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# Behavior of transition amplitude and evolution of the energy of quantum kicked rotator

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**Abstract**

I give certain analytical properties of quantum kicked rotators, both theoretic and observed from computation. I give an expression for the wave function between two successive impulses. An element of the transition matrix, the matrix that takes the wave function from one impulse to another, exponentially falls away from the main diagonal. The decay parameter is given for a few special cases of a periodic potential. Finally, the transition matrix is used to derive the time evolution of the energy for periodic cases and for fundamental resonances. The empirical expressions of Dorizzi et al. are compatible with our analytical results.

**Key Words:** Quantum chaos, quantum kicked rotator

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## 1. Introduction

I study quantum systems for Hamiltonians whose classical counterparts are chaotic. Consider the Hamiltonian

$$H = \frac{p^2}{2} + \kappa T V(\theta) \sum_n \delta(t - nT), \tag{1}$$

where $\theta$ and $p$ are the angle and angular momentum variables. This is the Hamiltonian of a rotator on which an impulse of potential $V(\theta)$ and of strength $\kappa T$ is applied at equal time intervals $T$. For $V(\theta) = \cos\theta$ the Hamiltonian describes the so-called standard map of chaos. The system, treated as classical, is chaotic for $\kappa T > 0.971635$, [1–6].

Different aspects of Hamiltonian (1) have been studied both classically and quantum mechanically by many authors theoretically [1–20] and experimentally [21–22]. This system is applicable in the very different branches of physics, e.g., laser [21], statistical physics [23, 24], and solid state physics [25].

Casati et al. [7] have considered the quantum mechanical counterpart of the standard map analytically and computationally and have compared their results with classical ones. There are important differences. They have observed that in fundamental resonances ($T = 4\pi m$, $m$ is an integer) the energy of the rotator grows quadratically with time. Izrailev and Shepelyansky [8] have extended the work of Casati et al. for resonances in general (that is, when $T$ is a rational multiple of $4\pi$) and observed a similar behavior of energy with time.

Dorizzi et al. [9] have studied the potential

$$V(\theta) = \frac{2}{k} tg^{-1}\left(\frac{k}{2}\cos\theta\right), \quad k = \kappa T. \tag{2}$$

Their numerical results indicate quadratic growth of energy with time in resonances.

In Section 2, I consider the potential of equation (2), quantize the kicked rotator, and calculate the unitary evolution matrix $U$, which takes the wave function from one impulse to another. In Section 3, I give some important properties of the evolution operator. In Section 4, I study the time evolution of the energy and obtain an analytical expression for fundamental resonances and periodic cases. Finally, I conclude my discussion and numerical results in Section 5.

## 2. Evolution operator

The wave functions at times $t_0$ and $t$ are related through the following integral form of Schrödinger's equation:

$$\psi(t) = e^{-i\int_{t_0}^{t} H(t')dt'}\psi(t_0), \tag{3}$$

where Plank's constant is taken to be 1. For $t_0 = nT^+$ and $t = (n+1)T^-$, where "$-$" and "$+$" signs indicate the times immediately before and after the impulses, respectively. Equations (1) and (3) gives

$$\psi((n+1)T^-) = e^{-iTH_0}\psi(nT^+), \tag{4a}$$

where $H_0 = p^2/2$. Similarly for $t_0$ and $t$ equal to $(n+1)T^-$ and $(n+1)T^+$, respectively one gets

$$\psi((n+1)T^+) = e^{-ikV(\theta)}\psi((n+1)T^-). \tag{4b}$$

Combining equations (4), the wave function after the $(n+1)^{st}$ impulse, in terms of that after the $n^{th}$, one obtains

$$\psi((n+1)T^+) = U(k,\theta,T)\psi(nT^+), \tag{5a}$$

where

$$U(k,\theta,T) = e^{-ikV(\theta)}e^{-iH_0T}. \tag{5b}$$

I expand this wave function in terms of the eigenfunctions of free particle, $\phi_n = (1/\sqrt{2\pi})e^{in\theta}$. Thus

