Realization of Frobenius manifolds as submanifolds in pseudo-Euclidean spaces

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Abstract

We introduce a class of k-potential submanifolds in pseudo-Euclidean spaces and prove that for an arbitrary positive integer k and an arbitrary nonnegative integer p, each N-dimensional Frobenius manifold can always be locally realized as an N-dimensional k-potential submanifold in $((k+1)N+p)$-dimensional pseudo-Euclidean spaces of certain signatures. For $k=1$ this construction was proposed by the present author in a previous paper (2006). The realization of concrete Frobenius manifolds is reduced to solving a consistent linear system of second-order partial differential equations.

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1 Introduction

In this paper we develop and significantly generalize the construction proposed earlier by the present author in [1] and generated by a deep nontrivial relationship discovered in [1] between the theory of Frobenius manifolds, the associativity equations of two-dimensional topological quantum field theories (the Witten–Dijkgraaf–Verlinde–Verlinde equations), and the Dubrovin–Frobenius structures on the one hand and the theory of submanifolds in pseudo-Euclidean spaces on the other hand. In this connection we construct new very natural integrable *k-potential* reductions of the fundamental Gauss–Codazzi–Ricci equations and new interesting integrable classes of submanifolds in pseudo-Euclidean spaces. These classes are important for applications. In particular, in this paper we introduce a new integrable class of *k-potential* submanifolds in pseudo-Euclidean spaces and prove that for an arbitrary positive integer k and an arbitrary nonnegative integer p, each N-dimensional Frobenius manifold can always be locally realized as an N-dimensional *k-potential* submanifold in $((k+1)N+p)$-dimensional pseudo-Euclidean spaces of certain signatures. For $k=1$ this construction was proposed by the present author in [1]. The realization of any concrete N-dimensional Frobenius manifold as an N-dimensional k-potential submanifold in $((k+1)N+p)$-dimensional pseudo-Euclidean spaces is reduced to

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solving a consistent linear system of second-order partial differential equations, which can be solved explicitly in elementary and special functions. First of all, we prove that for an arbitrary positive integer k and an arbitrary nonnegative integer p, the associativity equations of two-dimensional topological quantum field theories (the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations, see [2]–[5]) for a function (a *potential*) $\Phi = \Phi(u^1, \dots, u^N)$,

$$\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k}\eta^{kl}\frac{\partial^3 \Phi}{\partial u^l \partial u^m \partial u^n} = \sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^m \partial u^k}\eta^{kl}\frac{\partial^3 \Phi}{\partial u^l \partial u^j \partial u^n}, \tag{1}$$

where η^{ij} is an arbitrary constant nondegenerate symmetric matrix, $\eta^{ij} = \text{const}$, $\det(\eta^{ij}) \neq 0$, $\eta^{ij} = \eta^{ji}$, are very natural integrable k-potential reductions of the fundamental nonlinear equations in the submanifold theory (the corresponding Gauss–Codazzi–Ricci equations) that describe N-dimensional submanifolds in $((k+1)N+p)$-dimensional pseudo-Euclidean spaces. The WDVV equations give a natural and important nontrivial integrable class of *k-potential* N-dimensional submanifolds of codimension $kN + p$ in $((k + 1)N + p)$-dimensional pseudo-Euclidean spaces. For the special case $k = 1$ and $p = 0$ all these statements were formulated and proved in [1] and [6], where such N-dimensional submanifolds of codimension N were called *potential*. All k-potential submanifolds have natural differential-geometric special structures of Frobenius algebras (the Dubrovin–Frobenius structures) on their tangent spaces. These Dubrovin–Frobenius structures are generated by the corresponding flat first fundamental forms and some sets of the second fundamental forms of the submanifolds (the structural constants of a Frobenius algebra are given and duplicated by sets of the Weingarten operators of the k-potential submanifold).

We recall that each solution $\Phi(u^1, \dots, u^N)$ of the associativity equations (1) gives Dubrovin–Frobenius structures, i.e., specific N-parameter deformations of Frobenius algebras; in our case these algebras are commutative associative algebras equipped with nondegenerate invariant symmetric bilinear forms. Indeed, consider the algebras $A(u)$ in an N-dimensional vector space with the basis $e_1, \dots, e_N$ and multiplication (see [2])

$$e_i \circ e_j = c^k_{ij}(u) e_k, \qquad c^k_{ij}(u) = \eta^{ks}\frac{\partial^3 \Phi}{\partial u^s \partial u^i \partial u^j}. \tag{2}$$

For all values of the parameters $u = (u^1, \dots, u^N)$, the algebras $A(u)$ are commutative, $e_i \circ e_j = e_j \circ e_i$, and the associativity condition

$$(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k) \tag{3}$$

in the algebras $A(u)$ is equivalent to the WDVV equations (1). The inverse η_{ij} of the matrix η^{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$, defines a nondegenerate invariant symmetric bilinear form on the algebras $A(u)$,

$$\langle e_i, e_j \rangle = \eta_{ij}, \qquad \langle e_i \circ e_j, e_k \rangle = \langle e_i, e_j \circ e_k \rangle. \tag{4}$$

Recall that locally any Frobenius manifold has a Dubrovin–Frobenius structure (see [2]); namely, the tangent space at every point $u = (u^1, \ldots, u^N)$ of any Frobenius manifold possesses the structure of a Frobenius algebra (2)–(4), which is determined by a certain solution of the associativity equations (1) and smoothly depends on the point. In this paper we prove that for an arbitrary positive integer k and an arbitrary nonnegative integer p, each N-dimensional Frobenius manifold can always be locally represented as an N-dimensional k-potential submanifold in some $((k+1)N+p)$-dimensional pseudo-Euclidean spaces. The corresponding representations of any given Frobenius manifold are parametrized by the set of admissible Gram matrices of the scalar products of basic vectors in the normal spaces of the corresponding k-potential submanifolds; in particular, for $p = 0$ they are parametrized by the set of arbitrary nondegenerate symmetric constant $k \times k$ matrices determining all the admissible $kN \times kN$ Gram matrices of the scalar products of basic vectors in the normal spaces of the corresponding k-potential submanifolds. If for an arbitrary Frobenius manifold we fix a certain admissible $(kN+p)\times(kN+p)$ Gram matrix of the scalar products of basic vectors in the normal spaces of the corresponding k-potential submanifolds, then the corresponding k-potential submanifold realizing this Frobenius manifold and having the given Gram matrix of basic vectors in the normal spaces is determined by our construction uniquely up to motions in the corresponding ambient pseudo-Euclidean space. We note that an alternative approach to the description of the submanifolds realizing arbitrary Frobenius manifolds is developed by the present author in [7].

2 Frobenius algebras and Frobenius manifolds

2.1 Frobenius algebras

In the mathematical literature there are various widely spread approaches to the notion of Frobenius algebra and different definitions of Frobenius algebras not always requiring even associativity of the algebra, to say nothing of the requirement of presence of a unit in the algebra, symmetry of the corresponding bilinear form, and commutativity of the algebra. Therefore, we give here necessary definitions that will be used in this article. The presence of a special nondegenerate bilinear form that is compatible with the multiplication in the algebra is a common feature of all definitions of Frobenius algebras.

Let us consider a finite-dimensional algebra $\mathcal{A}$ (with multiplication $\circ$) over a field $\mathbb{K}$ (in this paper we consider algebras only over $\mathbb{R}$ or $\mathbb{C}$).

Definition 2.1 A bilinear form $f : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$ in an algebra $\mathcal{A}$ is called *invariant* (or *associative*) if

$$f(a \circ b, c) = f(a, b \circ c) \tag{5}$$

for all $a, b, c \in \mathcal{A}$.

Definition 2.2 A finite-dimensional associative algebra $\mathcal{A}$ over a field $\mathbb{K}$ that is equipped with a nondegenerate invariant symmetric bilinear form is called *Frobenius*.

Note that we do not require the presence of a unit in a Frobenius algebra.

Example 2.1 Matrix algebra $M_n(\mathbb{K})$.

Consider the algebra $M_n(\mathbb{K})$ (the algebra of $n \times n$ matrices over a field $\mathbb{K}$), the linear functional (trace of matrices)

$$\theta(a) = \mathrm{Tr}\,(a),\ \ a \in M_n(\mathbb{K}),$$

and the bilinear form $f(a, b) = \theta(ab)$. This bilinear form is invariant, since the matrix algebra is associative. It is easy to prove that this bilinear form is nondegenerate, and the matrix algebra $(M_n(\mathbb{K}), f)$ is a noncommutative Frobenius algebra with a unit over a field $\mathbb{K}$. Note that the bilinear form $f(a, b) = \theta(ab)$ is symmetric, $\theta(ab) = \theta(ba)$.

Example 2.2 Group algebra $\mathbb{K}G$.

