Turk J Phys
32 (2008) , 261 – 267.

Path Integral Quantization of Spinning Superparticle

H. A. ELEGLA, N. I. FARAHAT
Department of Physics, Islamic University of Gaza, P.O. Box 108 Gaza, PALESTINE
e-mail: helegla@iugaza.edu.ps • nfarahat@iugaza.edu.ps

Received 13.03.2007

Abstract

The Hamilton-Jacobi formalism is used to discuss the path integral quantization of a spinning superparticle model. The equations of motion are obtained as total differential equations in many variables. The equations of motion are integrable, and the path integral is obtained as an integration over the canonical phase space coordinates.

Key Words: Hamilton-Jacobi formalism, Singular Lagrangian, Path integral quantization.
PACS: 11.10.Ef
PACS: 03.65.-w

1. Introduction

Dirac initiated the path to locating all the constraints in the Hamiltonian formulation of singular theories [1]. The constraints are naturally classified according to the corresponding stages of this procedure. It is convenient to reorganize the constraints such that they are explicitly decomposed into first-class and second-class constraints. Information about the constraint structure is important for identification in the physical sector, for the study of classical and quantum symmetries, for quantization purposes, and so on. The main feature of Dirac's method is to consider primary constraints initially; then all constraints are obtained by using the consistency conditions. Hence one can obtain equations of motion of a singular Lagrangian system obtained via the consistency conditions [2, 3]. A variant method is the Hamilton-Jacobi formalism for constrained systems [4, 5], based on Carathéodory's equivalent the Lagrangian method [6]. In this formalism, those coordinates whose correspondent accelerations can not be solved as a function of the momenta are arbitrary variables of the theory.

We have obtained a set of Hamilton-Jacobi partial differential equations in terms of these variables, and from this set we obtained the equations of motion of the system as total differential equations for the characteristics. These total differential equations so obtained must satisfy integrability conditions, and for these conditions to be satisfied the nature of the constraints (first class or second class) will play an essential role. This formalism does not give distinction between the first and second class constraints, and does not need any gauge-fixing terms.

Recently, several interesting constrained systems were investigated by applying Hamilton-Jacobi treatment to treat the classical regulations, e.g. electromagnetic field, the Young-Mills field, the Einstein gravitational field, and massless Seigel superparticle with simple supersymmetry [7, 14].

The aim of this paper is to present a refined analysis of the path integral quantization of spinning superparticle, which involve extra fermionic variables to represent space-time degrees of freedom [15], by applying the Hamilton-Jacobi Formulation.

Now let us make a brief review on the Hamilton-Jacobi formulation of singular system (the canonical method). The Lagrangian function of any physical system with n degrees of freedom is a function of n generalized coordinates, n generalized velocities and parameter t, i.e. $L \equiv L(q_i, \dot{q}_i, t)$. The Hess matrix is defined as

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad i, j = 1, 2, \ldots, n. \tag{1}$$

If the rank of this matrix is n, then the Lagrangian is called regular, otherwise it is called singular. Systems which have singular Lagrangian are called singular systems or constrained systems.

One starts from the singular Lagrangian $L(q_i, \dot{q}_i, t)$ with Hess matrix of rank $(n-p)$, $p < n$. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, 2, \ldots, n-p, \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = n-p+1, \ldots, n. \tag{3}$$

Since the rank of the Hess matrix is $(n-p)$, one may solve (2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, p_b; t) \equiv \omega_a. \tag{4}$$

Substituting (4), into (3), we get

$$p_\mu = \left.\frac{\partial L}{\partial \dot{q}_\mu}\right|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\mu, p_a; t). \tag{5}$$

The canonical Hamiltonian H_o is defined as

$$H_0 = -L\big(q_i, \dot{q}_\nu, \dot{q}_a; t\big) + p_a \omega_a + p_\mu \dot{q}_\mu\big|_{p_\nu = -H_\nu}, \qquad \mu, \nu = n-p+1, \ldots, n. \tag{6}$$

The set of Hamilton-Jacobi partial differential equations (HJPDE) is expressed as

$$H'_\alpha\left(\tau,\ q_\nu,\ q_a,\ P_i = \frac{\partial S}{\partial q_i},\ P_0 = \frac{\partial S}{\partial \tau}\right) = 0, \qquad \alpha = 0, n-p+1, \ldots, n, \tag{7}$$

where

$$H'_0 = p_0 + H_0\ , \tag{8}$$

$$H'_\mu = p_\mu + H_\mu. \tag{9}$$

We define $P_0 = \dfrac{\partial S}{\partial \tau}$, and $P_i = \dfrac{\partial S}{\partial q_i}$, with $q_0 = t$, and S being the action.

