Semiclassical description of quantum perturbations

Diego A. Wisniacki[1],* Natalia Ares[1], and Eduardo Vergini [2]
[1]*Departamento de Física, FCEyN, UBA, Ciudad Universitaria, Buenos Aires C1428EGA, Argentina* [2]*Departamento de Física, CNEA, Av. Libertador 8250, Buenos Aires C1429BNP, Argentina*
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The effect of a perturbation over a quantum system is described by the local density of states (LDOS), a distribution of the overlaps square between the unperturbed and perturbed eigenstates. Its witdh measures the spread of an unperturbed state in the perturbed basis and it is related to fundamental problems such as the sensitivity of quantum evolutions or dissipation when the perturbation varies with time. We derive a semiclassical expression for the width of the LDOS, σ_{sc}, for generic chaotic systems. We show that σ_{sc} is very accurate to describe the width of the LDOS of paradigmatic systems of quantum chaos as the cat maps and the Stadium billiard.

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The action of a perturbation on eigenenergies and eigenfunction of a quantum system has been a subject of paramount importance since the beginning of quantum theory. Its understanding is in the heart of fundamental problems of quantum mechanics as dissipation, phase transition or irreversibility. The usual perturbation theory is a good starting point that describe successfully this effect when the perturbation is small. However, approximated theories usually fail for strong perturbations and highly demanding computational methods are needed to compute quantities that describe characteristics of the perturbed system.

The local density of states (LDOS) or the *strenght function* is a widely studied magnitude to characterize the effect of perturbations on quantum systems [1, 2, 3, 4]. It is a distribution of the overlaps square between the unperturbed and perturbed eigenstates. Furthermore, LDOS has been extensively computed for different systems and perturbations. Let consider a system that is described by a parameter dependent Hamiltonian $H(x)$ with eigenenergies $E_j(x)$ and eigenstates $\phi_j(x)$, the LDOS for an eigenstate i at $x = x_0$ (that we call unperturbed) is defined as,

$$\rho_i(E, \delta x) = \sum_j |\langle \phi_j(x)|\phi_i(x_0)\rangle|^2 \delta(E - \Delta E_{ij}), \tag{1}$$

where $\delta x = x - x_0$ and $\Delta E_{ij} = E_j(x) - E_i(x_0)$. To avoid a dependence on the particular characteristics of the unperturbed state it frequently refers to an average taken over a set of unperturbed states in a considered energy region.

The LDOS is related with important measures of irreversibility and sensitivity to perturbations in quantum systems as the survival probability and the Loschmidt echo (LE) [6, 7, 8]. In fact, the LDOS is the Fourier tranform of the survival probability [9] and its width gives the decay rate of the LE for a small enough strength of the perturbation [10, 11]. These relations are exploited in this letter to derive a semiclassical expression of the witdth σ_{sc} of the LDOS of chaotic systems. We test the ability of σ_{sc} to describe quantitatively the witdth of the LDOS in cat maps and the Bunimovich stadium billiard. We show that σ_{sc} works very well in both systems. Although σ_{sc} is derived for local perturbation we show that it also works for global perturbations. This unexpected behavior is explain using very reasonable assumptions.

We start with an average of Eq. 1 of M unperturbed states,

$$\bar{\rho}(E, \delta x) = \frac{1}{M} \sum_i \rho_i(E, \delta x). \tag{2}$$

The Fourier tranform of the previous equation gives,

$$\mathcal{F}[\bar{\rho}(E, \delta x)] = \frac{1}{M} \sum_i \langle \phi_i(x_0)| e^{iH(x)t/\hbar} e^{-iE_i(x_0)t/\hbar} |\phi_i(x_0)\rangle, \tag{3}$$

where the sum run over the so-called survival probability which is the amplitude fidelity of an eigenstate.