Let G be a finite group. Consider the group algebra $\mathbb{K}G$ over a field $\mathbb{K}$,

$$\mathbb{K}G = \{a \mid a = \sum_{g\in G} \alpha_g g,\ \ \alpha_g \in \mathbb{K}\}.$$

Obviously, $\mathbb{K}G$ is an associative algebra with unit over the field $\mathbb{K}$. Let e be the unit of the group G. Consider the linear functional

$$\theta(a) = \alpha_e(a), \quad a = \sum_{g\in G} \alpha_g(a) g \in \mathbb{K}G, \quad \alpha_g(a) \in \mathbb{K},$$

and the bilinear form $f(a, b) = \theta(ab)$. This bilinear form is invariant, since the group algebra is associative. It is easy to prove that this bilinear form is nondegenerate. Indeed, we have

$$f(g^{-1}, a) = \theta(g^{-1}a) = \alpha_g(a)$$

for all $g \in G$. Therefore, if $f(g, a) = \theta(ga) = 0$ for all $g \in G$, then $\alpha_g(a) = 0$ for all $g \in G$, i.e., $a = 0$. Hence the bilinear form f is nondegenerate, and the group algebra $(\mathbb{K}G, f)$ is a noncommutative Frobenius algebra with a unit over a field $\mathbb{K}$ (it is commutative only for Abelian groups G). Note that the bilinear form $f(a, b) = \theta(ab)$ is symmetric for any group G, $\theta(ab) = \theta(ba)$.

2.2 Frobenius manifolds

It would be quite natural to call a manifold *Frobenius* if each tangent space at any point of this manifold is equipped with a Frobenius algebra structure that depends smoothly on the point of the manifold. However, a remarkable and very fruitful theory of Frobenius manifolds with very special Frobenius structures was constructed by Dubrovin (see [2]) in connection with two-dimensional topological quantum field theories and quantum cohomology, and it is these manifolds that were called Frobenius. Such Frobenius manifolds play an important role in singularity theory, enumerative geometry, theory of Gromov–Witten invariants, quantum cohomology theory, topological quantum field theories, and in various other domains of modern differential geometry and mathematical and theoretical physics. In this paper we follow the definition of [2], but we do not impose some very severe Dubrovin's constraints on Frobenius manifolds (in particular, we do not require quasihomogeneity, the presence of a special Euler vector field, and the presence of a covariantly constant unit in the Frobenius algebra on the tangent spaces of the manifold). We will call the corresponding structures on manifolds *Dubrovin–Frobenius structures*.

Definition 2.3 [2] An N-dimensional pseudo-Riemannian manifold M^N with a metric g and an algebra structure $(T_u M, \circ)$, $T_u M \times T_u M \stackrel{\circ}{\to} T_u M$, that is defined on each tangent space $T_u M$ and depends smoothly on the point $u \in M^N$ is called *Frobenius* if

(I) the pseudo-Riemannian metric g is a nondegenerate invariant symmetric bilinear form on each tangent space $T_u M$,

$$g(X \circ Y, Z) = g(X, Y \circ Z) \tag{6}$$

for all vector fields X, Y, and Z on M^N;

(II) the algebra $(T_u M, \circ)$ is commutative at each point $u \in M^N$,

$$X \circ Y = Y \circ X \tag{7}$$

for all vector fields X and Y on M^N;

(III) the algebra $(T_u M, \circ)$ is associative at each point $u \in M^N$,

$$(X \circ Y) \circ Z = X \circ (Y \circ Z) \tag{8}$$

for all vector fields X, Y, and Z on M^N;

(IV) the metric g is flat;

(V) $A(X, Y, Z) = g(X \circ Y, Z)$ is a symmetric tensor on M^N such that the tensor $(\nabla_W A)(X, Y, Z)$ is symmetric with respect to all vector fields X, Y, Z, and W on M^N (∇ is the covariant differentiation generated by the Levi-Civita connection of the metric g).

It is obvious that conditions (I)–(III) mean that at each point $u \in M^N$ the algebra $(T_u M, \circ, g)$ is a commutative Frobenius algebra.

2.3 Associativity equations

Let us consider an arbitrary Frobenius manifold, i.e., an arbitrary manifold satisfying conditions (I)–(V). Let $u = (u^1, \ldots, u^N)$ be arbitrary flat coordinates of the flat metric g. In any flat local coordinates, the metric $g(u)$ is a constant nondegenerate symmetric matrix η_{ij}, $\eta_{ij} = \eta_{ji}$, $\det(\eta_{ij}) \neq 0$, $\eta_{ij} =$ const, $g(X, Y) = \eta_{ij} X^i(u) Y^j(u)$.

In these flat local coordinates, for structural functions $c^i_{jk}(u)$ of the Frobenius algebra on the manifold,

$$X \circ Y = W, \quad W^i(u) = c^i_{jk}(u) X^j(u) Y^k(u),$$

and for the symmetric tensor $A_{ijk}(u)$, we have

$$A(X,Y,Z) = A_{ijk}(u)X^i(u)Y^j(u)Z^k(u) = g(X \circ Y, Z) =$$
$$= g(W,Z) = \eta_{ij}W^i(u)Z^j(u) = \eta_{ij}c^i_{kl}(u)X^k(u)Y^l(u)Z^j(u);$$

i.e.,

$$A_{ijk}(u) = \eta_{sk}c^s_{ij}(u). \tag{9}$$

According to condition (V), $(\nabla_l A_{ijk})(u)$ is a symmetric tensor; i.e., in the flat local coordinates we also have

$$\frac{\partial A_{ijk}}{\partial u^l} = \frac{\partial A_{ijl}}{\partial u^k}.$$

Hence there locally exists a function (a *potential*) $\Phi(u)$ such that

$$A_{ijk}(u) = \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k}.$$

From relation (9) for the structural functions $c^i_{jk}(u)$ we obtain

$$c^i_{jk}(u) = \eta^{is}A_{sjk}(u) = \eta^{is}\frac{\partial^3 \Phi}{\partial u^s \partial u^j \partial u^k}, \tag{10}$$

where the matrix η^{ij} is the inverse of the matrix η_{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$.

For any values of the parameters $u = (u^1, \ldots, u^N)$, the structural functions (10) give a commutative algebra

$$\partial_i \circ \partial_j = c^k_{ij}(u)\partial_k = \eta^{ks}\frac{\partial^3 \Phi}{\partial u^s \partial u^i \partial u^j}\partial_k \tag{11}$$

equipped with a symmetric invariant nondegenerate bilinear form

$$\langle \partial_i, \partial_j \rangle = \eta_{ij} \tag{12}$$

for any constant nondegenerate symmetric matrix η_{ij} and for any function $\Phi(u)$, but, generally speaking, this algebra is not associative. All conditions (I)–(V) except the associativity condition (III) are obviously satisfied for all these N-parameter deformations of nonassociative algebras.

The associativity condition (III) is equivalent to a nontrivial overdetermined system of nonlinear partial differential equations for the potential $\Phi(u)$,

$$\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k}\eta^{kl}\frac{\partial^3 \Phi}{\partial u^l \partial u^m \partial u^n} = \sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^m \partial u^k}\eta^{kl}\frac{\partial^3 \Phi}{\partial u^l \partial u^j \partial u^n}, \tag{13}$$

which is well known as the system of associativity equations of two-dimensional topological quantum field theories (the WDVV equations, see [2]–[5], [8]–[11]). The system of associativity equations (13) is consistent, integrable by the inverse scattering method, and possesses a rich set of nontrivial exact solutions (see [2]).

It is obvious that each solution $\Phi(u^1, \dots, u^N)$ of the associativity equations (13) gives N-parameter deformations of commutative Frobenius algebras (11) equipped with nondegenerate invariant symmetric bilinear forms (12). These Dubrovin–Frobenius structures satisfy all conditions (I)–(V).

Further in this paper we show that the associativity equations (13) are natural reductions of the fundamental nonlinear equations in the theory of submanifolds in pseudo-Euclidean spaces and give a natural class of *k-potential* submanifolds. All k-potential submanifolds in pseudo-Euclidean spaces have natural differential-geometric structures of Frobenius algebras (11), (12) on their tangent spaces. These differential-geometric Dubrovin–Frobenius structures are generated by the flat first fundamental forms and the sets of the second fundamental forms on the submanifold (the structural constants of the Frobenius algebra are given by the Weingarten operators of the submanifold).

