Criterion for faithful teleportation with an arbitrary multiparticle channel

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We consider quantum teleportation when the given entanglement channel is an arbitrary multiparticle state. A general criterion is presented, which allows one to judge if the channel can be used to teleport faithfully an arbitrary quantum state of a given dimension. The general protocol proposed here is much easier to implement experimentally than the others found in the literature.

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Quantum teleportation is arguably the most novel application of quantum mechanics in quantum information science. This protocol provides a means of recreating an arbitrary quantum state at a remote site without the need of transferring any particles or a large amount of classical information. The magic of teleportation is made possible by prior quantum entanglement between the sender (Alice) and the receiver (Bob). It is well known that, in the original protocol proposed by Bennett et al. [1], if Alice and Bob share a two-qubit entangled state (Bell or EPR state), then Alice can teleport an arbitrary one-qubit state to Bob. This protocol is linear, that means if Alice and Bob shares N Bell states, then Alice will be able to teleport an arbitrary N-qubit state to Bob. In recent years, quantum teleportation has been experimentally realized in several different quantum systems [2, 3, 4, 5, 6].

To facilitate the ensuing discussions, we first demonstrate the teleportation of an arbitrary N-qubit state below. The four Bell states are given by

$$|\phi^1\rangle_{ab} = \frac{1}{\sqrt{2}}\Big(|01\rangle_{ab} - |10\rangle_{ab}\Big), \tag{1}$$

$$|\phi^2\rangle_{ab} = \frac{1}{\sqrt{2}}\Big(|01\rangle_{ab} + |10\rangle_{ab}\Big), \tag{2}$$

$$|\phi^3\rangle_{ab} = \frac{1}{\sqrt{2}}\Big(|00\rangle_{ab} - |11\rangle_{ab}\Big), \tag{3}$$

$$|\phi^4\rangle_{ab} = \frac{1}{\sqrt{2}}\Big(|00\rangle_{ab} + |11\rangle_{ab}\Big). \tag{4}$$

Suppose Alice shares a Bell state $|\phi^1\rangle_{ab}$ with Bob (Alice holds qubit a and Bob holds qubit b), and in addition she owns an arbitrary N-qubit pure state $|\Psi\rangle_{12\dots N}$ to be teleported to Bob. Note that if the N qubits are in a mixed state, then it can always be purified by introducing ancilla qubits [7]. However since the ancillas do not participate in the teleportation process, we will ignored their possible existence in the following discussions.

The product of $|\Psi\rangle_{12\dots N}$ and $|\phi^1\rangle_{ab}$ can be rewritten as

$$|\Psi\rangle_{12\dots N}|\phi^1\rangle_{ab} = -\frac{1}{2}\Big(\sum_{i=1}^{4}|\phi^i\rangle_{1a}U_b^i\Big)|\Psi\rangle_{b2\dots N}, \tag{5}$$

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where $\{U^i\} = \{I, \sigma^z, -\sigma^x, i\sigma^y\}$, and I is the 2×2 unit matrix. Therefore if Alice makes a Bell state measurement on the qubit pair $(1, a)$ and sends a 2-bit classical message to inform Bob of the outcome (i), then Bob can reconstruct the original N-qubit state by applying a local unitary operation U_b^i on his qubit. The only difference is that qubit 1 has been renamed b and is now in his possession. It is easy to see that, with more Bell states, this process can be repeated on the other qubits in $|\Psi\rangle_{12\ldots N}$. Therefore if Alice shares N pairs of Bell states with Bob, then she will be able to teleport perfectly the entire state $|\Psi\rangle_{12\ldots N}$ to Bob.

However in practice the channel state shared by Alice and Bob may not be a tensor product of N Bell states. Then one must consider each case individually to decide if it is useful for teleportation, and if so how to proceed; there exists no general rule. Some special cases have been studied in the literature [8, 9, 10, 11, 12, 13, 14], and most of them are concerned with four-qubit channels. For example, Yeo and Chua [8] introduced a so-called "genuinely four-qubit entangled state" which is not reducible to a pair of Bell states, and showed that it could be used to teleport an arbitrary two-qubit state. Chen et al. [9] generalized the results of [8] to N-qubit teleportation. Rigolin [11] constructed 16 four-qubit entangled state which are useful for two-qubit teleportation. Agrawal and Pati considered teleportation using asymmetric W states. Muralidharan and Panigrahi [13] employed a"genuinely entangled" channel of five qubits to teleport two qubits. A criterion has been proposed by Zha and Song [14] in terms of the unitarity property of a "transformation matrix", which tells if a four-qubit entanglement channel supports two-qubit teleportation. However no general results exist in the literature when the quantum channel in question is an arbitrary multiparticle entangled state.