$$\psi(\theta,t) = \sum_j C_j(t)\phi_j(\theta). \tag{6}$$

Inserting equation (6) in equations (5), multiplying both sides by $\phi_m^*$ and integrating over the interval $(0,\, 2\pi)$, gives

$$C_m((n+1)T) = U_{mj}C_j(nT) = A_{m\ell}\Omega_{\ell j}C_j(nT), \tag{7}$$

where $U_{mj}$, $A_{m\ell}$ and $\Omega_{\ell j}$ are the matrix elements of $U(k,\theta,T)$, $\exp[-ikV(\theta)]$ and $\exp(-iH_0T)$, respectively. Thus

$$U_{mj} =< \phi_m \left| e^{-ikV(\theta)} e^{-iH_0T} \right| \phi_j >, \tag{8a}$$

$$A_{m\ell} =< \phi_m \left| e^{-ikV(\theta)} \right| \phi_\ell >, \tag{8b}$$

$$\Omega_{\ell j} =< \phi_\ell \left| e^{-iH_0T} \right| \phi_j >= e^{-i\omega_j T}\delta_{\ell j}, \tag{8c}$$

where $\omega_j = \frac{1}{2}j^2$. Calculation of $A_{m\ell}$ for the potential of Equation (2) is elaborate. However, one obtains the following exact expression

$$A_{m\ell} = \frac{2}{\beta}Z^{|m-\ell|} - \delta_{m\ell}, \tag{9a}$$

where

$$Z = 2i(1-\beta)/k \text{ and } \beta = (1+k^2/4)^{\frac{1}{2}}. \tag{9b}$$

Inserting Equations (9), and (8c) in Equation (8a) gives

$$U_{mj} = \left(\frac{2}{\beta}Z^{|m-j|} - \delta_{mj}\right) e^{-ij^2T/2}. \tag{10}$$

Variation of absolute of $U$ with respect to matrix elements $U_{n\,m}$ and parameter $k$ can be observed in Figure 1. This figure shows that the matrix elements $U_{n\,m}$ decay exponentially by increasing the difference of elements $n = |m-j|$; but by varying the parameter $k$ from 0 to 15, first it decreases and goes to zero, then increases.

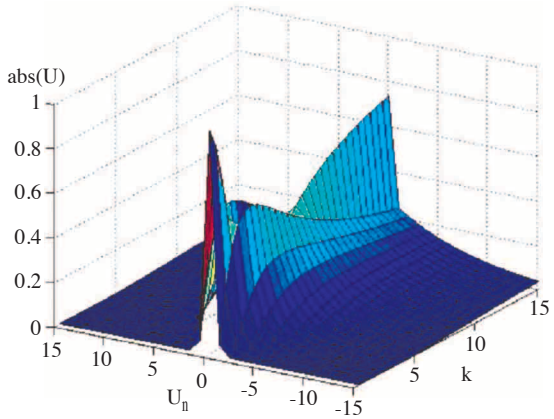


**Figure 1.** Variation of absolute of $U$ with respect to matrix elements $U_{n\,m}$ and parameter $k$.

Here after I shall use a matrix notation rather than the component notation. Let $C(0)$ and $C(n)$ be the initial wave vector and that after the $n^{th}$ impulse. Repeated application of Equation (7) gives

$$C(n) = U^nC(0) = (A\Omega)^nC(0). \tag{11}$$

# 3. Some properties of the evolution operator

From Equations (10) and (11) one may read the following properties of $U$ and $C(n)$.

## 3.1. Unitary property

$U$ is unitary. Hence if $C(0)$ is normalized, then $C(n)'s$ remain normalized for all $n$, $C^{+}(n)C(n) = 1$. I have used this as a criterion for checking the precision of our numerical computations.