The equations of motion are obtained as total differential equations and take the forms

$$dq_a = \frac{\partial H'_\alpha}{\partial P_a} dt_\alpha, \tag{10}$$

$$dP_r = \frac{\partial H'_\alpha}{\partial q_r} dt_\alpha, \qquad r = 0, 1, \ldots, n. \tag{11}$$

$$dZ = \left(-H_\alpha + P_a \frac{\partial H'_\alpha}{\partial P_a}\right) dt_\alpha, \tag{12}$$

where $Z = S(t_\alpha, q_a)$. These equations are integrable if and only if [16, 17]

$$dH'_0 = 0, \tag{13}$$

$$dH'_\mu = 0, \qquad\qquad \mu = n-p+1,\ldots,n. \tag{14}$$

If conditions (13) and (14) are not satisfied identically, one may consider them as new constraints and again test the integrability conditions. Thus repeating this procedure, one may obtain a set of constraints such that all the variations vanish, then we may solve equations (10) and (11) to get the canonical phase-space coordinates as

$$q_a \equiv q_a(t,t_\mu), \qquad p_a \equiv p_a(t,t_\mu), \qquad \mu = 1,\ldots,p. \tag{15}$$

Then the path integral representation may be written as

$$\langle \text{Out} \mid S \mid \text{In} \rangle = \int \prod_{a=1}^{n-r} dq^a dp^a \exp\left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right], \tag{16}$$

where $a = 1,\ldots,n-p$, and $\alpha = 0,\ n-p+1,\ldots,n$. We should notice that integral (16) is an integration over the canonical phase space coordinates (q_a, p_a).

2. Hamilton-Jacobi Formulation of Spinning Superparticle

The spinning superparticle model which possesses both (local) world-line and (rigid) target-space supersymmetry, is described by the action [13]:

$$\begin{aligned} S = \int \bigg\{ & \frac{1}{2e}\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big)^2 + \frac{i}{2}\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\frac{d}{d\tau}\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big) \\ & + \frac{i}{e}\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big)\bigg\} d\tau. \end{aligned} \tag{17}$$

In such a space-time one can define, in addition to the usual coordinate, a spinor of real fermionic supermultiplets (θ, h), which define an anti-commuting and a commuting Majorana spinor in the target space-time, respectively.

The singular Lagrangian is

$$\begin{aligned} L = & \frac{1}{2e}\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big)^2 + \frac{i}{2}\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\frac{d}{d\tau}\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big) \\ & + \frac{i}{e}\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big). \end{aligned} \tag{18}$$

The canonical momenta defined in (2) and (3) take the forms

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{e}\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big) + \frac{i}{e}\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big), \tag{19}$$

$$\pi_\theta = \frac{\partial L}{\partial \dot{\theta}} = -i\left(P_\mu\gamma^\mu\bar{\theta} + \frac{1}{2}\psi_\mu\gamma^\mu\bar{h}\right) = -H_\theta, \tag{20}$$

$$\pi_{\bar{\theta}} = \frac{\partial L}{\partial \dot{\bar{\theta}}} = 0 = -H_{\bar{\theta}}, \tag{21}$$

$$\pi_h = \frac{\partial L}{\partial \dot{h}} = 0 = -H_h, \tag{22}$$

$$\pi_{\bar{h}} = \frac{\partial L}{\partial \dot{\bar{h}}} = -\frac{i}{2}\psi_\mu\gamma_\mu\theta = -H_{\bar{h}}, \tag{23}$$

$$P_\psi^\mu = \frac{\partial L}{\partial \dot{\psi}_\mu} = \frac{i}{2}\left(\psi_\mu - \bar{h}\gamma_\mu\theta\right) = -H_\psi, \tag{24}$$

$$P_\chi = \frac{\partial L}{\partial \dot{\chi}} = 0 = -H_\chi, \tag{25}$$

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_e. \tag{26}$$

We can solve (19) for $\dot{x}_\mu$ in terms of P_μ and other coordinates to get

$$\dot{x}_\mu = eP_\mu - i\chi\left(\psi_\mu - \bar{h}\gamma_\mu\theta\right) + i\bar{\theta}\gamma_\mu\dot{\theta} + e\bar{h}\gamma_\mu h. \tag{27}$$

