target space symmetries, namely an abelian subgroup of the group of area-preserving diffeomorphisms on target space, see [20]. The profile function f can be derived from the equation

$$-\partial_z\left[\frac{f'f^2}{(1+f^2)^2}\Omega\right] + \left(\frac{ff'^2}{(1+f^2)^2}\Omega\right) + \tilde{\lambda} f = 0, \tag{12}$$

where we introduced

$$\Omega = m_1^2 z + m_2^2(1-z) \tag{13}$$

and

$$\tilde{\lambda} = \frac{\lambda R_0^4}{128\beta}. \tag{14}$$

In order to get a solution with nontrivial topological Hopf charge one has to impose boundary conditions which guarantee that the configuration covers the whole $\mathbb{S}^2$ target space at least once

$$f(z=0) = \infty, \quad f(z=1) = 0. \tag{15}$$

The equation for f can be further simplified leading to

$$f\left(\partial_z\left[\frac{f'f}{(1+f^2)^2}\Omega\right] - \tilde{\lambda}\right) = 0. \tag{16}$$

This expression is obeyed by the trivial, vacuum solution $f = 0$ or by a nontrivial configuration satisfying

$$\partial_z\left[\frac{f'f}{(1+f^2)^2}\Omega\right] = \tilde{\lambda} \quad \Rightarrow \quad \frac{f'f}{(1+f^2)^2}\Omega = \tilde{\lambda}(z+z_0). \tag{17}$$

This formula may be also integrated giving finally

$$\frac{1}{1+f^2} = -\frac{\tilde{\lambda}}{2}\int dz \frac{z+z_0}{m_1^2 z + m_2^2(1-z)} + C, \tag{18}$$

where C and z_0 are real integration constants, whose values can be found from the assumed boundary conditions.

One can also easily calculate the energy density

$$\varepsilon = \frac{32\beta}{R_0^4}\frac{4f^2f'^2}{(1+f^2)^4}\left(m_1^2 z + m_2^2(1-z)\right) + \frac{\lambda f^2}{1+f^2} \tag{19}$$

and the total energy

$$E = \frac{(2\pi)^2 R_0^3}{2}\int_0^1 dz\varepsilon. \tag{20}$$

2.2 Compact hopfions

It follows from the results of [21], [22], [23] that one should expect the appearance of compactons in the pure SFN model with the old baby Skyrme potential. As suggested by its name, a compacton is a solution with a finite support, reaching the vacuum value at a finite distance [24]. Thus, compactons do not possess exponential tails but approach the vacuum in a power-like manner.

An especially simple situation occurs for the $m_1 = \pm m_2 \equiv m$ case. Then, the equation of motion for the profile function reduces to

$$\partial_z^2 g = \frac{2\tilde{\lambda}}{m^2}, \tag{21}$$

where

$$g = 1 - \frac{1}{1+f^2}. \tag{22}$$

Observe that $g \geq 0$ by the definition of the function g. The pertinent boundary conditions for compact hopfions are $f(0) = \infty$ and $f(z = z_R) = 0$, where $z_R \leq 1$ is the radius of a compacton. In addition, as one wants to deal with a globally defined solution, the compact hopfion must be glued with the trivial vacuum configuration at z_R, i.e., $f'(z = z_R) = 0$. In terms of the function g we have $g(0) = 1$, $g(z = z_R) = 0$ and $g_z(z = z_R) = 0$. Thus, the compacton solution is

$$g(z) = \begin{cases} \left(1 - \frac{z\sqrt{\tilde{\lambda}}}{m}\right)^2 & z \leq z_R \\ 0 & z \geq z_R. \end{cases} \tag{23}$$

We remark that the energy density in terms of the function g may be expressed like

$$\varepsilon = \frac{128\beta}{R_0^4}\left(\frac{1}{4}g'^2 + \tilde{\lambda} g\right) \tag{24}$$

which makes it obvious that the vacuum configuration $g \equiv 0$ minimizes the energy functional. The size of the compact soliton is

$$z_R = \frac{m}{\sqrt{\tilde{\lambda}}}.$$

As the z coordinate is restricted to the interval $[0, 1]$, we get a limit for the topological charge for possible compact solitons. Namely

$$m \leq \sqrt{\tilde{\lambda}} = \frac{\lambda R_0^2}{\sqrt{128\beta}}. \tag{25}$$

In other words, one can derive a compact hopfion solution provided that its topological charge does not exceed a maximal value $Q_{max} = \lfloor \tilde{\lambda} \rfloor$, which is fixed once λ, β, R_0 are given.