## 3.2. Eigenvalues of $U$

The eigenvalues of $U$ are of the form $\exp(i\alpha_n)$. If all $\alpha_n s$ are rational multiples of $2\pi$ then there is the possibility of complete periodicity for the system. That is, the system may return to its initial value after certain number of impulses. This is a quantum phenomenon with no classical counterparts. I shall examine an example of period two in Section IV.

## 3.3. Periodicity of the elements of $U$

The elements of $U$ are periodic in $T$. There follows that systems with impulse intervals $T_n = T + 4\pi n$, $n$ is integer, behave the same. On the other hand, $k = \kappa T$ is a measure of the strength of the impulse. By the same argument, only $k' = [\kappa T - \mathrm{mod}(4\pi n)]$ will enter equations (8)–(11) chaotic and/or no chaotic regimes will recur as $k$ increases indefinitely. This is in contrast to the classical case where there is a threshold to the strength of potential beyond which chaos sets in and persists thereafter. Of course this remark does not apply to the fixed points of classical chaotic systems.

# 4. Evolution of the energy of quantum systems

The expectation value of the energy at $t = nT^{+}$ is

$$E(n) = \int_0^{2\pi} \psi^*(\theta, n)\frac{p^2}{2}\psi(\theta, n)d\theta = \frac{1}{2}\sum_{j=-\infty}^{\infty} j^2 \left|C_j(n)\right|^2 = C^{+}(n)HC(n), \tag{12}$$

where the elements of $H$ are $H_{\kappa j} = (1/2)j^2\delta_{kj}$. The classical counterpart of Equation (12) has chaotic behavior for $k > k_c$ where $k_c$ is some critical value depending on the impulse potential. But in a quantum system three different cases can be recognized.

## 4.1. The periodic case

For $T/4\pi = 1/2$ the matrix elements of $\Omega$ are $\pm 1$, and Equation (10) reduces to

$$U_{mj} = \left[\frac{2}{\beta}\left|Z\right|^{|m-j|} - \delta_{mj}\right] i^{|m-j|}(-1)^j. \tag{13}$$

From this it can be observed that $U_{mj} = U^*_{jm}$ or $U = U^{+} = U^{-1}$, or $U^2 = 1$. Thus, the system returns to its initial state after every two impulses, and for all values of $k$ there are only two states in which the system may

assume, $C(0)$ and $UC(0)$. Correspondingly, the energy also alternates between two values. For example, for $C_j(0) = \delta_{oj}\,,\ j = 0, \pm 1, ...$ one finds

$$E_{2n} = 0, E_{2n+1} = \frac{(1/4)k^2}{1 + (1/4)k^2}. \tag{14}$$

This analytical feature has been checked numerically by using Equation (12), as it is shown in Figures 2a and 2b for $T/4\pi = 1/2$, with the number of kicks $n = 50$ and $k = 0.5\,,\ k = 2\,,$ respectively.