In this paper, we consider the most general situation in which Alice and Bob share an arbitrary bipartite state $|X\rangle_{A_1\ldots A_m B_1\ldots B_n}$ of $(m+n)$ qubits, of which m qubits belong to Alice and n to Bob. We shall first assume $m \geq n$; the $m < n$ case can later be included in a straightforward manner. As mentioned before, entanglement is the key ingredient which makes quantum teleportation possible. For an arbitrary bipartite state $|X\rangle_{A_1\ldots A_m B_1\ldots B_n}$ (or $|X\rangle_{AB}$ for short), the degree of entanglement between Alice's and Bob's subsystems can be quantified by the von Neumann entropy of either of two subsystems [15, 16], which is given by

$$E_{AB} = -\mathrm{Tr}\,(\rho_A \log_2 \rho_A) = -\mathrm{Tr}\,(\rho_B \log_2 \rho_B), \tag{6}$$

where ρ_A and ρ_B are the reduced density matrices of the subsystems,

$$\begin{aligned} \rho_A &= \mathrm{Tr}_B(|X\rangle_{AB}\langle X|_{AB}), &(7)\\ \rho_B &= \mathrm{Tr}_A(|X\rangle_{AB}\langle X|_{AB}). &(8)\end{aligned}$$

Since we assume $m \geq n$, therefore

$$E_{AB} \leq n. \tag{9}$$

Consider first the case

$$E_{AB} = n \tag{10}$$

so that the entanglement between the two subsystems is maximal. Then we must have

$$\rho_B = I_B/2^n, \tag{11}$$

where I_B is the $2^n \times 2^n$ identity matrix in Bob's Hilbert space H_B. Now consider another situation in which Alice and Bob share n pairs of Bell states, and in addition Alice owns an arbitrary pure state $|\mathcal{O}\rangle$ of $(m-n)$ qubits. The combined $(m+n)$-qubit state is

$$|\Lambda\rangle_{AB} = \Big(\prod_{i=1}^{n} |\phi^k\rangle_{A_i B_i}\Big)|\mathcal{O}\rangle_{A_{n+1}\ldots A_m}, \tag{12}$$

where $k \in \{1,2,3,4\}$. Although the properties of $|\mathcal{O}\rangle_{A_{n+1}\ldots A_m}$ are irrelevant, for definiteness and without loss of generality, we may take

$$|\mathcal{O}\rangle_{A_{n+1}\ldots A_m} = \prod_{j=n+1}^{m} |0\rangle_{A_j}. \tag{13}$$

The reduced density matrix on Bob's side is given by

$$\tilde{\rho}_B = \mathrm{Tr}_A\, |\Lambda\rangle_{AB}\langle\Lambda|_{AB} = I_B/2^n. \tag{14}$$

It follows that $|\Lambda\rangle_{AB}$ and $|X\rangle_{AB}$ are two purifications of the state $I_B/2^n$ in H_B to the joint space $H_B \otimes H_A$. By the freedom of purification [17], these two pure states are related by an unitary transformation on Alice's side. In other words, Alice can transform $|X\rangle_{AB}$ into $|\Lambda\rangle_{AB}$ by applying a local unitary operation $\mathcal{U}_A$ on her qubits,

$$|\Lambda\rangle_{AB} = \mathcal{U}_A\, |X\rangle_{AB}, \tag{15}$$

where $\mathcal{U}_A$ is a m-qubit unitary operator in H_A. It is important to note that Alice can carry out this transformation by herself, and there is no need for Bob to do anything. Note that, if $m < n$, then maximal entanglement means $E_{AB} = m$ and Bob must carry out the corresponding n-qubit transformation $\mathcal{U}_B$. And if $m = n$, either party can do it.

Eq. (15) essentially establishes that, for any arbitrary bipartite state $|X\rangle_{A_1\ldots A_m B_1\ldots B_n}$, if the von Neumann entropy of either of the subsystems is $n(\leq m)$, then it can be used to teleport faithfully an arbitrary n-qubit state. Conversely, by applying arbitrary unitary operators $\mathcal{U}_A^{-1}$ to the state $|\Lambda\rangle_{AB}$ given in Eq. (12), one can generate any number of states which support n-qubit teleportation. Indeed, all of the special channels proposed in the literature can be obtained this way [8, 9, 10, 11, 12, 13, 14].