Further, the energy density onshell is

$$\varepsilon = 2\lambda g \tag{26}$$

and the total energy

$$E = (2\pi)^2 \lambda R_0^3 \int_0^{\frac{m}{\sqrt{\tilde{\lambda}}}} dz \left(1 - \frac{z\sqrt{\tilde{\lambda}}}{m}\right)^2 = (2\pi)^2 \lambda R_0^3 \frac{m}{\sqrt{\tilde{\lambda}}}\frac{1}{3} = \frac{32\sqrt{2}\pi^2}{3}\sqrt{\lambda\beta}\, m R_0. \tag{27}$$

Taking into account the expression for the Hopf index

$$Q = m_1 m_2 = m^2.$$

we get

$$E = \frac{32\sqrt{2}\pi^2}{3}\sqrt{\lambda\beta}R_0 \ \ |Q|^{\frac{1}{2}}, \quad |Q| \le |Q_{max}|. \tag{28}$$

For a generic situation, when $m_1^2 \neq m_2^2$, we find the exact solutions

$$g(z) = 1 + \frac{2\tilde{\lambda}}{m_1^2 - m_2^2}\left[z - \left(z_R + \frac{m_2^2}{m_1^2 - m_2^2} \right) \ln\left(1 + z\frac{m_1^2 - m_2^2}{m_2^2} \right) \right]. \tag{29}$$

In this case, the size of the compacton z_R is given by a solution of the non-algebraic equation

$$z_R - \left(z_R + \frac{m_2^2}{m_1^2 - m_2^2} \right) \ln\left(1 + z_R\frac{m_1^2 - m_2^2}{m_2^2} \right) + \frac{m_1^2 - m_2^2}{2\tilde{\lambda}} = 0. \tag{30}$$

2.3 Non-compact hopfions

Let us again consider the profile function equation for $m_1 = \pm m_2$ (21) but with non-compacton boundary conditions. Namely, $g(0) = 1$, $g(z = 1) = 0$, i.e., the solutions nontrivially cover the whole $\mathbb{S}^3$ base space. The pertinent solution reads

$$g(z) = \frac{\tilde{\lambda}}{m^2}z^2 - \left(1 + \frac{\tilde{\lambda}}{m^2} \right) z + 1. \tag{31}$$

However, this solution makes sense only if the image of g is not negative. This is the case if

$$\frac{\tilde{\lambda}}{m^2} \le 1 \ \ \Rightarrow \ \ m \ge \sqrt{\tilde{\lambda}} \tag{32}$$

and we found a lower limit for the Hopf charge. Thus, such non-compact hopfions occur if their topological charge is larger than a minimal charge $Q_{min} = \lceil \tilde{\lambda} \rceil$.
The corresponding energy is

$$E = \frac{(2\pi)^2}{2}\lambda R_0^3 \ \ \left[\frac{32\beta}{R_0^4}|Q| \left(1 - \frac{\lambda R_0^4}{128\beta|Q|} \right)^2 + \lambda \left(1 - \frac{1}{3}\frac{R_0^4\lambda}{128\beta|Q|} \right) \right], \tag{33}$$

for $|Q| \ge |Q_{min}|$.
Finally we are able to write down a formula for the total energy for a soliton solution with a topological charge Q

$$E = \begin{cases} \frac{32\sqrt{2}\pi^2}{3}\sqrt{\lambda\beta}R_0 \ \ |Q|^{\frac{1}{2}} & |Q| \le \lfloor \frac{\lambda R_0^4}{128\beta} \rfloor \\ \frac{(2\pi)^2}{2}\lambda R_0^3 \ \ \left[\frac{32\beta}{R_0^4}|Q| \left(1 - \frac{\lambda R_0^4}{128\beta|Q|} \right)^2 + \lambda \left(1 - \frac{1}{3}\frac{R_0^4\lambda}{128\beta|Q|} \right) \right] & |Q| \ge \lceil \frac{\lambda R_0^4}{128\beta} \rceil, \end{cases} \tag{34}$$

where the first line describes the compact hopfions and the second one the standard non-compact solitons.