A great number of concrete examples of Frobenius manifolds and solutions of the associativity equations are given in [2]. Consider here one simple but important example from [2]. Let $N = 3$, let the metric η_{ij} be antidiagonal,

$$(\eta_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{14}$$

and let e_1 be a unit in the Frobenius algebra (11), (12). In this case, the function (the potential) $\Phi(u)$ has the form

$$\Phi(u) = \frac{1}{2}(u^1)^2 u^3 + \frac{1}{2} u^1 (u^2)^2 + f(u^2, u^3),$$

and the associativity equations (13) for the function $\Phi(u)$ are equivalent to the following remarkable equation for the function $f(u^2, u^3)$ (see [2]):

$$\frac{\partial^3 f}{\partial (u^3)^3} = \left(\frac{\partial^3 f}{\partial (u^2)^2 \partial u^3} \right)^2 - \frac{\partial^3 f}{\partial (u^2)^3} \frac{\partial^3 f}{\partial u^2 \partial (u^3)^2}. \tag{15}$$

This equation is connected with the quantum cohomology of projective plane and classical problems of enumerative geometry (see [12]). In particular, all nontrivial polynomial quasihomogeneous solutions of equation (15) are described in [2]:

$$f = \frac{1}{4}(u^2)^2 (u^3)^2 + \frac{1}{60}(u^3)^5, \quad f = \frac{1}{6}(u^2)^3 u^3 + \frac{1}{6}(u^2)^2 (u^3)^3 + \frac{1}{210}(u^3)^7, \tag{16}$$

$$f = \frac{1}{6}(u^2)^3 (u^3)^2 + \frac{1}{20}(u^2)^2 (u^3)^5 + \frac{1}{3960}(u^3)^{11}. \tag{17}$$

As shown by the author in [13] (see also [14]–[16]), equation (15) is equivalent to the integrable nondiagonalizable homogeneous system of hydrodynamic type

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_{u^3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & 2b & -a \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{u^2}, \tag{18}$$

$$a = \frac{\partial^3 f}{\partial (u^2)^3}, \quad b = \frac{\partial^3 f}{\partial (u^2)^2 \partial u^3}, \quad c = \frac{\partial^3 f}{\partial u^2 \partial (u^3)^2}. \tag{19}$$

In this case, the Weingarten operators of potential submanifolds that realize the corresponding Frobenius manifolds have the form

$$(w_1)^i_j(u) = \delta^i_j, \quad (w_2)^i_j(u) = \begin{pmatrix} 0 & b & c \\ 1 & a & b \\ 0 & 1 & 0 \end{pmatrix}, \quad (w_3)^i_j(u) = \begin{pmatrix} 0 & c & b^2 - ac \\ 0 & b & c \\ 1 & 0 & 0 \end{pmatrix}. \tag{20}$$

For concrete solutions of the associativity equation (15), in particular, for (16) and (17), or for any concrete solutions of the system of hydrodynamic type (18), the corresponding linear systems that provide explicit realizations of the corresponding Frobenius manifolds as k-potential submanifolds in pseudo-Euclidean spaces can be solved explicitly in elementary and special functions.

3 General fundamental equations of the local theory of totally nonisotropic submanifolds in pseudo-Euclidean spaces

Let us consider an arbitrary totally nonisotropic smooth N-dimensional submanifold M^N in an $(N+L)$-dimensional pseudo-Euclidean space $\mathbb{E}^{N+L}_q$ of arbitrary signature q, $M^N \subset \mathbb{E}^{N+L}_q$. Recall that a submanifold of a pseudo-Euclidean space is called *totally nonisotropic* if it is not tangent to isotropic cones of the ambient pseudo-Euclidean space at any of its points. A submanifold of a pseudo-Euclidean space is totally nonisotropic if and only if the metric induced on the submanifold from the ambient pseudo-Euclidean space (the first fundamental form of the submanifold) is nondegenerate. Note that in this paper we consider only the local theory of submanifolds.

Let $(z^1, \ldots, z^{N+L})$ be pseudo-Euclidean coordinates in $\mathbb{E}^{N+L}_q$, and let the submanifold M^N be given locally by a smooth vector function $r(u^1, \ldots, u^N)$ of N independent variables $u^1, \ldots, u^N$ (local coordinates on the submanifold M^N), $r(u^1, \ldots, u^N) = (z^1(u^1, \ldots, u^N), \ldots, z^{N+L}(u^1, \ldots, u^N))$, $\operatorname{rank}(\partial z^i / \partial u^j) = N$, $1 \le i \le N+L$, $1 \le j \le N$. In this case $\partial r / \partial u^i = r_i(u)$, $1 \le i \le N$, is a basis of the tangent space $\mathbb{T}_u$ at each point $u = (u^1, \ldots, u^N)$ of the submanifold M^N, and $g_{ij}(u) = (r_i, r_j)$, $1 \le i, j \le N$, is the first fundamental form of the submanifold M^N, where $(\cdot, \cdot)$ is the pseudo-Euclidean scalar product in $\mathbb{E}^{N+L}_q$. In the normal space $\mathbb{N}_u$ of the submanifold M^N at each point u, we fix an arbitrary basis $n_1(u), \ldots, n_L(u)$ that depends smoothly on the point u. Consider the corresponding matrix of scalar products of the basis vectors in the normal spaces on the submanifold M^N (we will also call it the Gram matrix in the normal spaces of the submanifold M^N, although in this case the scalar product is, generally speaking, pseudo-Euclidean), i.e., the matrix of functions $h_{\alpha\beta}(u) = (n_\alpha, n_\beta)$, $1 \le \alpha, \beta \le L$. For totally nonisotropic submanifolds we always have $\det g_{ij}(u) \ne 0$ and $\det h_{\alpha\beta}(u) \ne 0$. Note that usually in the local theory of submanifolds some orthonormal bases in the normal spaces $\mathbb{N}_u$ are considered, but it is fundamentally important for our approach to consider arbitrary bases in the normal spaces $\mathbb{N}_u$. Therefore, we develop such a general approach here and present in detail the corresponding general fundamental relations, formulae, and equations of the local theory of totally nonisotropic submanifolds in pseudo-Euclidean spaces in the form necessary for us.

3.1 Gauss and Weingarten decompositions

Since the set of vectors $(r_1(u), \ldots, r_N(u), n_1(u), \ldots, n_L(u))$ forms a basis in $\mathbb{E}^{N+L}_q$ at each point of the submanifold M^N, we can decompose each of the vectors on the submanifold M^N with respect to this basis, in particular, the vectors $\partial^2 r/\partial u^i \partial u^j$, $1 \le i, j \le N$, and the vectors $\partial n_\alpha/\partial u^i$, $1 \le \alpha \le L$, $1 \le i \le N$, getting the Gauss decomposition

$$\frac{\partial^2 r}{\partial u^i \partial u^j} = a^k_{ij}(u) \frac{\partial r}{\partial u^k} + b^\beta_{ij}(u) n_\beta(u) \tag{21}$$

and the Weingarten decomposition

$$\frac{\partial n_\alpha}{\partial u^j} = c^k_{\alpha j}(u) \frac{\partial r}{\partial u^k} + d^\beta_{\alpha j}(u) n_\beta(u), \tag{22}$$

where the coefficients $a^k_{ij}(u)$, $b^\beta_{ij}(u)$, $c^k_{\alpha j}(u)$, and $d^\beta_{\alpha j}(u)$ are smooth functions on the submanifold M^N.

For each submanifold, there is a number of fundamental relations, including the Gauss, Codazzi, and Ricci equations, between the metric $g_{ij}(u)$, the functions $h_{\alpha\beta}(u)$, and the coefficients $a^k_{ij}(u)$, $b^\beta_{ij}(u)$, $c^k_{\alpha j}(u)$, and $d^\beta_{\alpha j}(u)$. If $g_{ij}(u)$, $h_{\alpha\beta}(u)$, $a^k_{ij}(u)$, $b^\beta_{ij}(u)$, $c^k_{\alpha j}(u)$, and $d^\beta_{\alpha j}(u)$ satisfy locally all these relations, then by the Bonnet theorem there always exists a unique (up to motion in the ambient pseudo-Euclidean space) submanifold with these differential-geometric objects.

It follows immediately from the Gauss decomposition (21) that the coefficients $a^k_{ij}(u)$ and $b^\beta_{ij}(u)$ are symmetric with respect to the lower indices:

$$a^k_{ij}(u) = a^k_{ji}(u), \tag{23}$$

$$b^\beta_{ij}(u) = b^\beta_{ji}(u). \tag{24}$$

In addition, taking the scalar product of the Gauss decomposition (21) with the tangent vectors $r_l(u)$, $1 \le l \le N$, we have

$$\left(\frac{\partial^2 r}{\partial u^i \partial u^j}, \frac{\partial r}{\partial u^l} \right) = a^k_{ij}(u) g_{kl}(u). \tag{25}$$

Differentiating the relation

$$\left(\frac{\partial r}{\partial u^i}, \frac{\partial r}{\partial u^j} \right) = g_{ij}(u) \tag{26}$$

with respect to u^s, we get

$$\left(\frac{\partial^2 r}{\partial u^i \partial u^s}, \frac{\partial r}{\partial u^j}\right) + \left(\frac{\partial r}{\partial u^i}, \frac{\partial^2 r}{\partial u^j \partial u^s}\right) = \frac{\partial g_{ij}}{\partial u^s} \tag{27}$$

or, taking into account (25),

$$a^k_{is}(u) g_{kj}(u) + a^k_{js}(u) g_{ki}(u) = \frac{\partial g_{ij}}{\partial u^s}. \tag{28}$$

In addition, rearranging the indices and taking into account the symmetry of $a^k_{ij}(u)$ with respect to the lower indices (23), we have

$$a^k_{ij}(u) g_{ks}(u) + a^k_{js}(u) g_{ki}(u) = \frac{\partial g_{is}}{\partial u^j}, \tag{29}$$

$$a^k_{is}(u) g_{kj}(u) + a^k_{ij}(u) g_{ks}(u) = \frac{\partial g_{sj}}{\partial u^i}. \tag{30}$$