Next we consider the non-maximally entangled case

$$E_{AB} < n. \tag{16}$$

In this case, perfect teleportation of an arbitrary n-qubit state will succeed only probabilistically. Nevertheless it may still be used to teleport a state of $d(< n)$ qubits. Let

$$|X'\rangle_{AB} = \mathcal{U}_B|X\rangle_{AB}, \tag{17}$$

where is $\mathcal{U}_B$ is an unitary operator in H_B which maximizes the value of $d(\leq E_{AB})$ in the following expression,

$$\rho'_B = \mathrm{Tr}_A|X'\rangle_{AB}\langle X'|_{AB} = \eta_{B'}\frac{1}{2^d}\prod_{i=1}^{d} I_{B_i}, \tag{18}$$

where I_{B_i} is the 2×2 identity operator for qubit B_i, and $\eta_{B'}$ is the density matrix of the qubits in $B' = \{B_{d+1}, ..., B_n\}$. (Note that some relabelling may be required.) Then using an unitary operator $\mathcal{U}_A$, Alice can transform $|X'\rangle_{AB}$ into a state

$$|\lambda\rangle_{AB} = \Big(\prod_{i=1}^{d} |\phi^k\rangle_{A_iB_i}\Big)|\varphi\rangle_{A'B'}, \tag{19}$$

where $k \in \{1,2,3,4\}$, $|\phi^k\rangle$ are Bell states, $A' = \{A_{d+1}, ..., A_m\}$, and

$$\mathrm{Tr}_{A'}|\varphi\rangle_{A'B'}\langle\varphi|_{A'B'} = \eta_{B'}. \tag{20}$$

This is so because $|X'\rangle_{AB}$ and $|\lambda\rangle_{AB}$ have the same reduced density matrix ρ'_B on Bob's side [17]. Hence

$$|\lambda\rangle_{AB} = \mathcal{U}_A\mathcal{U}_B|X\rangle_{AB}, \tag{21}$$

and Alice can employ the d shared Bell states in $|\lambda\rangle_{AB}$ to teleport an arbitrary d-qubit state to Bob. When $d = 0$, not even a single qubit can be teleported perfectly.

In the maximally entangled case ($E_{AB} = n$), we have $\mathcal{U}_B = I$, and $d = n$ in Eq. (18). So the general condition for faithful teleportation can be stated as follows. If there exists an unitary operator $\mathcal{U}_B$ such that Eq. (18) holds, then $|X\rangle_{AB}$ can be used to teleport faithfully an arbitrary state containing d qubits. This condition is also necessary. The following general protocol works for any arbitrary channel state $|X\rangle_{AB}$ satisfying Eq. (18): (1)Bob calculates $\mathcal{U}_B$ which determines the maximum number (d) of qubits that can be teleported, and Alice calculates $\mathcal{U}_A$. (2)Alice and Bob apply $\mathcal{U}_A$ and $\mathcal{U}_B$ respectively to the qubits in their control. (3)Then Alice can use the resulting d shared Bell states to teleport an arbitrary d-qubit state to Bob.

Note that, if $E_{AB} = n$, then Alice is required to perform one m-qubit operation and n Bell state measurements, and Bob is required to make n single qubit operations at most. In other protocols proposed for multiqubit teleportation, Alice is required to perform operations which are more complex than ours. For example, in [8, 9, 11, 12, 13], Alice is required to perform a joint operation involving $(m + n)$ qubits. In our case this operation is broken down to a series of operations involving smaller number of qubits, which are much easier to perform experimentally. On the other hand, if $E_{AB} < n$, then Bob may be required to perform a n-qubit operation $\mathcal{U}_B$ as well. The generalization to $m < n$ should be straightforward by now.