Remark 1. The pure Skyrme-Faddeev-Niemi model with potential (3) can be mapped, after the dimension reduction, on the signum-Gordon model [21].

Indeed, if we rewrite the energy functional using our Ansatz with $m_1 = \pm m_2$, and take into account the definition of the function g, then we get the energy for the real signum-Gordon model

$$E = \frac{(2\pi)^2 R_0^3}{2} \int_0^1 dz \left(\frac{32\beta m^2}{R_0^4} g_z^2 + \lambda g \right). \tag{35}$$

The signum-Gordon model is well-known to support compact solutions, so this map is one simple way to understand their existence. The same is true on two-dimensional Euclidean base space, explaining the existence of compactons in the model of Ref. [22] (to our knowledge, compactons in a relativistic field theory have been first discussed in that reference).

Remark 2. Compact hopfions saturate the BPS bound, whereas non-compact hopfions do not saturate it.
This follows immediately from the last expression and the fact that all solitons are solutions of a first order ordinary differential equation. Namely,

$$E = \frac{(2\pi)^2 R_0^3}{2} \int_0^1 dz \left[\left(\sqrt{\frac{32\beta m^2}{R_0^4}} g_z + \sqrt{\lambda} g^{1/2} \right)^2 - 2\sqrt{\frac{32\beta m^2}{R_0^4}} g_z \sqrt{\lambda} g^{1/2} \right]. \tag{36}$$

Then,

$$E \geq -2\frac{(2\pi)^2 R_0^3}{2} \sqrt{\frac{32\beta\lambda m^2}{R_0^4}} \int_{g(0)}^{g(z_R)} dz g_z g^{1/2} \tag{37}$$

and

$$E \geq \frac{32\sqrt{2}\pi^2}{3} \sqrt{\lambda\beta} R_0 (g(0)^{3/2} - g(z_R)^{3/2}) = \frac{32\sqrt{2}\pi^2}{3} \sqrt{\lambda\beta} R_0, \tag{38}$$

as $g(0) = 1$ and $g(z_R) = 0$. The inequality is saturated if the first term in Eq. (36) vanishes i.e.,

$$\frac{32\beta m^2}{R_0^4} g_z^2 = \lambda g, \tag{39}$$

which is exactly the first order equation obeyed by the compact hopfions. On the other hand, the non-compact solitons satisfy

$$\frac{32\beta m^2}{R_0^4} g_z^2 = \lambda g + C, \tag{40}$$

where C is a non-zero constant

$$C = \left(1 - \frac{\tilde{\lambda}}{m^2} \right)^2.$$

2.4 More general potentials

The generalization to the models with the potentials

$$V_s = \lambda \left(\frac{1}{2}(1 - n^3) \right)^s, \tag{41}$$

where $s \in (0,2)$ leads to similar compact solutions. Namely,

$$g(z) = \left\{ \begin{array}{ll} \left(1 - \frac{z\sqrt{\tilde{\lambda}}(2-s)}{m}\right)^{\frac{2}{2-s}} & z \leq z_R \\ 0 & \geq z_R. \end{array} \right. \tag{42}$$

Now, the size of the compacton is

$$z_R = \frac{m}{z\sqrt{\tilde{\lambda}}(2-s)}, \tag{43}$$

and the limit for the maximal allowed topological charge (in the $m_1 = \pm m_2$ case) is

$$m \leq \sqrt{\tilde{\lambda}}(2-s). \tag{44}$$

For a bigger value of the Hopf index one gets a non-compact hopfion. The energy-charge relation remains (up to a multiplicative constant) unchanged.
In the limit when $s = 2$, i.e.,

$$V_2 = \lambda \left(\frac{1}{2}(1-n^3)\right)^2, \tag{45}$$

we get only non-compact hopfions

$$g(z) = \cosh\left(\frac{2z\sqrt{\tilde{\lambda}}}{m}\right) - \coth\left(\frac{2\sqrt{\tilde{\lambda}}}{m}\right)\sinh\left(\frac{2z\sqrt{\tilde{\lambda}}}{m}\right). \tag{46}$$