Let us evaluate the previous sum semiclassically. Vanícek has proposed a semiclassical approximation of the amplitude fidelity, called *dephasing representation* [13], by assuming a classically small perturbation in such a way that the *shadowing* theorem [12] is valid. Using such an approximation, Eq. 3 resulting in,

$$\mathcal{F}[\bar{\rho}(E, \delta x)] \approx \int dq dp W \exp[-i\Delta S_t(q, p)/\hbar], \tag{4}$$

where $\Delta S_t(q, p)$ is the action difference evaluated along the unperturbed orbit starting at (q, p) that evolves a time t. Moreover, $W = (1/M)\sum_i W_i(q, p)$, being $W_i(q, p)$ the Wigner function of $\phi_i(x)$. In chaotic systems, W takes an uniform value.

In the case of a local perturbations, the right hand of Eq. 4 has been evaluated on a Poincaré surface of section by Goussev *et. al* [14] resulting,

$$\mathcal{F}[\bar{\rho}(E, \delta x)] = e^{-\gamma|t|} \tag{5}$$

*Electronic address: wisniacki@df.uba.ar

where,

$$\gamma \equiv \alpha(1 - (\langle e^{i\Delta S(q,p)/\hbar}\rangle)). \quad (6)$$

The average is evaluated in the region of the surface of section where the local perturbation acts. For numerical comparation we will consider that,

$$\langle e^{-i\Delta S^{\delta x}(q,p)/\hbar}\rangle \equiv \frac{1}{\alpha}\int_{p_1}^{p_2}\int_{q_1}^{q_2} e^{-i\Delta S^{\delta x}(q_0,p_0)/\hbar} dq dp, \quad (7)$$

where p_1, p_2, q_1 and q_2 stands for the limits of the perturbed portion in the surface of section of area α . Finally, the Fourier tranform of the exponentially decaying $\mathcal{F}[\bar{\rho}(E, \delta x)]$ is the Lorentzian function,

$$L(\beta, x) = \frac{\beta}{\pi(x^2 + \beta^2)} \quad (8)$$

with $\beta = \Re[\gamma]$. Then, the semiclassical approximation of the width of the LDOS results,

$$\sigma_{sc} \equiv \tan(\frac{7}{20}\pi)\Re[\gamma] \approx 1.963\Re[\gamma] \quad (9)$$

where the fraction 7/20 is related with our definition of the width of a lorentzian probability distribution.

We stress that this semiclassical approximation σ_{sc} was derived in the limit of $\alpha \to 0$. However, we will see that by using a very razonable assumtions it can be extended to arbitrary values of α. Let us consider the following basis sets: $\{\phi_i^{(0)}\}$ be the set of unperturbed eigenfunctions, $\{\phi_i^{(1)}\}$ be the set resulting after applying a perturbation α_1, and $\{\phi_i^{(2)}\}$ be the set resulting after applying the perturbation α_2 to the previous system (the one with the perturbation α_1). Assuming that α_1 and α_2 are small perturbations over disjoined regions of phase space, we conclude their LDOS as been uncorrelated. Therefore, the LDOS connecting the first and third basis sets is obtained by the convolution of the other two ones. The resulting LDOS is also a Lorentzian function taken into account that they are Lorentzian functions,

$$\int L(\beta_1, y)L(\beta_2, x-y)\mathrm{d}y = L(\beta_1+\beta_2, x), \quad (10)$$

with $\beta_1 = \Re[\gamma_1]$ being the parameter resulting from the first perturbation (idem for β_2). By noticing that γ is proportional to α we arrive to the conclusion that eq. (9) works for global perturbations too.