From (28), (29), and (30) we obtain

$$a^k_{ij}(u) g_{ks}(u) = \frac{1}{2}\left(\frac{\partial g_{sj}}{\partial u^i} + \frac{\partial g_{is}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^s}\right), \tag{31}$$

or

$$a^k_{ij}(u) = \frac{1}{2} g^{ks}(u)\left(\frac{\partial g_{sj}}{\partial u^i} + \frac{\partial g_{is}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^s}\right), \tag{32}$$

where $g^{ks}(u)$ is the contravariant metric that is the inverse of the first fundamental form $g_{ij}(u)$: $g^{ik}(u) g_{kj}(u) = \delta^i_j$; i.e., the coefficients $a^k_{ij}(u)$ are the coefficients of the Levi-Civita connection $\Gamma^k_{ij}(u)$ of the metric $g_{ij}(u)$:

$$\Gamma^k_{ij}(u) = \frac{1}{2} g^{ks}(u)\left(\frac{\partial g_{sj}}{\partial u^i} + \frac{\partial g_{is}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^s}\right) = a^k_{ij}(u). \tag{33}$$

Taking the scalar product of the Weingarten decomposition (22) with the normal vectors $n_\beta(u)$, $1 \le \beta \le L$, we have

$$\left(\frac{\partial n_\alpha}{\partial u^i}, n_\beta\right) = d^\gamma_{\alpha i}(u) h_{\gamma\beta}(u). \tag{34}$$

The expressions

$$\varkappa_{\alpha\beta i}(u) = \left(\frac{\partial n_\alpha}{\partial u^i}, n_\beta\right) = d^\gamma_{\alpha i}(u) h_{\gamma\beta}(u), \qquad 1 \le \alpha, \beta \le L, \tag{35}$$

for any α and β give components of a covariant vector field and are called the *torsion coefficients of the submanifold*, and the 1-forms $\varkappa_{\alpha\beta i}(u) du^i$, $1 \le \alpha, \beta \le L$, are called the *torsion forms of the submanifold*.

Differentiating the relation

$$(n_\alpha, n_\beta) = h_{\alpha\beta}(u) \tag{36}$$

with respect to u^i, we get

$$\left(\frac{\partial n_\alpha}{\partial u^i}, n_\beta\right) + \left(n_\alpha, \frac{\partial n_\beta}{\partial u^i}\right) = \frac{\partial h_{\alpha\beta}}{\partial u^i}, \tag{37}$$

and taking into account (34),

$$d^\gamma_{\alpha i}(u) h_{\gamma\beta}(u) + d^\gamma_{\beta i}(u) h_{\gamma\alpha}(u) = \frac{\partial h_{\alpha\beta}}{\partial u^i}, \tag{38}$$

or

$$\varkappa_{\alpha\beta i}(u) + \varkappa_{\beta\alpha i}(u) = \frac{\partial h_{\alpha\beta}}{\partial u^i}. \tag{39}$$

Taking the scalar product of the Gauss decomposition (21) with the normal vectors $n_\alpha(u)$, $1 \leq \alpha \leq L$, we have

$$\left(\frac{\partial^2 r}{\partial u^i \partial u^j}, n_\alpha(u)\right) = b^\beta_{ij}(u) h_{\beta\alpha}(u), \tag{40}$$

and taking the scalar product of the Weingarten decomposition (22) with the tangent vectors $r_j(u)$, $1 \leq j \leq N$, we get

$$\left(\frac{\partial n_\alpha}{\partial u^i}, \frac{\partial r}{\partial u^j}\right) = c^k_{\alpha i}(u) g_{kj}(u). \tag{41}$$

The expressions

$$\omega_{\alpha ij}(u) = \left(\frac{\partial^2 r}{\partial u^i \partial u^j}, n_\alpha(u)\right) = b^\beta_{ij}(u) h_{\beta\alpha}(u), \qquad 1 \leq \alpha \leq L, \tag{42}$$

for any α give components of a symmetric covariant tensor and are called the *second fundamental forms of the submanifold*. To each basis vector $n_\alpha(u)$, $1 \leq \alpha \leq L$, of the normal space of the submanifold there corresponds its own second fundamental form.

Differentiating the relation

$$\left(\frac{\partial r}{\partial u^i}, n_\alpha(u)\right) = 0 \tag{43}$$

with respect to u^j, we get

$$\left(\frac{\partial^2 r}{\partial u^i \partial u^j}, n_\alpha(u)\right) + \left(\frac{\partial r}{\partial u^i}, \frac{\partial n_\alpha}{\partial u^j}\right) = 0, \tag{44}$$

or, taking into account (40) and (41),

$$b^\beta_{ij}(u) h_{\beta\alpha}(u) + c^k_{\alpha j}(u) g_{ki}(u) = 0. \tag{45}$$

Thus, the following relation always holds:

$$c^k_{\alpha i}(u) = -g^{ks}(u) b^\beta_{si}(u) h_{\beta\alpha}(u) = -g^{ks}(u) \omega_{\alpha si}(u). \tag{46}$$

For any α the coefficients $c^k_{\alpha i}(u)$, $1 \leq \alpha \leq L$, give components of a tensor of rank (1, 1) (an affinor). These affinors are called the *Weingarten operators of the submanifold*.

3.2 Consistency conditions

The consistency conditions for the Gauss decomposition (21) and the Weingarten decomposition (22) are represented by fundamental equations in submanifold theory, namely, by the Gauss equations, the Codazzi equations, and the Ricci equations. Here, we consider the consistency conditions for the Gauss decomposition (21) and the Weingarten decomposition (22) in the form necessary for us.

Differentiating the Gauss decomposition (21) with respect to u^s, we find

$$\frac{\partial^3 r}{\partial u^i \partial u^j \partial u^s} = \frac{\partial \Gamma^k_{ij}}{\partial u^s}\frac{\partial r}{\partial u^k} + \Gamma^k_{ij}(u)\frac{\partial^2 r}{\partial u^k \partial u^s} + \frac{\partial b^\beta_{ij}}{\partial u^s} n_\beta(u) + b^\beta_{ij}(u)\frac{\partial n_\beta}{\partial u^s}. \tag{47}$$

Using the Gauss decomposition (21) and the Weingarten decomposition (22), we obtain

$$\frac{\partial^3 r}{\partial u^i \partial u^j \partial u^s} = \frac{\partial \Gamma^k_{ij}}{\partial u^s}\frac{\partial r}{\partial u^k} + \Gamma^k_{ij}(u)\left(\Gamma^l_{ks}(u)\frac{\partial r}{\partial u^l} + b^\beta_{ks}(u) n_\beta(u)\right) +$$
$$+\frac{\partial b^\beta_{ij}}{\partial u^s} n_\beta(u) + b^\beta_{ij}(u)\left(c^k_{\beta s}(u)\frac{\partial r}{\partial u^k} + d^\gamma_{\beta s}(u) n_\gamma(u)\right). \tag{48}$$

The condition of symmetry with respect to the indices j and s yields

$$\frac{\partial \Gamma^k_{ij}}{\partial u^s}\frac{\partial r}{\partial u^k} + \Gamma^l_{ij}(u)\left(\Gamma^k_{ls}(u)\frac{\partial r}{\partial u^k} + b^\beta_{ls}(u) n_\beta(u)\right) +$$
$$+\frac{\partial b^\beta_{ij}}{\partial u^s} n_\beta(u) + b^\gamma_{ij}(u)\left(c^k_{\gamma s}(u)\frac{\partial r}{\partial u^k} + d^\beta_{\gamma s}(u) n_\beta(u)\right) =$$
$$= \frac{\partial \Gamma^k_{is}}{\partial u^j}\frac{\partial r}{\partial u^k} + \Gamma^l_{is}(u)\left(\Gamma^k_{lj}(u)\frac{\partial r}{\partial u^k} + b^\beta_{lj}(u) n_\beta(u)\right) +$$
$$+\frac{\partial b^\beta_{is}}{\partial u^j} n_\beta(u) + b^\gamma_{is}(u)\left(c^k_{\gamma j}(u)\frac{\partial r}{\partial u^k} + d^\beta_{\gamma j}(u) n_\beta(u)\right). \tag{49}$$

The coefficients of $\partial r/\partial u^k$ give the Gauss equations

$$\frac{\partial \Gamma^k_{ij}}{\partial u^s} - \frac{\partial \Gamma^k_{is}}{\partial u^j} + \Gamma^l_{ij}(u)\Gamma^k_{ls}(u) - \Gamma^l_{is}(u)\Gamma^k_{lj}(u) = b^\gamma_{is}(u) c^k_{\gamma j}(u) - b^\gamma_{ij}(u) c^k_{\gamma s}(u), \tag{50}$$

or

$$R^k_{isj}(u) = b^\gamma_{is}(u) c^k_{\gamma j}(u) - b^\gamma_{ij}(u) c^k_{\gamma s}(u), \tag{51}$$

where $R^k_{isj}(u)$ is the Riemannian curvature tensor of the metric $g_{ij}(u)$.