The total energy is found to be

$$E = \frac{(2\pi)^2}{2}\lambda R_0^3 \frac{m}{4\sqrt{\tilde{\lambda}}}\left(\coth\frac{2\sqrt{\tilde{\lambda}}}{m} + \frac{\frac{2\sqrt{\tilde{\lambda}}}{m}}{\sinh^2\left(\frac{2\sqrt{\tilde{\lambda}}}{m}\right)}\right). \tag{47}$$

Asymptotically, for large topological charge $Q = \pm m^2$ we get

$$E = \frac{(2\pi)^2}{2}\lambda R_0^3\left(\frac{128\beta}{\lambda R_0^4}|Q| + \frac{1}{45}\frac{\lambda R_0^4}{32\beta|Q|}\right). \tag{48}$$

Finally, let us comment that for $s > 2$ there are no finite energy compact hopfions, at least as long as the Ansatz is assumed. Indeed, the Bogomolny equation for g in this case is

$$g_z^2 = \frac{4\tilde{\lambda}}{m^2}g^s$$

and the power-like approach to the vacuum $g \sim (z - z_R)^\alpha$ leads to

$$\alpha = \frac{2}{2-s}$$

which is negative for $s > 2$. There may, however, exist non-compact hopfions. In the case $s = 4$, for instance (the so-called holomorphic potential in the baby Skyrme model), the resulting first order equation for g is

$$g_z^2 = \frac{4\tilde{\lambda}}{m^2}(g^4 + g_0^4)$$

the general solution of which is given by the eliptic integral

$$\int_{g=0}^{g=g(z)} \frac{dg}{(g^4+g_0^4)^{1/2}} = -\frac{2}{|m|}\sqrt{\tilde{\lambda}}(z-z_0)$$

(we chose the negative sign of the root because g is a decreasing function of z), and we have to impose the boundary conditions

$$g(z=1)=0 \quad \Rightarrow z_0=1$$

and $g(z=0)=1$ which leads to

$$\int_0^1 \frac{dg}{(g^4+g_0^4)^{1/2}} = \frac{2}{|m|}\sqrt{\tilde{\lambda}}.$$

The last condition can always be fulfilled because the l.h.s. becomes arbitrarily large for sufficiently small values of g_0 and vice versa.

2.5 Double vacuum potential

Another popular potential often considered in the context of the baby skyrmions, and referred to as the new baby Skyrme potential, is given by the following expression

$$V = 1-(n^3)^2. \tag{49}$$

In contrast to the cases considered before, this potential has two vacua at $n^3 = \pm 1$. After taking into account the Ansatz and the definition of the function g, the equation of motion reads

$$\frac{1}{2}\partial_z(\Omega g_z) = \tilde{\lambda}4(1-2g), \tag{50}$$

leading, for $m_1 = \pm m_2$, to the general solution

$$g(z) = \frac{1}{2}\left(1-\sqrt{1+4C}\sin\left(\frac{4\sqrt{\tilde{\lambda}}(z-z_0)}{m}\right)\right), \tag{51}$$

where C, z_0 are constants.
Here, we start with the non-compact solitons. Then, assuming the relevant boundary conditions we find

$$g(z) = \frac{1}{2}\left[1-\frac{\sin\frac{4\sqrt{\tilde{\lambda}}}{m}(z-\frac{1}{2})}{\sin\frac{2\sqrt{\tilde{\lambda}}}{m}}\right]. \tag{52}$$

This configuration describes a single soliton if g is a monotonous function from 1 to 0. This implies that the sine has to be a single-valued function on the interval $z \in [0,1]$, i.e.,

$$\frac{4\sqrt{\tilde{\lambda}}}{m} \leq \pi \quad \Rightarrow \quad |Q| \geq \frac{16\tilde{\lambda}}{\pi^2}. \tag{53}$$

Exactly as before, the non-compact solutions do not saturate the corresponding Bogomolny bound.