A straightforward calculation shows that the canonical Hamiltonian H_0 is obtained as

$$\begin{aligned} H_0 &= -L + P_\mu\dot{x^\mu} + \pi_\theta\dot{\theta} + \pi_{\bar{\theta}}\dot{\bar{\theta}} + \pi_h\dot{h} + \pi_{\bar{h}}\dot{\bar{h}} + P_\psi\dot{\psi} + P_\chi\dot{\chi} + P_e\dot{e} \\ &= \frac{1}{2e}\Big\{eP_\mu - i\chi\big(\psi_\mu - \bar{h}\gamma_\mu\theta\big)\Big\}^2 + eP_\mu\big(\bar{h}\gamma_\mu h\big). \end{aligned} \tag{28}$$

The set of HJPDE's is thus

$$H_0' = P_0 + \frac{1}{2e}\Big\{eP_\mu - i\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big\}^2 + eP_\mu\Big(\bar{h}\gamma_\mu h\Big), \tag{29}$$

$$H_\theta' = \pi_\theta + i\Big(P_\mu\gamma^\mu\bar{\theta} + \frac{1}{2}\psi_\mu\gamma^\mu\bar{h}\Big), \tag{30}$$

$$H_{\bar{\theta}}' = \pi_{\bar{\theta}}, \tag{31}$$

$$H_h' = \pi_h, \tag{32}$$

$$H_{\bar{h}}' = \pi_{\bar{h}} + \frac{i}{2}\psi_\mu\gamma_\mu\theta, \tag{33}$$

$$H_\psi' = P_\psi^\mu - \frac{i}{2}\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big), \tag{34}$$

$$H_\chi' = P_\chi, \tag{35}$$

and

$$H_e' = P_e. \tag{36}$$

The equations of motion (10) and (11) may now be concisely written in the forms

$$dx_\mu = \Big\{eP_\mu - i\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big) + e\Big(\bar{h}\gamma_\mu h\Big)\Big\}d\tau + i\bar{\theta}\gamma_\mu d\theta, \tag{37}$$

$$dP_0 = 0, \tag{38}$$

$$dP_\mu = 0, \tag{39}$$

$$d\pi_\theta = \Big\{-\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\bar{h}\gamma_\mu\chi\Big\}d\tau - \frac{i}{2}\psi_\mu\gamma_\mu d\bar{h} - \frac{i}{2}\bar{h}\gamma_\mu d\psi_\mu, \tag{40}$$

$$d\pi_{\bar{\theta}} = \Big(-iP_\mu\gamma_\mu\Big)d\theta, \tag{41}$$

$$d\pi_h = \Big(-eP_\mu\bar{h}\gamma_\mu\Big)d\tau, \tag{42}$$

$$\begin{aligned} d\pi_{\bar{h}} = &-\Big\{\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\chi\bar{h}\gamma_\mu + eP_\mu\gamma_\mu h\Big\}d\tau - \frac{i}{2}\psi_\mu\gamma_\mu d\theta \\ &- \frac{i}{2}\gamma_\mu\theta d\psi_\mu, \end{aligned} \tag{43}$$

$$\begin{aligned} dP_\psi^\mu = &\Big\{\frac{i}{e}\Big(eP_\mu - i\chi\left(\psi_\mu - \bar{h}\gamma_\mu\theta\right)\Big)\chi\Big\}d\tau - \frac{i}{2}\bar{h}\gamma_\mu d\theta - \frac{i}{2}\gamma_\mu\theta d\bar{h} \\ &- \frac{i}{2}\gamma^\mu d\psi_\mu, \end{aligned} \tag{44}$$

$$dP_\chi = \Big\{\frac{i}{e}\Big(eP_\mu - i\chi\left(\psi_\mu - \bar{h}\gamma_\mu\theta\right)\Big)\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big\}d\tau, \tag{45}$$

and

$$dP_e = \Big\{\frac{1}{2}P^2 + \frac{1}{e^2}\chi^2\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)^2 + P_\mu\big(\bar{h}\gamma_\mu h\big)\Big\}d\tau. \tag{46}$$