The coefficients of $n_\beta(u)$ give the Codazzi equations

$$\Gamma^l_{ij}(u) b^\beta_{ls}(u) + \frac{\partial b^\beta_{ij}}{\partial u^s} + b^\gamma_{ij}(u) d^\beta_{\gamma s}(u) = \Gamma^l_{is}(u) b^\beta_{lj}(u) + \frac{\partial b^\beta_{is}}{\partial u^j} + b^\gamma_{is}(u) d^\beta_{\gamma j}(u). \tag{52}$$

Differentiating the Weingarten decomposition (22) with respect to u^s, we have

$$\frac{\partial^2 n_\alpha}{\partial u^j \partial u^s} = \frac{\partial c^k_{\alpha j}}{\partial u^s}\frac{\partial r}{\partial u^k} + c^k_{\alpha j}(u)\frac{\partial^2 r}{\partial u^k \partial u^s} + \frac{\partial d^\beta_{\alpha j}}{\partial u^s} n_\beta(u) + d^\beta_{\alpha j}(u)\frac{\partial n_\beta}{\partial u^s}. \tag{53}$$

Using the Gauss decomposition (21) and the Weingarten decomposition (22), we get

$$\begin{aligned}\frac{\partial^2 n_\alpha}{\partial u^j \partial u^s} &= \frac{\partial c^k_{\alpha j}}{\partial u^s}\frac{\partial r}{\partial u^k} + c^k_{\alpha j}(u)\left(\Gamma^l_{ks}(u)\frac{\partial r}{\partial u^l} + b^\beta_{ks}(u) n_\beta(u)\right) + \\ &+ \frac{\partial d^\beta_{\alpha j}}{\partial u^s} n_\beta(u) + d^\beta_{\alpha j}(u)\left(c^k_{\beta s}(u)\frac{\partial r}{\partial u^k} + d^\gamma_{\beta s}(u) n_\gamma(u)\right).\end{aligned} \tag{54}$$

The condition of symmetry with respect to the indices j and s gives

$$\begin{aligned}&\frac{\partial c^k_{\alpha j}}{\partial u^s}\frac{\partial r}{\partial u^k} + c^l_{\alpha j}(u)\left(\Gamma^k_{ls}(u)\frac{\partial r}{\partial u^k} + b^\beta_{ls}(u) n_\beta(u)\right) + \\ &+ \frac{\partial d^\beta_{\alpha j}}{\partial u^s} n_\beta(u) + d^\gamma_{\alpha j}(u)\left(c^k_{\gamma s}(u)\frac{\partial r}{\partial u^k} + d^\beta_{\gamma s}(u) n_\beta(u)\right) = \\ &= \frac{\partial c^k_{\alpha s}}{\partial u^j}\frac{\partial r}{\partial u^k} + c^l_{\alpha s}(u)\left(\Gamma^k_{lj}(u)\frac{\partial r}{\partial u^k} + b^\beta_{lj}(u) n_\beta(u)\right) + \\ &+ \frac{\partial d^\beta_{\alpha s}}{\partial u^j} n_\beta(u) + d^\gamma_{\alpha s}(u)\left(c^k_{\gamma j}(u)\frac{\partial r}{\partial u^k} + d^\beta_{\gamma j}(u) n_\beta(u)\right).\end{aligned} \tag{55}$$

The coefficients of $n_\beta(u)$ give the Ricci equations

$$c^l_{\alpha j}(u) b^\beta_{ls}(u) + \frac{\partial d^\beta_{\alpha j}}{\partial u^s} + d^\gamma_{\alpha j}(u) d^\beta_{\gamma s}(u) = c^l_{\alpha s}(u) b^\beta_{lj}(u) + \frac{\partial d^\beta_{\alpha s}}{\partial u^j} + d^\gamma_{\alpha s}(u) d^\beta_{\gamma j}(u). \tag{56}$$

The coefficients of $\partial r/\partial u^k$ give the Codazzi equations

$$\frac{\partial c^k_{\alpha j}}{\partial u^s} + c^l_{\alpha j}(u)\Gamma^k_{ls}(u) + d^\gamma_{\alpha j}(u) c^k_{\gamma s}(u) = \frac{\partial c^k_{\alpha s}}{\partial u^j} + c^l_{\alpha s}(u)\Gamma^k_{lj}(u) + d^\gamma_{\alpha s}(u) c^k_{\gamma j}(u). \tag{57}$$

If relations (46), (23), (28), (33), and (38) hold, then equations (57) are equivalent to the Codazzi equations (52). Indeed, substituting $c^k_{\alpha i}(u)$ from (46) into (57), we obtain

$$\begin{aligned}&-\frac{\partial g^{ks}}{\partial u^j} b^\beta_{si}(u) h_{\beta\alpha}(u) - g^{ks}(u)\frac{\partial b^\beta_{si}}{\partial u^j} h_{\beta\alpha}(u) - g^{ks}(u) b^\beta_{si}(u)\frac{\partial h_{\beta\alpha}}{\partial u^j} - \\ &- g^{ls}(u) b^\beta_{si}(u) h_{\beta\alpha}(u)\Gamma^k_{lj}(u) - d^\beta_{\alpha i}(u) g^{ks}(u) b^\gamma_{sj}(u) h_{\gamma\beta}(u) = \\ &= -\frac{\partial g^{ks}}{\partial u^i} b^\beta_{sj}(u) h_{\beta\alpha}(u) - g^{ks}(u)\frac{\partial b^\beta_{sj}}{\partial u^i} h_{\beta\alpha}(u) - g^{ks}(u) b^\beta_{sj}(u)\frac{\partial h_{\beta\alpha}}{\partial u^i} - \\ &- g^{ls}(u) b^\beta_{sj}(u) h_{\beta\alpha}(u)\Gamma^k_{li}(u) - d^\beta_{\alpha j}(u) g^{ks}(u) b^\gamma_{si}(u) h_{\gamma\beta}(u).\end{aligned} \tag{58}$$

From the compatibility condition of the connection $\Gamma^r_{pj}(u)$ with the metric $g_{rp}(u)$ (or from relations (23), (28), and (33)), for the derivative of the contravariant metric $g^{ks}(u)$ we have

$$\frac{\partial g^{ks}}{\partial u^j} = -g^{kp}(u)\left(\Gamma^r_{pj}(u)g_{rl}(u) + \Gamma^r_{lj}(u)g_{rp}(u)\right)g^{ls}(u) =$$
$$= -\Gamma^s_{pj}(u)g^{kp}(u) - \Gamma^k_{lj}(u)g^{ls}(u). \tag{59}$$

Using relations (59) and (38), from (58) we get

$$g^{kp}(u)\left(\Gamma^r_{pj}(u)g_{rl}(u) + \Gamma^r_{lj}(u)g_{rp}(u)\right)g^{ls}(u)b^\beta_{si}(u)h_{\beta\alpha}(u) -$$
$$-g^{ks}(u)\frac{\partial b^\beta_{si}}{\partial u^j}h_{\beta\alpha}(u) - g^{ks}(u)b^\beta_{si}(u)\left(d^\gamma_{\alpha j}(u)h_{\gamma\beta}(u) + d^\gamma_{\beta j}(u)h_{\gamma\alpha}(u)\right) -$$
$$-g^{ls}(u)b^\beta_{si}(u)h_{\beta\alpha}(u)\Gamma^k_{lj}(u) - d^\beta_{\alpha i}(u)g^{ks}(u)b^\gamma_{sj}(u)h_{\gamma\beta}(u) =$$
$$= g^{kp}(u)\left(\Gamma^r_{pi}(u)g_{rl}(u) + \Gamma^r_{li}(u)g_{rp}(u)\right)g^{ls}(u)b^\beta_{sj}(u)h_{\beta\alpha}(u) -$$
$$-g^{ks}(u)\frac{\partial b^\beta_{sj}}{\partial u^i}h_{\beta\alpha}(u) - g^{ks}(u)b^\beta_{sj}(u)\left(d^\gamma_{\alpha i}(u)h_{\gamma\beta}(u) + d^\gamma_{\beta i}(u)h_{\gamma\alpha}(u)\right) -$$
$$-g^{ls}(u)b^\beta_{sj}(u)h_{\beta\alpha}(u)\Gamma^k_{li}(u) - d^\beta_{\alpha j}(u)g^{ks}(u)b^\gamma_{si}(u)h_{\gamma\beta}(u), \tag{60}$$

or

$$\Gamma^s_{pj}(u)b^\beta_{si}(u) - \frac{\partial b^\beta_{pi}}{\partial u^j} - b^\gamma_{pi}(u)d^\beta_{\gamma j}(u) = \Gamma^s_{pi}(u)b^\beta_{sj}(u) - \frac{\partial b^\beta_{pj}}{\partial u^i} - b^\gamma_{pj}(u)d^\beta_{\gamma i}(u), \tag{61}$$

i.e., the Codazzi equations (52).