Let us now show the power of the semiclassical approximation σ_{sc} to describe the width of the LDOS. We consider first the cat map, a canonical example in classical and quantum chaos studies. Cat maps are linear automorphisms of the torus that exhibit hard chaos. In particular, we use the cat map perturbed with a nonlinear shear in momentum,

$$\begin{array}{rcl} q' & = & 2q + p \\ p' & = & 3q + 2p + \epsilon(q) \end{array} \pmod 1,$$

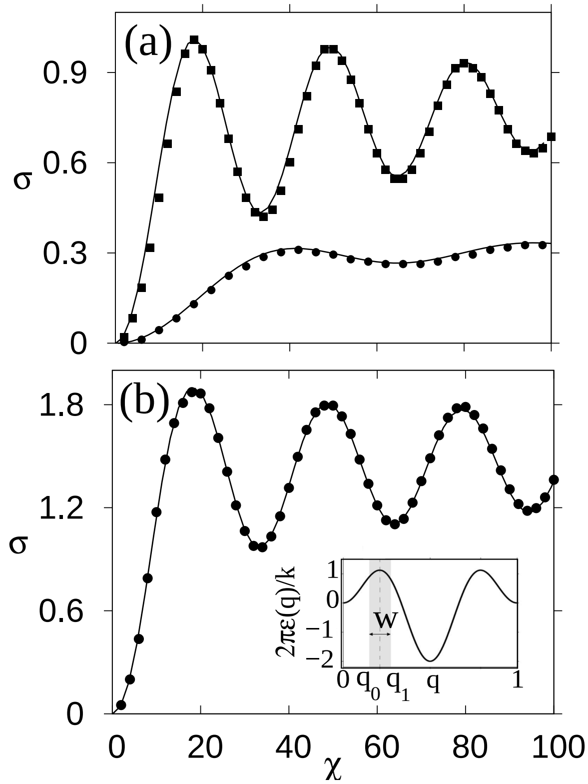

FIG. 1: Width σ of LDOS as a function of the scaled perturbation strength $\chi \equiv \frac{k}{2\pi\hbar} = kN$ for a local perturbation. Solid symbols corresponds to the quantum case and the solid line corresponds to the semiclassical calculation σ_{sc}. The size of the Hilbert space $N = 800$. In all the cases $q_0 = 0.01$. In panel (a) the shear in momentum is applied in a region of width $w = 0.2$ (circles) and $w = 0.4$ (squares). In panel (b) $w = 0.7$. Inset: Schematic figure showing the local perturbation that was used. The scaled shear $[2\pi\epsilon(q)/k]$ (k is the strength of the perturbation) is plotted as a function of the q coordinate. q_1 and q_0 indicates the limits of the interval that was perturbed and w its width.

where $\epsilon(q) = \frac{k}{2\pi}(cos(2\pi q) - cos(4\pi q))$, being k the strength of the perturbation. We note that $k < 0.11$ for the perturbation strength to satisfy the Anosov theorem [15].

We start testing σ_{sc} in case of local perturbations. For these reason, we have applied the already introduced shear in momentum to a window of coordinates of the phase space [16]. The perturbation is applied from q_0 to q_1 with width $w = q_1 - q_0$ [see inset of Fig. 1]. Note that $\alpha = w$ which is needed for the semiclassical σ_{sc} [Eq. 6 and Eq. 9]. Also, to compute σ_{sc}, the action difference between perturbed and unperturbed orbits for one iteration of the map is required. It is given by

$$\Delta^k S(q) = -\int \epsilon(q)dq = \frac{k}{4\pi^2}[sin(2\pi q) - \frac{1}{2}sin(4\pi q)]. \quad (11)$$