For a sufficiently small value of the topological charge we obtain a one-parameter family of compact hopfions

$$g(z) = \begin{cases} 1 & 0 \le z \le z_r \\ \frac{1}{2}\left[1 - \sin\frac{4\sqrt{\lambda}}{m}(z - z_0)\right] & z_r \le z \le z_R \\ 0 & z \ge z_R \end{cases}, \tag{54}$$

where the boundary conditions have been specified as $g(z_r) = 1, g(z_R) = 0$ and $g'(z_r) = g'(z_R) = 0$. The initial and end point of the compacton are

$$z_r = z_0 + \frac{\pi m}{8\sqrt{\tilde{\lambda}}}, \quad z_R = z_0 + \frac{3\pi m}{8\sqrt{\tilde{\lambda}}} \tag{55}$$

and z_0 is a free parameter restricted to

$$z_0 \in [-\frac{\pi m}{8\sqrt{\tilde{\lambda}}}, 1 - \frac{3\pi m}{8\sqrt{\tilde{\lambda}}}]. \tag{56}$$

We remark that in this case the energy density in terms of the function g may be expressed like

$$\varepsilon = \frac{128\beta}{R_0^4}\left(\frac{1}{4}g'^2 + \tilde{\lambda}g(1-g)\right) \tag{57}$$

which makes it obvious again that both vacuum configurations $g = 0, 1$ minimize the energy functional.

As we see, compact solutions in the model with the new baby Skyrme potential are shell-like objects. In fact, there is a striking qualitative resemblance between the baby skyrmions and the compact hopfions in the pure Skyrme-Faddeev-Niemi model with potentials (3), (49). Namely, it has been observed that the old baby skyrmions are rather standard solitons with or without rotational symmetry, whereas the new baby skyrmions possess a ring-like structure [18]. Here, in the case of the new baby potential, we get a higher dimensional generalization of ring structures, i.e., shells.

The energy-charge relation again takes the form of the square root dependence for compactons,

$$E = \frac{\pi^3}{2}R_0\sqrt{128\beta\lambda}\ \ |Q|^{1/2}, \tag{58}$$

where we used the fact that the compact solutions saturate the Bogomolny bound.

2.6 Free model case

To have a better understanding of the role of the potential let us briefly consider the case without potential, i.e., $\lambda = 0$. In this case one can easily find the hopfions [15]

$$g(z) = 1 - \frac{\ln\left(1 + z\frac{m_1^2 - m_2^2}{m_2^2}\right)}{\ln\left(1 + \frac{m_1^2 - m_2^2}{m_2^2}\right)} \tag{59}$$

for $m_1^2 \neq m_2^2$ and

$$g(z) = 1 - z \tag{60}$$

for $m_1 = \pm m_2$. As we see, all solitons are of the non-compact type, which differs profoundly from the previous situation.
The energy-charge formula reads

$$E = \frac{(2\pi)^2\beta}{4R_0} \frac{m_1^2 - m_2^2}{\ln m_1 - \ln m_2} \tag{61}$$

or for $m_1^2 = m_2^2$

$$E = \frac{(2\pi)^2\beta}{2R_0}|Q|. \tag{62}$$

Again, the difference is quite big as we re-derived the standard linear dependence.
Remark: There exists a significant difference between models which have the quartic, pure Skyrme term as the only kinetic term (containing derivatives) on the one hand, and models which have a standard quadratic kinetic term (either in addition to or instead of the quartic Skyrme term), on the other hand. Models with a quadratic kinetic term have the typical vortex type behaviour

$$u \sim r^m e^{im\phi}$$

near the zeros of u. Here r is a generic radial variable, ϕ is a generic angular variable wrapping around the zero, and m is the winding number. In other words, configurations with higher winding about a zero of u are higher powers of the basic u with winding number one, where both the modulus and the phase part of u are taken to a higher power. This behaviour is, in fact, required by the finiteness of the Laplacian Δu at $r = 0$. Models with only a quartic pure Skyrme kinetic term (both with and without potential), however, show the behaviour

$$u \sim re^{im\phi}$$

i.e., only the phase is taken to a higher power for higher winding. For our concrete model on base space $\mathbb{S}^3$, and for the simpler case $m_1 = m_2 \equiv m$, we have $u \sim z^{-1/2}e^{im(\phi_1+\phi_2)}$ near $z = 0$ (both with and without a potential term), but with the help of the symmetries $u \rightarrow (1/u)$ and $u \rightarrow \bar{u}$ this may be brought easily to the form