3.3 Bonnet theorem

For totally nonisotropic submanifolds in pseudo-Euclidean spaces, an analog of the classical Bonnet theorem holds. Let a pseudo-Riemannian metric $g_{ij}(u)du^idu^j$, symmetric 2-forms $\omega_{\alpha ij}(u)du^idu^j$, $\omega_{\alpha ij}(u) = \omega_{\alpha ji}(u)$, $1 \le \alpha \le L$, 1-forms $\varkappa_{\alpha\beta i}(u)du^i$, $1 \le \alpha, \beta \le L$, and functions $h_{\alpha\beta}(u)$, $1 \le \alpha, \beta \le L$, such that $\det h_{\alpha\beta}(u) \neq 0$ and $h_{\alpha\beta}(u) = h_{\beta\alpha}(u)$, $1 \le \alpha, \beta \le L$, be locally given. If in this case relations (39) as well as the Gauss equations (51), the Codazzi equations (52) and the Ricci equations (56) are satisfied for the forms $g_{ij}(u)$, $\omega_{\alpha ij}(u)$, $\varkappa_{\alpha\beta i}(u)$ and the functions $h_{\alpha\beta}(u)$ (the coefficients $b^\beta_{ij}(u)$, $c^k_{\alpha i}(u)$, and $d^\beta_{\alpha j}(u)$ are uniquely determined by formulae (42), (46), and (35), respectively), then there exists a unique (up to motions in the ambient pseudo-Euclidean space) smooth totally nonisotropic N-dimensional submanifold M^N with the first fundamental form $ds^2 = g_{ij}(u)du^idu^j$, the Gram matrix $h_{\alpha\beta}(u)$, $1 \le \alpha, \beta \le L$, of scalar products of the basis vectors in the normal spaces, the second fundamental forms $\omega_{\alpha ij}(u)du^idu^j$, and the torsion forms $\varkappa_{\alpha\beta i}(u)du^i$ in an $(N+L)$-dimensional pseudo-Euclidean space, the signature of which is determined by the sum of the signatures of the metrics $g_{ij}(u)$, $1 \le i, j \le N$, and $h_{\alpha\beta}(u)$, $1 \le \alpha, \beta \le L$.

4 Submanifolds with zero torsion in pseudo-Euclidean spaces

Let us consider the class of totally nonisotropic smooth N-dimensional submanifolds with *zero torsion* in $(N+L)$-dimensional pseudo-Euclidean spaces; i.e., all the torsion forms $d^{\beta}_{\alpha i}(u)du^i$, $1 \le \alpha, \beta \le L$, of submanifolds of this class vanish, $d^{\beta}_{\alpha i}(u) = 0$, $1 \le \alpha, \beta \le L$, $1 \le i \le N$, for the chosen bases in the normal spaces. In this case it follows immediately from relations (38) that the functions $h_{\alpha\beta}(u)$, $1 \le \alpha, \beta \le L$, must be constant: $h_{\alpha\beta}(u) = \mu_{\alpha\beta}$, $\mu_{\alpha\beta} =$ const, where $\mu_{\alpha\beta} = \mu_{\beta\alpha}$ and $\det(\mu_{\alpha\beta}) \neq 0$ by virtue of the definition of these functions. Relations (38) hold in this case. Note that if the functions $h_{\alpha\beta}(u)$, $1 \le \alpha, \beta \le L$, are constant, then relations (38) are equivalent to the condition

$$d^{\gamma}_{\alpha i}(u)h_{\gamma\beta}(u) + d^{\gamma}_{\beta i}(u)h_{\gamma\alpha}(u) = 0, \tag{62}$$

i.e., the skew-symmetry condition of the torsion 1-forms $\varkappa_{\alpha\beta i}(u)du^i = d^{\gamma}_{\alpha i}(u)h_{\gamma\beta}(u)du^i$ with respect to the indices α and β: $\varkappa_{\alpha\beta i}(u) = -\varkappa_{\beta\alpha i}(u)$. The converse is also true; i.e., the functions $h_{\alpha\beta}(u)$, $1 \le \alpha, \beta \le L$, are constant if and only if the torsion 1-forms $\varkappa_{\alpha\beta i}(u)du^i$ are skew-symmetric with respect to the indices α and β.

For submanifolds with zero torsion, the following relations hold:

$$c^k_{\alpha i}(u) = -g^{ks}(u)b^{\beta}_{si}(u)\mu_{\beta\alpha}, \qquad \omega_{\alpha ij}(u) = b^{\beta}_{ij}(u)\mu_{\beta\alpha}, \tag{63}$$

the Gauss equations

$$R^k_{isj}(u) = b^{\gamma}_{is}(u)c^k_{\gamma j}(u) - b^{\gamma}_{ij}(u)c^k_{\gamma s}(u), \tag{64}$$

the Codazzi equations

$$\Gamma^l_{ij}(u)b^{\beta}_{lk}(u) + \frac{\partial b^{\beta}_{ij}}{\partial u^k} = \Gamma^l_{ik}(u)b^{\beta}_{lj}(u) + \frac{\partial b^{\beta}_{ik}}{\partial u^j}, \tag{65}$$

and the Ricci equations

$$c^l_{\alpha j}(u)b^{\beta}_{ls}(u) = c^l_{\alpha s}(u)b^{\beta}_{lj}(u). \tag{66}$$

The Codazzi equations (65) can be rewritten in the form

$$\nabla_k b^{\alpha}_{ij} = \nabla_j b^{\alpha}_{ik} \tag{67}$$

or

$$\nabla_k \omega_{\alpha ij} = \nabla_j \omega_{\alpha ik}, \tag{68}$$

where ∇ is the covariant derivative generated by the Levi-Civita connection of the first fundamental form $g_{ij}(u)$.

Using relation (63), one can rewrite the Gauss equations (64) in the form

$$R_{ijkl}(u) = b^{\alpha}_{ik}(u)\mu_{\alpha\beta}b^{\beta}_{lj}(u) - b^{\alpha}_{jk}(u)\mu_{\alpha\beta}b^{\beta}_{li}(u) \tag{69}$$

or

$$R^{ik}_{sj}(u) = c^i_{\alpha j}(u)\mu^{\alpha\beta}c^k_{\beta s}(u) - c^i_{\alpha s}(u)\mu^{\alpha\beta}c^k_{\beta j}(u), \tag{70}$$

and also

$$R_{ijkl}(u) = \omega_{\alpha ik}(u)\mu^{\alpha\beta}\omega_{\beta lj}(u) - \omega_{\alpha jk}(u)\mu^{\alpha\beta}\omega_{\beta li}(u), \tag{71}$$

where the matrix $\mu^{\alpha\beta}$ is the inverse of the matrix $\mu_{\alpha\beta}$: $\mu^{\alpha\gamma}\mu_{\gamma\beta} = \delta^\alpha_\beta$, and the Ricci equations (66) take the form

$$b^\alpha_{ik}(u)g^{kl}(u)b^\beta_{lj}(u) - b^\alpha_{jk}(u)g^{kl}(u)b^\beta_{li}(u) = 0 \tag{72}$$

or

$$c^i_{\alpha j}(u)g_{ik}(u)c^k_{\beta s}(u) - c^i_{\alpha s}(u)g_{ik}(u)c^k_{\beta j}(u) = 0, \tag{73}$$

and also

$$\omega_{\alpha ik}(u)g^{kl}(u)\omega_{\beta lj}(u) - \omega_{\alpha jk}(u)g^{kl}(u)\omega_{\beta li}(u) = 0. \tag{74}$$

4.1 Flat submanifolds with zero torsion

Now we consider flat submanifolds with zero torsion in pseudo-Euclidean spaces, i.e., torsionless submanifolds with flat metrics, namely, with flat first fundamental forms $g_{ij}(u)$ on the submanifolds. In this case, we can consider that $u = (u^1, \ldots, u^N)$ are some flat coordinates of the metric $g_{ij}(u)$. In flat coordinates the metric is a constant nondegenerate symmetric matrix η_{ij}, $\eta_{ij} =$ const, $\det(\eta_{ij}) \neq 0$, $\eta_{ij} = \eta_{ji}$, and the Codazzi equations (65), (67) take the form

$$\frac{\partial b^\alpha_{ij}}{\partial u^k} = \frac{\partial b^\alpha_{ik}}{\partial u^j} \tag{75}$$

or

$$\frac{\partial \omega_{\alpha ij}}{\partial u^k} = \frac{\partial \omega_{\alpha ik}}{\partial u^j}. \tag{76}$$

Hence, there locally exist some functions $\psi_\alpha(u)$, $1 \leq \alpha \leq L$, such that

$$\omega_{\alpha ij}(u) = \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^j}. \tag{77}$$

We have thus proved the following important lemma.

Lemma 4.1 [17] *All the second fundamental forms of each flat torsionless submanifold in a pseudo-Euclidean space are locally Hessians in any flat coordinates on the submanifold.*

It follows from Lemma 4.1 that in any flat coordinates the Gauss equations (71) have the form

$$\sum_{\alpha=1}^{L}\sum_{\beta=1}^{L} \mu^{\alpha\beta}\left(\frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^k}\frac{\partial^2 \psi_\beta}{\partial u^j \partial u^l} - \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^l}\frac{\partial^2 \psi_\beta}{\partial u^j \partial u^k}\right) = 0, \tag{78}$$

and the Ricci equations (74) have the form

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\eta^{ij}\left(\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^k}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^l}-\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^l}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^k}\right)=0, \tag{79}$$

where the matrix η^{ij} is the inverse of the matrix η_{ij}: $\eta^{is}\eta_{sj}=\delta^i_j$.