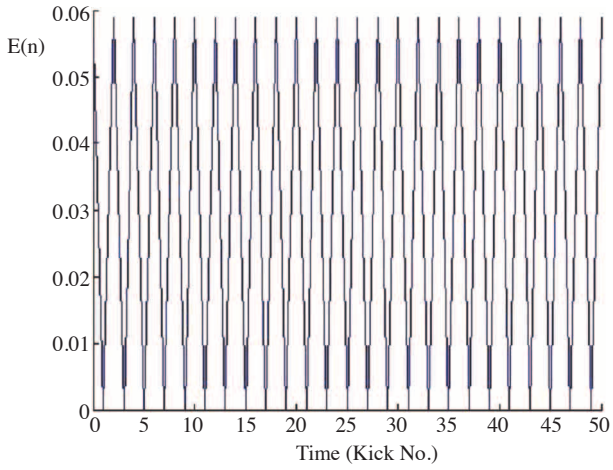


**Figure 2a.** Evolution of the energy with respect to kick number, for $k = 0.5\,,\ T/4\pi = 1/2$and $n = 50$.

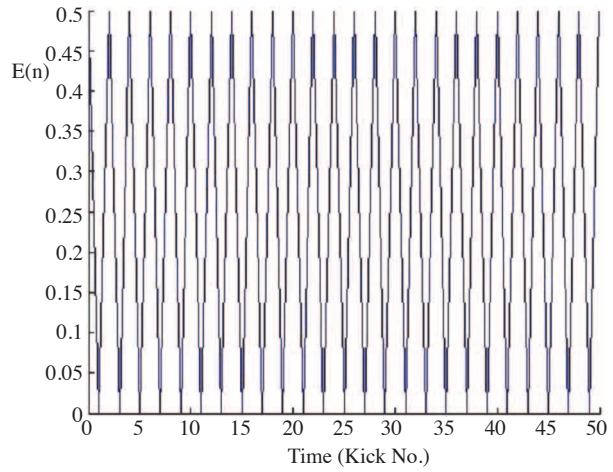


**Figure 2b.** The same as figure 2a, but for $k = 2$.

## 4.2. Resonances

The cases $T/4\pi = p/q \neq 1/2$, $p$ and $q$ are integers, are known as resonances. The special cases of $T/4\pi = m, m$ is an integer, are the fundamental resonances. It has been verified numerically by me and others [5] that, in all resonances, the energy grows quadratically with time. Here, I prove this feature for fundamental resonances analytically.

For $T/4\pi = m, \Omega$, the matrix representation of $\exp(-iH_0T)$ becomes a unit matrix, see equation (8c). Correspondingly, the evolution operator of equation (5b) reduces to $\exp[-ikV(\theta)]$. For the wave function after the $n^{th}$ impulse one finds

$$\psi(\theta, n) = e^{-inkV(\theta)}\psi(\theta, 0). \tag{15}$$

The expectation value of the energy is

$$E(n) = \int \psi^*(\theta, n)(\frac{1}{2}p^2)\psi(\theta, n)d\theta \quad = \int \left|\frac{\partial\psi(\theta, n)}{\partial\theta}\right|^2 d\theta$$

$$= \eta(k)n^2 + \xi(k)n + E(0), \tag{16}$$

where

$$\eta(k) = \frac{1}{2}k^2 \int \left|\frac{\partial V}{\partial\theta}\psi(\theta, 0)\right|^2 d\theta, \tag{17a}$$

$$\xi(k) = -\frac{1}{2}ik \int \frac{\partial V}{\partial\theta}\left[\psi^*(\theta, 0)\frac{\partial\psi(\theta, 0)}{\partial\theta} - \frac{\partial\psi^*(\theta, 0)}{\partial\theta}\psi(\theta, 0)\right] d\theta, \tag{17b}$$

$$E(0) = \frac{1}{2}\int \left|\frac{\partial\psi(\theta, 0)}{\partial\theta}\right|^2 d\theta. \tag{17c}$$

Equations (15)–(17) hold for any potential. For the potential of Equation (2) one has

$$\eta(k) = \frac{1}{2}k^2 \int_0^{2\pi} \frac{\sin^2\theta}{(1 + \frac{k^2}{4}\cos^2\theta)^2} |\psi(\theta, 0)|^2 d\theta, \tag{18a}$$

$$\xi(k) = -\frac{1}{2}k \int_0^{2\pi} \frac{\sin\theta}{1 + \frac{k^2}{4}\cos^2\theta}\left[\psi^*(\theta, 0)\frac{\partial\psi(\theta, 0)}{\partial\theta} - \psi(\theta, 0)\frac{\partial\psi^*(\theta, 0)}{\partial\theta}\right] d\theta. \tag{18b}$$