$$u \sim \sqrt{z}e^{im(\phi_1+\phi_2)},$$

as above. As said, the Laplacian acting on this field is singular at $z = 0$, so the field has a conical singularity at this point. One may wonder whether this singularity shows up in the field equation and requires the introduction of a delta-like source term. The answer to this question is no. Thanks to the specific form of the quartic kinetic term, the second derivatives in the field equation show up in such a combination that the singularity cancels and the field equation is well-defined at the zero of u. As this behaviour is generic and only depends on the Skyrme term and on the existence of topological solutions (and not on the base space) we show it for the simplest case with base space $\mathbb{R}^2$ (i.e., the model of Gisiger and Paranjape), where r and ϕ are just polar coordinates in this space. A compact soliton centered about the origin behaves like $u \sim re^{im\phi}$ near the origin, and has the singular Laplacian

$$\Delta u = (1 - m^2)r^{-1}e^{-m\phi}.$$

On the other hand, the field equation (9) is finite at $r = 0$, because the vector $\vec{\mathcal{K}}$ behaves like

$$\vec{\mathcal{K}} = 8\beta\frac{m^2\hat{e}_r - im\hat{e}_\phi}{(1+r^2)^2}e^{-im\phi} \equiv \mathcal{K}_r\hat{e}_r + \mathcal{K}_\phi\hat{e}_\phi$$

(here $\hat{e}_r$ and $\hat{e}_\phi$ are the unit vectors along the corresponding coordinates), and its divergence (which enters into the field equation) is

$$\nabla \cdot \vec{\mathcal{K}} \equiv \frac{1}{r}\partial_r(r\mathcal{K}_r) + \frac{1}{r}\partial_\phi \mathcal{K}_\phi = \frac{32\beta r}{(1+r^2)^3} e^{-im\phi}$$

and a potential singular $(1/r)$ contribution cancels between the first and the second term. As said, this behaviour is completely generic for models with the Skyrme term as the only kinetic term. These fields, therefore, solve the field equations also at the singular points $u = 0$ and are, consequently, true solutions of the corresponding variational problem.

3 Compact strings in Minkowski space

In the (3+1) dimensional standard Minkowski space-time we are not able to find analytic soliton solutions with finite energy, because the symmetries of the model do not allow for a symmetry reduction to an ordinary differential equation in this case. We may, however, derive static and time-dependent solutions with a compact string geometry with the string oriented, e.g. along the z direction. These strings have finite energy per unit length in the z direction. Further, the pertinent topological charge is the winding number $Q = n$. In this section (x, y, z) refer to the standard cartesian coordinates in flat Euclidean space. Further, we use the old baby Skyrme potential of Section 2.1. The Ansatz we use reads

$$u = f(r)e^{in\phi}e^{i(\omega t + kz)}, \tag{63}$$

where ω, k are real parameters, $r^2 \equiv x^2 + y^2$, $\phi = \arctan(y/x)$, and n fixes the topological content of the configuration. It gives the following equation for the profile function f

$$f\left(\frac{1}{r}\partial_r\left[r\frac{f'f}{(1+f^2)^2}\Omega\right] - \tilde{\lambda}\right) = 0, \tag{64}$$

where $\tilde{\lambda} = \lambda/32\beta$ and

$$\Omega = k^2 - \omega^2 + \frac{n^2}{r^2}. \tag{65}$$

The simplest solutions may be obtained for $\omega^2 = k^2$. Then, after introducing

$$x = \frac{r^2}{2}, \quad \text{and} \quad g = 1 - \frac{1}{1+f^2} \tag{66}$$

we get

$$g_{xx} = \frac{2\tilde{\lambda}}{n^2}. \tag{67}$$

The compact solution reads

$$g(r) = \begin{cases} \left(1 - r^2\frac{\sqrt{\tilde{\lambda}}}{n\sqrt{2}}\right)^2 & r \leq \frac{\sqrt{n}\sqrt[4]{2}}{\sqrt[4]{\tilde{\lambda}}} \\ 0 & r \geq \frac{\sqrt{n}\sqrt[4]{2}}{\sqrt[4]{\tilde{\lambda}}}. \end{cases} \tag{68}$$