Theorem 4.1 [1], [17], [18] *The class of N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces is described (in flat coordinates) by the system of nonlinear equations* (78), (79) *for functions $\psi_\alpha(u)$, $1\leq\alpha\leq L$. Here, η^{ij}, $1\leq i,j\leq N$, and $\mu^{\alpha\beta}$, $1\leq\alpha,\beta\leq L$, are arbitrary constant nondegenerate symmetric matrices, $\eta^{ij}=\eta^{ji}$, $\eta^{ij}=\mathrm{const}$, $\det(\eta^{ij})\neq 0$, $\mu^{\alpha\beta}=\mathrm{const}$, $\mu^{\alpha\beta}=\mu^{\beta\alpha}$, $\det(\mu^{\alpha\beta})\neq 0$; the signature of the ambient $(N+L)$-dimensional pseudo-Euclidean space is the sum of the signatures of the metrics η^{ij} and $\mu^{\alpha\beta}$; $\mathbf{I}=ds^2=\eta_{ij}du^idu^j$ is the first fundamental form, where η_{ij} is the inverse of the matrix η^{ij}, $\eta^{is}\eta_{sj}=\delta^i_j$, and $\mathbf{II}_\alpha=(\partial^2\psi_\alpha/(\partial u^i\partial u^j))du^idu^j$, $1\leq\alpha\leq L$, are the second fundamental forms given by the Hessians of the functions $\psi_\alpha(u)$, $1\leq\alpha\leq L$, for the corresponding flat torsionless submanifold determined by an arbitrary solution of the system of nonlinear equations* (78), (79).

According to the Bonnet theorem, any solution $\psi_\alpha(u)$, $1\leq\alpha\leq L$, of the nonlinear system (78), (79) determines a unique (up to motion in the ambient pseudo-Euclidean space) totally nonisotropic N-dimensional flat torsionless submanifold in the corresponding $(N+L)$-dimensional pseudo-Euclidean space with the first fundamental form $\eta_{ij}du^idu^j$ and the second fundamental forms $\omega_\alpha(u)=(\partial^2\psi_\alpha/(\partial u^i\partial u^j))du^idu^j$, $1\leq\alpha\leq L$, given by the Hessians of the functions $\psi_\alpha(u)$, $1\leq\alpha\leq L$, and the constant Gram matrix $\mu_{\alpha\beta}$, $1\leq\alpha,\beta\leq L$, of the scalar products of basic vectors in the normal spaces. It is obvious that we can always add arbitrary terms linear in the coordinates $u^1,\ldots,u^N$ to any solution of the system (78), (79), but the set of the second fundamental forms and the corresponding submanifold will remain the same. Moreover, any two sets of the second fundamental forms of the form $\omega_{\alpha ij}(u)=\partial^2\psi_\alpha/(\partial u^i\partial u^j)$, $1\leq\alpha\leq L$, coincide if and only if the corresponding functions $\psi_\alpha(u)$, $1\leq\alpha\leq L$, coincide up to terms linear in the coordinates; hence we must not distinguish here solutions of the nonlinear system (78), (79) that differ by terms linear in the coordinates $u^1,\ldots,u^N$.

We consider the following linear problem with parameters for vector functions $\partial a(u)/\partial u^i$, $1\leq i\leq N$, and $b_\alpha(u)$, $1\leq\alpha\leq L$:

$$\frac{\partial^2 a}{\partial u^i\partial u^j}=\lambda\,\mu^{\alpha\beta}\omega_{\alpha ij}(u)b_\beta(u),\qquad \frac{\partial b_\alpha}{\partial u^i}=\rho\,\eta^{kj}\omega_{\alpha ij}(u)\frac{\partial a}{\partial u^k}, \tag{80}$$

where η^{ij}, $1\leq i,j\leq N$, and $\mu^{\alpha\beta}$, $1\leq\alpha,\beta\leq L$, are arbitrary constant nondegenerate symmetric matrices, $\eta^{ij}=\eta^{ji}$, $\eta^{ij}=\mathrm{const}$, $\det(\eta^{ij})\neq 0$, $\mu^{\alpha\beta}=\mathrm{const}$, $\mu^{\alpha\beta}=\mu^{\beta\alpha}$, $\det(\mu^{\alpha\beta})\neq 0$; λ and ρ are arbitrary constants (parameters) [1]. (In fact, only one of the parameters λ and ρ is essential.) It is obvious that the coefficients $\omega_{\alpha ij}(u)$, $1 \leq \alpha \leq L$, here must be symmetric matrix functions, $\omega_{\alpha ij}(u) = \omega_{\alpha ji}(u)$.

The consistency conditions for the linear system (80) are equivalent to the nonlinear system (78), (79) describing the class of N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces.

Theorem 4.2 [1] *The nonlinear system* (78), (79) *is integrable by the inverse scattering method.*

Definition 4.1 A class of submanifolds in a Euclidean or pseudo-Euclidean space is called *integrable* if the system of the fundamental Gauss–Codazzi–Ricci equations giving this class of submanifolds is integrable.

Essentially, the theory of integrable classes of surfaces in $\mathbb{E}^3$ goes back to the classical differential geometry of the XIX century, when there were established remarkable properties of some nonlinear partial differential equations (in particular, the sine-Gordon equation and the Liouville equation) describing some important classes of surfaces in $\mathbb{E}^3$. From the modern viewpoint, after the methods of the soliton theory were discovered and worked out and the theory of integrable nonlinear partial differential equations was developed in the second half of the XX century, it is clear that these properties indicate the integrability of these nonlinear equations by the inverse scattering method. In connection with the rapid and intensive development of the theory of integrable systems, integrable classes of surfaces have been considered and studied in many papers; in particular, we mention the cycle of Sym's papers (see [19], [20]) and also the papers [21]–[24]. We also note that the considered notion of integrability concerns only classes of surfaces or submanifolds and makes no sense for a single surface or submanifold. In particular, the definition of an *integrable surface* via the integrability of *its* Gauss–Codazzi–Ricci equations in [23] and [24] is quite absurd, since for any surface *its* Gauss–Codazzi–Ricci equations are always satisfied identically. In the context of the integrability of the Gauss–Codazzi–Ricci equations, one can only speak of integrable classes of surfaces and of whether a given surface belongs to a certain integrable class of surfaces, but not of integrable Gauss–Codazzi–Ricci equations of a concrete surface. Of course, a concrete surface (or a submanifold) can belong to various integrable and nonintegrable classes.

Theorem 4.3 *The class of flat torsionless submanifolds in any Euclidean or pseudo-Euclidean space is integrable.*

5 k-potential reductions and k-potential submanifolds in pseudo-Euclidean spaces

Consider the case when $L = kN + p$, where k is an arbitrary positive integer and p is an arbitrary nonnegative integer, $p \geq 0$.

In this case, the Gauss equations (78) and the Ricci equations (79) can be rewritten in the following form:

$$\sum_{r=1}^{k}\sum_{s=1}^{k}\sum_{\alpha=(r-1)N+1}^{rN}\sum_{\beta=(s-1)N+1}^{sN}\mu^{\alpha\beta}\left(\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^q}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^l}-\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^l}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^q}\right)+$$
$$+\sum_{\alpha=kN+1}^{L}\sum_{\beta=1}^{L}\mu^{\alpha\beta}\left(\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^q}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^l}-\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^l}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^q}\right)+$$
$$+\sum_{\alpha=1}^{kN}\sum_{\beta=kN+1}^{L}\mu^{\alpha\beta}\left(\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^q}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^l}-\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^l}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^q}\right)=0, \tag{81}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\eta^{ij}\left(\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^q}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^l}-\frac{\partial^2\psi_\alpha}{\partial u^i\partial u^l}\frac{\partial^2\psi_\beta}{\partial u^j\partial u^q}\right)=0. \tag{82}$$

We consider a special *k-potential* ansatz for the functions $\psi_\alpha(u)$, $1\le\alpha\le L$. Consider an arbitrary function $\Phi(u)$ and define the functions $\psi_\alpha(u)$, $1\le\alpha\le L$, as follows:

$$\psi_{(s-1)N+i}=\frac{\partial\Phi}{\partial u^i},\qquad 1\le s\le k,\ 1\le i\le N, \tag{83}$$

and $\psi_\alpha(u)$, $kN+1\le\alpha\le L$, are arbitrary functions that are linear in the coordinates (the corresponding second fundamental forms vanish). In this case, the Gauss equations (81) can be rewritten in the form

$$\sum_{r,s=1}^{k}\sum_{\alpha=(r-1)N+1}^{rN}\sum_{\beta=(s-1)N+1}^{sN}\mu^{\alpha\beta}\left(\frac{\partial^3\Phi}{\partial u^{\alpha-(r-1)N}\partial u^i\partial u^q}\frac{\partial^3\Phi}{\partial u^{\beta-(s-1)N}\partial u^j\partial u^l}-\right.$$
$$\left.-\frac{\partial^3\Phi}{\partial u^{\alpha-(r-1)N}\partial u^i\partial u^l}\frac{\partial^3\Phi}{\partial u^{\beta-(s-1)N}\partial u^j\partial u^q}\right)=0 \tag{84}$$

or

$$\sum_{r,s=1}^{k}\sum_{m,n=1}^{N}\mu^{(r-1)N+m,(s-1)N+n}\left(\frac{\partial^3\Phi}{\partial u^m\partial u^i\partial u^q}\frac{\partial^3\Phi}{\partial u^n\partial u^j\partial u^l}-\frac{\partial^3\Phi}{\partial u^m\partial u^i\partial u^l}\frac{\partial^3\Phi}{\partial u^n\partial u^j\partial u^q}\right)=0. \tag{85}$$

The Ricci equations (82) in this case take the form

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\eta^{ij}\left(\frac{\partial^3\Phi}{\partial u^m\partial u^i\partial u^q}\frac{\partial^3\Phi}{\partial u^n\partial u^j\partial u^l}-\frac{\partial^3\Phi}{\partial u^m\partial u^i\partial u^l}\frac{\partial^3\Phi}{\partial u^n\partial u^j\partial u^q}\right)=0; \tag{86}$$

i.e., they coincide with the associativity equations of two-dimensional topological quantum field theories (13) (the WDVV equations, see [2]–[5]).