Moreover, we have to take into account that the spectra

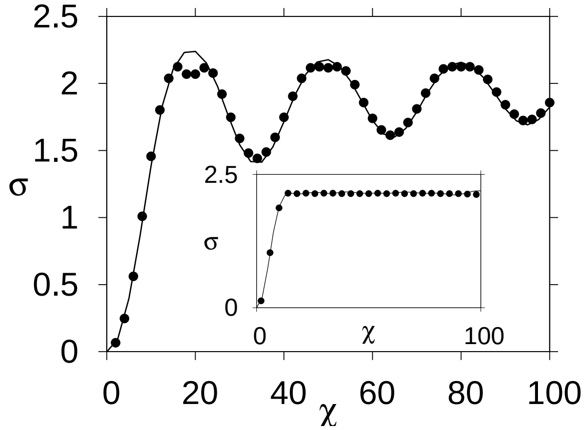

FIG. 2: Width σ of LDOS as a function of the scaled perturbation strength χ for global perturbations. Solid line is for the quantum case and the dashed line corresponds to σ_{sc}. The size of the Hilbert space is $N = 800$. In the main plot, the perturbation is the sameshear in momentum as Fig. 1 and in the inset the perturbation is a shear in momentum and position (see text for details).

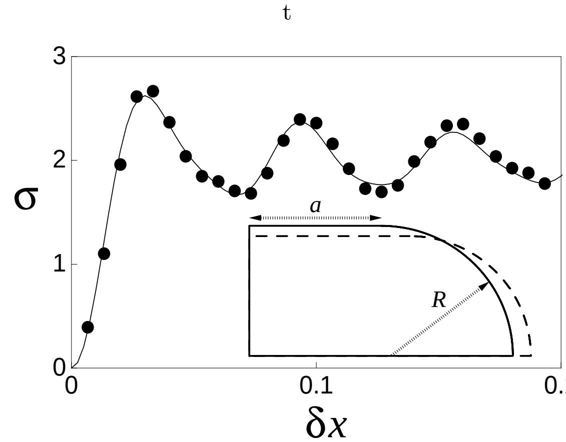

FIG. 3: Width σ of LDOS as a function of the perturbation strenght δx for the stadium billiard in the region of wave number $k = 300$ (solid circles). The semiclassical approximation σ_{sc} using Eq. 11 is plotted wis solid line. Inset: Schematic gure showing the curvilinear coordinate system on the boundary of the stadium billiard. On the dashed line a deformation of the billiard is shown

of the cat map is periodic because of a compact phase space. This periodicity change the form of the LDOS because the probability that leaves from one border will return to the other. Therefore the semiclassical width of the LDOS for the cat map results in

$$\sigma^p_{sc} \approx \sigma_{sc}[1 + 0.24\sigma_{sc} - (\frac{\sigma_{sc}}{\pi})^2], \qquad (12)$$

where the last term is due to the periodization of the Lorentzian distribution, and the linear term has been included because the leaving probability also affect the mean value position, producing an increase of the width. The value of 0.24 included in the equation has been obtained by a fitting procedure.

In Fig.1 we show the width σ of the LDOS and its semiclassical approximation for three different values of the window in positions where the perturbation is applied. In panel (a) of Fig. 1 we have used $w = 0.2$ and $w = 0.4$, and in panel (b) $w = 0.7$. To compute the width of the LDOS σ the quantum propagator was obtained from Ref. [15]. We see that the semiclassical approximation σ_{sc} works very well for all widths w of the perturbed region.

We have now taken one step forward by perturbing the cat map in all phase space which is a global perturbation. In this case, $\alpha = 1$, and the integral of Eq. 7 is done in all phase space. In Fig. 2 we show how the semiclassical σ_{sc} works to describe the width of the LDOS in the case of global perturbations. In the main panel, the perturbation is the same shear in momentum of Fig. 1 and in the inset the cat map is perturbed with a shears in momentum and positions. In the inset the shear in momentum is the same as in the main panel and the shear in position is $\bar{\epsilon}(p) = -\frac{k}{2\pi}[\frac{1}{3}sin(6\pi p) + \frac{1}{2}cos(4\pi p)]$. In this case the action difference between perturbed and unperturbed orbits for one iteration of the map results,

$$\Delta^k S(q,p) = \frac{k}{4\pi^2}[sin(2\pi q) - \frac{1}{2}sin(4\pi q) + cos(6\pi p) - \frac{1}{2}sin(4\pi p))]. \qquad (13)$$

We can clearly see that also in case of global perturbations, σ_{sc} describes very well the witdth of the LDOS. The small differences in values near the peaks at $\chi \approx 20$ and $\chi \approx 50$ are related to the maximum value, 7pi/20, of a uniform distribution.