Integrability conditions (13) and (14) imply that the variation of the constraints (29–36) should be identically zero. One notices that

$$dH'_0 = 0 \tag{47}$$

and

$$dH'_{\bar{\theta}} = 0 \tag{48}$$

are identically zero; whereas the variation of

$$dH'_\theta = \Big\{ -\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\bar{h}\gamma_\mu\chi\Big\}d\tau + iP_\mu\gamma_\mu d\theta \equiv H''_\theta d\tau, \tag{49}$$

$$dH'_h = \Big(-iP_\mu\bar{h}\gamma_\mu\Big)d\tau \equiv H''_h d\tau, \tag{50}$$

$$dH'_{\bar{h}} = -\Big\{\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\chi\bar{h}\gamma_\mu + eP_\mu\gamma_\mu h\Big\}d\tau \equiv H''_{\bar{h}} d\tau, \tag{51}$$

$$dH'_\psi = \Big\{\frac{i}{e}\Big(eP_\mu - i\chi(\psi_\mu - \bar{h}\gamma_\mu\theta)\Big)\chi\Big\}d\tau \equiv H''_\psi d\tau, \tag{52}$$

$$dH'_\chi = \Big\{\frac{i}{e}\Big(eP_\mu - i\chi(\psi_\mu - \bar{h}\gamma_\mu\theta)\Big)\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big\}d\tau \equiv H''_\chi d\tau, \tag{53}$$

and

$$dH'_e = \Big\{\frac{1}{2}P^2 + \frac{1}{e^2}\chi^2\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)^2 + P_\mu\Big(\bar{h}\gamma_\mu h\Big)\Big\}d\tau \equiv H''_e d\tau, \tag{54}$$

are not. Therefore we obtain the following set of additional constraints:

$$H''_\theta = \Big\{ -\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\bar{h}\gamma_\mu\chi\Big\}, \tag{55}$$

$$H''_h = \Big(-iP_\mu\bar{h}\gamma_\mu\Big), \tag{56}$$

$$H''_{\bar{h}} = -\Big\{\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\chi\bar{h}\gamma_\mu + eP_\mu\gamma_\mu h\Big\}, \tag{57}$$

$$H''_\psi = \Big\{\frac{i}{e}\Big(eP_\mu - i\chi(\psi_\mu - \bar{h}\gamma_\mu\theta)\Big)\chi\Big\}, \tag{58}$$

$$H''_\chi = \Big\{\frac{i}{e}\Big(eP_\mu - i\chi(\psi_\mu - \bar{h}\gamma_\mu\theta)\Big)\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big\}, \tag{59}$$

and

$$H''_e = \Big\{\frac{1}{2}P^2 + \frac{1}{e^2}\chi^2\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)^2 + P_\mu\Big(\bar{h}\gamma_\mu h\Big)\Big\}. \tag{60}$$

Now the total differential of H''_θ, H''_h, $H''_{\bar{h}}$, H''_ψ, H''_χ and H''_e vanish identically, i.e.

$$dH''_\theta = 0, \tag{61}$$

$$dH''_h = 0, \tag{62}$$

$$dH''_{\bar{h}} = 0, \tag{63}$$

$$dH''_\psi = 0, \tag{64}$$

$$dH''_\chi = 0, \tag{65}$$

and

$$dH''_e = 0. \tag{66}$$

Thus the equations of motion (37)–(46) and the new constraints (55)–(60) represent an integrable system.