We consider now a special ansatz for the constant Gram matrices $\mu_{\alpha\beta}$, $1\le\alpha,\beta\le L$, in the normal spaces of the submanifolds. Consider the case when

$$\mu^{(r-1)N+m,(s-1)N+n}=c^{rs}\eta^{mn},\qquad 1\le r,s\le k,\ 1\le m,n\le N, \tag{87}$$

where c^{rs} is an arbitrary nondegenerate symmetric constant matrix: $c^{rs} = c^{sr}$, $\det(c^{rs}) \neq 0$, $c^{rs} =$ const, and the other elements of the matrix $\mu^{\alpha\beta}$ (for $\alpha \geq kN+1$ or $\beta \geq kN+1$) are arbitrary constants such that the matrix $\mu^{\alpha\beta}$ is symmetric and nondegenerate. For such special constant Gram matrices $\mu_{\alpha\beta}$ (see (87)) the Gauss equations (85) take the form

$$\sum_{r,s=1}^{k} \sum_{m,n=1}^{N} c^{rs} \eta^{mn} \left(\frac{\partial^3 \Phi}{\partial u^m \partial u^i \partial u^q} \frac{\partial^3 \Phi}{\partial u^n \partial u^j \partial u^l} - \frac{\partial^3 \Phi}{\partial u^m \partial u^i \partial u^l} \frac{\partial^3 \Phi}{\partial u^n \partial u^j \partial u^q} \right) = 0; \qquad (88)$$

i.e., in this case, the Gauss equations (88) are a linear combination of the Ricci equations (86) (the associativity equations (13)). Thus, in this case, all the fundamental relations and equations of submanifold theory reduce to the associativity equations (13). We will call such special reductions of the fundamental Gauss–Codazzi–Ricci equations and relations of submanifold theory *k-potential*.

Theorem 5.1 *The associativity equations of two-dimensional topological quantum field theories* (13) *are natural k-potential reductions of the fundamental equations of submanifold theory.*

By the Bonnet theorem, for any nondegenerate symmetric constant matrix c^{rs}, $1 \leq r, s \leq k$, $c^{rs} = c^{sr}$, $\det(c^{rs}) \neq 0$, $c^{rs} =$ const, any nondegenerate symmetric constant matrix η^{ij}, $1 \leq i, j \leq N$, $\eta^{ij} = \eta^{ji}$, $\eta^{ij} =$ const, $\det(\eta^{ij}) \neq 0$, and any nondegenerate symmetric constant matrix $\mu^{\alpha\beta}$, $1 \leq \alpha, \beta \leq L$, $\mu^{\alpha\beta} =$ const, $\mu^{\alpha\beta} = \mu^{\beta\alpha}$, $\det(\mu^{\alpha\beta}) \neq 0$, such that relations (87) hold, any solution $\Phi(u)$ of the associativity equations (13) that is determined up to quadratic terms gives a unique (up to motion in the ambient pseudo-Euclidean space) totally nonisotropic N-dimensional flat torsionless submanifold with the first fundamental form $ds^2 = \eta_{ij} du^i du^j$, the second fundamental forms

$$\omega_{(s-1)N+m, ij}(u) du^i du^j = \frac{\partial^3 \Phi}{\partial u^m \partial u^i \partial u^j} du^i du^j, \qquad 1 \leq s \leq k, \ 1 \leq i, j, m \leq N,$$

$$\omega_{p\,ij}(u) du^i du^j = 0, \qquad kN+1 \leq p \leq L, \ 1 \leq i, j \leq N,$$

and the constant Gram matrix $\mu_{\alpha\beta}$, $1 \leq \alpha, \beta \leq L$, of the scalar products of basic vectors in the normal spaces in an $(N+L)$-dimensional pseudo-Euclidean space whose signature is the sum of the signatures of the metrics η_{ij}, $1 \leq i, j \leq N$, and $\mu_{\alpha\beta}$, $1 \leq \alpha, \beta \leq L$.

We will call such submanifolds parametrized by the special constant Gram matrices $\mu_{\alpha\beta}$, $1 \leq \alpha, \beta \leq L$ (see (87)), and solutions of the associativity equations (13) *k-potential*.

Theorem 5.2 *The class of k-potential submanifolds in any Euclidean or pseudo-Euclidean space is integrable.*

Theorem 5.3 *On each k-potential submanifold in a pseudo-Euclidean space, there are k natural identical structures of Frobenius algebras (k identical Dubrovin–Frobenius structures) given (in flat coordinates) for each s, $1 \leq s \leq k$, by the first fundamental form η_{ij} and the Weingarten operators $(A_{(s-1)N+m})^i_j(u) = -\eta^{il}\omega_{(s-1)N+m,lj}(u)$, $1 \leq i, j, l, m \leq N$:*

$$\langle e_i, e_j \rangle = \eta_{ij}, \qquad e_i \circ e_j = c^l_{ij}(u) e_l, \qquad e_i = \frac{\partial}{\partial u^i},$$
$$c^l_{mj}(u^1, \ldots, u^N) = -(A_{(s-1)N+m})^l_j(u) = \eta^{li}\omega_{(s-1)N+m,ij}(u^1, \ldots, u^N), \qquad (89)$$

where $\omega_{n\,ij}(u) du^i du^j$, $1 \leq n \leq kN$, are the second fundamental forms of the submanifold.

Theorem 5.4 *Each N-dimensional Frobenius manifold can be locally represented as a k-potential N-dimensional submanifold in a $((k+1)N + p)$-dimensional pseudo-Euclidean space (for an arbitrary positive integer k and an arbitrary nonnegative integer p).*

We note that the set of admissible signatures of the ambient pseudo-Euclidean space can be easily determined by the signature of the metric η_{ij}, $1 \leq i, j \leq N$, of the Frobenius manifold and by the given integers k and p (this set is never empty). Let $2s - N$ be the signature of the metric η_{ij} of the Frobenius manifold, where s, $0 \leq s \leq N$, is the positive index of inertia of the metric. Then the set of admissible signatures of the ambient pseudo-Euclidean space is determined by the formula $(2s - N)(2r - k + 1) + 2t - p$, $0 \leq r \leq k$, $0 \leq t \leq p$. In particular, in the simplest case when $p = 0$ and $k = 1$, only the signatures $2(2s - N)$ and 0 are admissible.

Theorem 5.5 *For an arbitrary Frobenius manifold that is given locally by a solution $\Phi(u^1, \ldots, u^N)$ of the associativity equations (13), the corresponding k-potential submanifolds in $((k+1)N + p)$-dimensional pseudo-Euclidean spaces that realize this Frobenius manifold are determined by any $((k+1)N + p)$-component vector function $r(u^1, \ldots, u^N)$ satisfying the following consistent linear system of second-order partial differential equations:*

$$\frac{\partial^2 r}{\partial u^i \partial u^j} = \sum_{r,s=1}^{k} \sum_{m,l=1}^{N} c^{rs}\eta^{ml} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^m} n_{(s-1)N+l}(u) +$$
$$+ \sum_{\beta=kN+1}^{kN+p} \sum_{r=1}^{k} \sum_{m=1}^{N} \mu^{(r-1)N+m,\beta} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^m} n_\beta(u), \qquad 1 \leq i, j \leq N, \qquad (90)$$

$$\frac{\partial n_{(r-1)N+m}}{\partial u^i} = -\sum_{l,j=1}^{N} \eta^{jl} \frac{\partial^3 \Phi}{\partial u^l \partial u^m \partial u^i} \frac{\partial r}{\partial u^j}, \qquad 1 \leq r \leq k, \; 1 \leq i, m \leq N, \qquad (91)$$

$$\frac{\partial n_\alpha}{\partial u^i} = 0, \qquad kN + 1 \leq \alpha \leq kN + p, \; 1 \leq i \leq N, \qquad (92)$$

where $n_\alpha(u^1, \dots, u^N)$, $1 \le \alpha \le kN + p$, *are some* $((k+1)N + p)$*-component vector functions. The consistency conditions for the linear system* (90)–(92) *are equivalent to the associativity equations* (13) *for the function* $\Phi(u)$.

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