The total energy (per unit length in z-direction) is

$$E = \int d^2x \frac{8\beta}{(1+|u|^2)^4}[(\nabla u \nabla \bar{u})^2 - (\nabla u)^2(\nabla \bar{u})^2] \tag{69}$$

$$+ \frac{8\beta}{(1+|u|^2)^4}[2u_0\bar{u}_0(\nabla u \nabla \bar{u}) - u_0^2(\nabla \bar{u})^2 - \bar{u}_0^2(\nabla u)^2] + \lambda \frac{|u|^2}{1+|u|^2}, \tag{70}$$

or after inserting our Ansatz

$$E = 2\pi \int_0^\infty rdr \left(\frac{32\beta f^2 f'^2}{(1+f^2)^4} \left(\frac{n^2}{r^2} + \omega^2 + k^2 \right) + \frac{\lambda f^2}{1+f^2} \right) \tag{71}$$

and finally

$$E = \frac{2\pi}{3} \left[12\sqrt{\lambda\beta}|Q| + 32\beta\omega^2 \right]. \tag{72}$$

A more complicated case is for $\delta^2 \equiv k^2 - \omega^2 > 0$. Then, $\Omega = \delta^2 + \frac{n^2}{r^2}$, and the equation for g is

$$\partial_x \left(g_x(n^2 + 2\delta^2 x) \right) - 2\tilde{\lambda} = 0. \tag{73}$$

The compacton solution (with the compacton boundary conditions) is

$$g(x) = 1 + \frac{\tilde{\lambda}}{\delta^2} \left[x - (\frac{n^2}{2\delta^2} + x_R) \ln \left(1 + \frac{2\delta^2 x}{n^2} \right) \right], \tag{74}$$

where x_R is given by

$$1 + \frac{\tilde{\lambda}}{\delta^2} \left[x_R - (\frac{n^2}{2\delta^2} + x_R) \ln \left(1 + \frac{2\delta^2 x_R}{n^2} \right) \right] = 0. \tag{75}$$

4 Conclusions

It has been the main purpose of the present paper to demonstrate and explicitly construct compact soliton solutions of the pure Skyrme–Faddeev–Niemi model (with only a quartic kinetic term) with a potential. These compact solutions are natural generalizations of the compact solutions of the purely quartic baby Skyrme model which have first been reported by Gisiger and Paranjape [22], and further investigated recently [23]. As we wanted to present exact analytical solutions, we chose the base space (spacetime) $\mathbb{S}^3 \times \mathbb{R}$ for finite energy solutions, because Minkowski spacetime does not offer sufficient symmetries to reduce the field equations to ordinary differential equations. Only in the case of spinning string-like solutions with a finite energy per length unit along the string the symmetry reduction in Minkowski space is possible (Section 3). For the case of $\mathbb{S}^3 \times \mathbb{R}$ spacetime, we found two rather different classes of finite energy soliton solutions, namely compactons (which cover only a finite fraction of the three-sphere) on the one hand, and non-compact solitons (which cover the full three-sphere) on the other hand. Both classes of solutions are topological, but their energies are quite different. The compacton energies behave like $E_c \sim R_0|Q|^{1/2}$ (where R_0 is the radius of the three-sphere, and Q is the topological charge), whereas the energies of the non-compact solitons behave like $E_s \sim R_0^3|Q|$. Further, the compactons only exist up to a certain maximum value of the topological charge, whereas the non-compact solitons start to exist from this value onwards. The different behaviours of the energies in the compact and non-compact case may be easily understood from the observation that the compactons obey a Bogomolny equation, whereas the non-compact solitons obey a "Bogomolny equation up to a constant". Indeed, if for an energy density of the type $\mathcal{E} = \mathcal{E}_4 + \mathcal{E}_0$ (here the subindices refer to the power of first derivatives in each term) a Bogomolny equation holds, then the energy density for solutions may be expressed like $\mathcal{E} \sim (\mathcal{E}_4\mathcal{E}_0)^{1/2}$. If we now take into account the scaling dimensions $\mathcal{E}_4 \sim R_0^{-4}$, $\mathcal{E}_0 \sim R_0^0$ and $\int d^3x \sim R_0^3$, then the behaviour $E_c \sim R_0$ easily follows. Physically this means that the compacton solutions are localised near the north pole of the three-sphere, and the localisation becomes more pronounced for larger radii R_0. On the other hand, the energy density of the non-compact solitons remains essentially delocalised and evenly distributed over the whole three-sphere. We remark that the behaviour of the compacton energies $E_c \sim R_0|Q|^{1/2}$ poses an apparent paradox, because it can be proven that already the quartic part of the energy alone can be bound from below by $|Q|$, that is, $E_4 \equiv \int d^3x\mathcal{E}_4 \geq \alpha R_0^{-1}|Q|$, where α is an unspecified constant. The proof was given in [25] for $R_0 = 1$, but the generalization for arbitrary radius is trivial using the scaling behaviour of the corresponding terms. The apparent paradox is of course resolved by the observation that compactons exist only for not too large values of $|Q|$, such that the lower bound is compatible with the energies of the explicit solutions. Finally, if the potential has more than one vacuum, then compactons of the shell type exist, such that the field takes two different vacuum values inside the inner and outside the outer compact shell boundary. Except for their different shape, these compact shells behave quite similarly to the compact balls in the one-vacuum case (e.g. the relation between energy and topological charge or the linear growth of the energy with the three-sphere radius is the same).