Since the equations of motion are integrable, the action can be written as

$$
\begin{aligned}
dZ &= -H_0 d\tau - H_\theta d\theta - H_{\bar\theta} d\bar\theta - H_h dh - H_{\bar h} d\bar h \\
&\quad - H_\psi d\psi_\mu - H_\chi d\chi - H_e de + P_\mu dx_\mu \\
&= \left\{ \frac{1}{2}\, eP^2 + \frac{1}{2e}\chi^2 \Big(\psi_\mu - \bar h \gamma_\mu \theta\Big)^2 - \frac{i}{2}\psi_\mu \Big(\bar h \gamma_\mu \dot\theta - \dot{\bar h}\gamma_\mu \theta\Big) \right. \\
&\quad \left. + \frac{i}{2}\Big(\psi_\mu - \bar h\gamma_\mu\theta\Big)\dot\psi_\mu \right\} d\tau.
\end{aligned}
\tag{67}
$$

We now present a phase-space action for the spinning superparticle,

$$
\begin{aligned}
S = \int &\left\{ \frac{1}{2}\, eP^2 + \frac{1}{2e}\chi^2 \Big(\psi_\mu - \bar h \gamma_\mu \theta\Big)^2 - \frac{i}{2}\psi_\mu \Big(\bar h \gamma_\mu \dot\theta - \dot{\bar h}\gamma_\mu \theta\Big) \right. \\
&\left. + \frac{i}{2}\Big(\psi_\mu - \bar h\gamma_\mu\theta\Big)\dot\psi_\mu \right\} d\tau.
\end{aligned}
\tag{68}
$$

Making use of (68), the path integral defined in (16) becomes

$$
\begin{aligned}
\langle x_\mu, \tau\,; x'_\mu, \tau' \rangle = \int dx_\mu\, dp_\mu \exp \Bigg[i \int &\left\{ \frac{1}{2} eP^2 + \frac{1}{2e}\chi^2 \Big(\psi_\mu - \bar h \gamma_\mu \theta\Big)^2 \right. \\
&\left. - \frac{i}{2}\psi_\mu \Big(\bar h \gamma_\mu \dot\theta - \dot{\bar h}\gamma_\mu \theta\Big) + \frac{i}{2}\Big(\psi_\mu - \bar h\gamma_\mu\theta\Big)\dot\psi_\mu \right\} d\tau \Bigg].
\end{aligned}
\tag{69}
$$

3. Conclusions

In this paper the spinning superparticle has been quantized by constructing a path integral quantization within the Hamilton-Jacobi approach to constrained systems. The equations of motion are obtained as total differential equations in many variables. All the constraints coming from the Hamiltonian procedure and the integrability conditions have been derived. The path integral quantization is performed using the action given by Hamilton-Jacobi formulation, and the integration is taken over the canonical phase space.

References

[1] P. A. M. Dirac, *Generalized Hamiltonian Dynamics*, *Can. J. Math. Phys.*, **2**, (1950), 129.

[2] K. Sundermeyer, *Constrained Dynamics*, (Springer, Verlag.1982).

[3] D. M. Gitman and I. V. Tyutin, *Quantization of Fields with Constraints*, (Springer, Verlag, 1990).

[4] Y. Güler, *Nuovo Cimento*, **B107**, (1992), 1389.

[5] Y. Güler, *Nuovo Cimento*, **B107**, (1992), 1143.

[6] C. Carathéodory, *Calculus of Variations and Partial Differential Equations of the First Order Part II*, (Holden-Day, 1967).

[7] Y. Güler, *Nuovo Cimento*, **B 109**, (1994), 341.

[8] N. I. Farahat, and Y. Güler, *Nuovo Cimento*, **B 111**, (1996), 513.

[9] N. I. Farahat and Z. Nassar, *Hadronic Journal*, **25**, (2002), 239.

[10] S. I. Muslih and Y. Güler, *Nuovo Cimento*, **B112**, (1997), 97.

[11] S. I. Muslih and Y. Güler, *Nuovo Cimento*, **B113**, (1998), 277.

[12] S. I. Muslih, *Nuovo Cimento*, **B115**, (2000), 1.

[13] S. I. Muslih, *Nuovo Cimento*, **B115**, (2000), 7.

[14] N. I. Farahat and H. A. Elegla, *Turk. J. Phys.*, **30**, (2006), 473.

[15] S. Aoyama, J. Kowalski-Glikman, J. Lukierski and J. W. Van Holten, *Phys. Lett.,* **B 201**, (1988), 487.

[16] S. I. Muslih, *Nuovo Cimento*, **B188**, (2003), 505.

[17] S. I. Muslih, *Mod. Phys. Letters A*, **19**, (2004), 863.