If the initial wave functions is the ground state, $\psi(\theta, 0) = 1/\sqrt{2\pi}$, then $E(0)$ and $\xi(k)$ vanish. The integral (18a) can be evaluated by the change of variable $k/(2\cos\theta) = \tan u$. Equation (16) then becomes

$$E(n) = \frac{k^2/4}{\sqrt{1 + k^2/4}}n^2 \tag{19}$$

$$\approx \begin{cases} \frac{1}{4}k^2n^2, & \text{where} \quad k << 1, \\ \frac{1}{2}kn^2, & \text{where} \quad k >> 1. \end{cases}$$

I have verified equations (12) for initial conditions $T/4\pi = p/q \neq 1/2$. All of them confirm equation (19) for a wide range of parameter $k$ and a variety of initial wave functions. For example, for $k = 0.5$ and $T/4\pi = 1$, the variation of energy in terms of kick number is shown in Figure 3.

The limiting values of analytical computation of Equation (19), for small and large $k$'s, were predicted numerically by Dorizzi et al. [9].

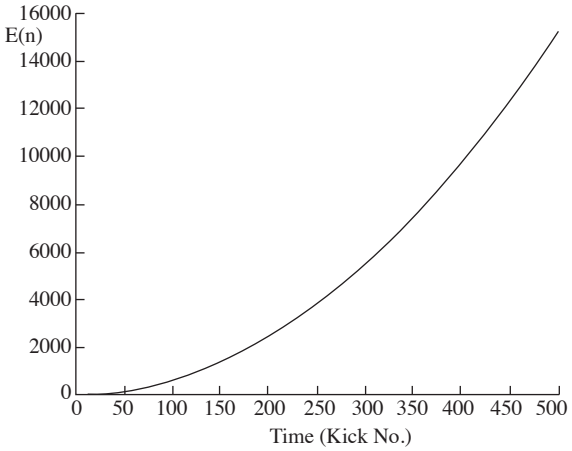


**Figure 3.** The variation of energy as a function of kick number for $k = 0.5$ and $T/4\pi = 1$.

## 4.3. Non resonant and non-periodic cases

Non resonant and non-periodic cases occur for $T/4\pi$ = irrational. An analytical study of the system is complicated. For small $k$'s, however, I have verified numerically the recurrence phenomenon reported by Hogg and Huberman [14]. With recurrence, one means the return of the expectation value of a dynamical quantity (in this case, the energy) into the close neighborhood of its initial value. In a numerical study the larger the $k$ the larger the dimension of the $U$ matrix. The numerical precision is rapidly lost. It becomes difficult to decide whether the system becomes chaotic or not.

## 4.4. Conclusion and numerical results

For the potential of equation (2) I have derived analytically the wave function of the system after each impulse, equations (6) and (11). In Section III, I have shown some properties of the evolution operator $U$: (1) $U$ is unitary; (2) the eigenvalues of $U$ are of the form $\exp(i\alpha)$. If all $\alpha's$ are rational multiples of $2\pi$ then the system is periodic. (3) The elements of $U$ are periodic in $T$. This is in contrast to the classical case.

The time evolution of the energy of the quantum kicked rotator with potential (2) is obtained for various cases. For $T/4\pi = 1/2, U^2 = 1$. The system returns to its initial state after every two impulses. For $T/4\pi = p/q \neq 1/2$, the energy grows quadratically in time, as expressed in equation (16). The coefficients of the quadratic expression for $T/4\pi =$integer, is explicitly given; see equation (19). For $T/4\pi =$irrational, the behavior of the system depends on the strength of the impulses, $k$. For small $k$ $(\leq 10)$, numerical computations show an upper bound for the energy.

In numerical calculations, wave vectors of 1000 components and matrices of $1000 \times 1000$ were considered in double precision. Due to power decay of elements of $U$ away from the main diagonal, terms of absolute values lest than $10^{-30}$ were dropped. The initial wave function was taken to be the ground state $\psi(\theta, 0) = 1/\sqrt{2\pi}$. The unitarity of the truncated matrix $U$ was used to check the accuracy of the numerical results in various stages of the computations.

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