One interesting question clearly is whether analogous compacton solutions with finite energy exist in Minkowski space. An exact calculation is probably not possible in this case, but we think that we have found already some indirect evidence for the existence of such solutions. The first argument is, of course, the fact that they exist in one dimension lower (in the baby Skyrme model). The second argument is related to the behaviour of our solutions for large radius R_0. The compacton solutions are localized and, therefore, their energies grow only moderately with R_0 (linearly in R_0). Further, the allowed range of topological charges for compactons grows like the fourth power of R_0. These are clear indications that compacton solutions might also exist in Minkowski space. Certainly this question requires some further investigation. If these compactons in Minkowski space exist, then an interesting question is which energy-charge relation will result. Will the energies grow like $E_c \sim |Q|^{1/2}$, like on the three-sphere, or will they obey the three-quarter law $E_c \sim |Q|^{3/4}$, like for the full SFN model without potential in Minkowski space? All we can say at the moment is that an upper bound for the energy in flat space can be derived. The derivation is completely analogous to the cases of the full SFN, Nicole or AFZ models (the choice of trial functions which explicitly saturate the bound), and also the result is the same, $E_c \leq \alpha|Q|^{3/4}$, see [9]. The attempt to derive a lower bound, analogous to the Vakulenku-Kapitanski bound for the SFN model, meets the same obstacles as for the Nicole or AFZ models, see Appendix C of the second reference in [9].

Assuming for the moment the existence of compactons in Minkowski space, another interesting proposal is to use the compacton solutions of the pure quartic model (with potential) as a lowest order approximation to soliton solutions of the full SFN model and try to approximate the full solitons by a kind of generalized expansion. If such an approximate solution is possible, it would have several advantages.

• The pure quartic model is much easier than the full theory. In the case of the baby Skyrme model (both with old and new potentials) one gets even solvable models (as long as the rotational symmetry is assumed).

• The lowest order solution is already a non-perturbative configuration, i.e., a compacton, which captures the topological properties of the full solution. Due to the compact nature of the lowest order solution we have a kind of "localization" of the topo-

logical properties in a finite volume.
• One can easily construct multi-compacton solutions which, if sufficiently separated, do not interact. They form something which perhaps may be called *a fake Bogomolny sector* as they are solutions of a first order equation (usually saturating a corresponding energy-charge inequality) and may form multi-soliton noninteracting complexes.
Of course, it remains to be seen whether such an approximate solution is possible at all. What can be said so far is that in the simpler case of a scalar field theory with a potential which is smooth if a certain parameter μ is non-zero and approaches a V-shaped potential in the $\mu \to 0$ limit, then the compacton is the $\mu \to 0$ limit of the non-compact soliton, see [26].

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