To further demonstrate the powerfull of the proposed semiclassical approximation for computing the witdth of the LDOS in a more realistic system, we consider the desymmetrized Bunimovich stadium billiard with radius r and straight line of lenght a [see inset of Fig 3]. This system is fully chaotic and has great theoretical and experimental relevance [17, 18, 19]. The system is perturbed by a boundary deformation displayed in the inset of Fig 3]. The area of the billiard is fixed to the value $1 + \pi/4$, so the boundary only depends on the shape parameter $l = a/r$ [20].

The changes of the boundary are parametrized by,

$$\mathbf{r}(s, \delta x) = \mathbf{r_0}(s) + z(s, \delta x)\ \mathbf{n}\ , \qquad (14)$$

with s a coordinate arround the unperturbed boundary $\mathcal{C}$, $\mathbf{r_0}(s)$ the parametric equation of $\mathcal{C}$, and $\mathbf{n}$ the outward normal unit vector to $\mathcal{C}$ at $\mathbf{r}_0(s)$. For the considered perturbation around $l_0 = 1$, $z(s, \delta x) \approx \delta x z^{'}(s) + \delta^2 x z^{''}(s)/2$, with

$$z'(s) = \begin{cases} -\frac{1}{2\,A} & \text{if } s \leq 1 \ , \\ \\ (1-\frac{1}{2A})\sin(s-1) - \frac{1}{2A} & \text{if } s > 1 \ \ . \end{cases} \tag{15}$$

and

$$z''(s) = \begin{cases} \frac{3}{4\,A^2} & \text{if } s \leq 1 \ , \\ \\ \frac{1}{A} - 1 + (1-\frac{1}{2A})^2 \sin^2(s-1) + & \\ \frac{1}{2\,A^2} - \frac{1}{A}(1-\frac{3}{4A}) \ \sin(s-1) & \text{if } s > 1 \ \ . \end{cases} \tag{16}$$

We consider the usual Birkhoff coordinates to describe the classical dynamics of the particle. The variable $q = s$ and $p = \sin(\theta)$ with θ the impinging angle with $\mathbf{n}$. To compute σ_{sc} we need the action difference between the unperturbed and perturbed orbit which results in billiard systems [14],

$$\Delta^{\delta x} S(s) = |\mathbf{p}|\, \Delta L = \hbar k z(s, \delta x), \tag{17}$$

where ΔL is the lenght difference between the unperturbed and perturbed orbits and $\mathbf{p}$ the momentum of the particle. Using Eq. 7, 17 and the approximation of $z(s, \delta x)$ up to second order [Eq. 15 and 16], we compute σ_{sc}. The result is shown in Fig. 3. The witdh σ computed with the exact eigenstates is plotted with full circles and the semiclassical aproximation is plotted in full line. The calculations shown if Fig. 3 were done in the region around the wave number $k = 300$. The agreement between the quantum and the semiclassical calculation is excelent. The eigenstates of the billiard were computed using the scaling method [21]. It is important to stress out that whilst the computation of full quantum σ of Fig. 3 is very time consuming [$t \approx 7 \times 10^7 seg$ in an CPU Intel Core 2 6400] our semiclassical calculation is a simple one variable integral.

In summary, we have obtained a semiclassical approximation of the width of the LDOS of chaotic systems. We have derived this formula based on the *dephasing representation* of fidelity and similar ideas that was applied in a recent study of the Loschmidt echo in locally perturbed systems [13, 14]. Our semiclassical expression was tested in two of the most popular systems in chaotic studies: the cat map and the Bunimovich stadium billiard. Moreover, we have shown that this formula works very well for locally and globally perturbed systems.

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