Using similar arguments, one can show that teleportation could also be performed without making Bell state measurements. For simplicity, we will show how it works for teleporting a one-qubit state $|\Psi\rangle_1$. The original procedure corresponds to Eq. (5) with $N = 1$. Let us replace the four Bell states $\{|\phi^1\rangle, |\phi^2\rangle, |\phi^3\rangle, |\phi^4\rangle\}$ by the four orthogonal product states $\{|\chi^i\rangle\} = \{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ respectively. The result is

$$|\xi\rangle_{1ab} = -\frac{1}{2}\Big(\sum_{i=1}^{4} |\chi^i\rangle_{1a} U_b^i\Big)|\Psi\rangle_b. \tag{22}$$

The reduced density matrix on Bob's side is

$$\mathrm{Tr}_{1a}(|\xi\rangle_{1ab}\langle\xi|_{1ab}) = I_b/2, \tag{23}$$

which is the same as that of the joint initial state $|\Psi\rangle_1|\phi^1\rangle_{ab}$. Therefore, again by the freedom of purification [17], $|\xi\rangle_{1ab}$ is related to $|\Psi\rangle_1|\phi^1\rangle_{ab}$ by an unitary transformation $\mathcal{U}_{1a}$ on Alice's side,

$$|\xi\rangle_{1ab} = \mathcal{U}_{1a}\Big(|\Psi\rangle_1|\phi^1\rangle_{ab}\Big). \tag{24}$$

It can be shown that (apart from an unimportant phase),

$$\mathcal{U}_{1a} = H_a C_1^a, \tag{25}$$

where H_a is the Hadamard operator, and C_1^a is the controlled-not operator with qubit 1 as the target. Hence the standard teleportation procedure can be replaced by the following alternative scheme: (1)Alice applies the unitary operator $\mathcal{U}_{1a}$ on qubits 1 and a. (2)She then individually measures the states of these qubits in the basis $\{|0\rangle, |1\rangle\}$, and sends the outcomes (i) to Bob. (3)Bob recovers the original state by applying the correctional unitary operator U_b^i to his qubit. Thus the Bell measurement in the original protocol [1] can be replaced by the unitary operation U_{1a} plus two single-qubit measurements. It turns out that this protocol is equivalent to the mysterious looking quantum computing circuit devised by Brassard et al. [18]. In a way, our derivation explains why there must exist an unitary quantum circuit for teleportation, even thought the original proposal of Bennett et al. used Bell state measurement.

Finally, as a simple demonstration, let us take the channel state to be the N-qubit GHZ state

$$|\mathrm{GHZ}\rangle_{1\ldots N} = \frac{1}{\sqrt{2}}(|0\ldots0\rangle_{1\ldots N} + |1\ldots1\rangle_{1\ldots N}). \tag{26}$$

No matter how the qubits are partitioned between Alice and Bob (provided that each party gets at least one qubit), the entropy of entanglement $E_{AB} = 1$, so it can be used to teleport a one-qubit state at most. If Alice holds qubits $\{1, ..., N-1\}$ and Bob holds the last qubit N, then the entanglement between the two subsystems is maximal. It follows from Eq. (15) that there exists an unitary operator $\mathcal{U}_A$ such that

$$\mathcal{U}_A|\mathrm{GHZ}\rangle_{1...N} = |\phi^4\rangle_{1N} \prod_{i=2}^{N-1} |0\rangle_i. \tag{27}$$

It is easy to show that $\mathcal{U}_A$ is just a series of controlled-not operators C_i^1:

$$\mathcal{U}_A = \prod_{i=2}^{N-1} C_i^1. \tag{28}$$

In this special case, Alice alone can transform $|\mathrm{GHZ}\rangle_{1...N}$ into the desired form given in Eq. (12). In general, if Alice holds qubits $\{1, ..., m\}$ and Bob holds qubits $\{m+1, ..., N\}$, the two subsystems are not maximally entangled. Then according to Eq. (21), both $\mathcal{U}_A$ and $\mathcal{U}_B$ are required, i.e.,

$$\mathcal{U}_A\mathcal{U}_B|\mathrm{GHZ}\rangle_{1...N} = |\phi^4\rangle_{1N} \prod_{i=2}^{N-1} |0\rangle_i, \tag{29}$$

where

$$\mathcal{U}_A = \prod_{i=2}^{m} C_i^1, \tag{30}$$

$$\mathcal{U}_B = \prod_{i=m+1}^{N-1} C_i^N. \tag{31}$$

In both cases, Alice and Bob share one Bell state, so the given channel can be used to teleport a single-qubit state only.

In summary, we have considered issues related to faithful teleportation when the given channel is an arbitrary $(m+n)$-qubit state $|X\rangle_{AB}$ partitioned between Alice and Bob in the (m, n) manner. A general condition, Eq. (18), is presented which allows one to judge if this channel can support faithful teleportation of an arbitrary d-qubit state, where $d \leq \min(m, n)$. We construct a general protocol which is applicable for any channel states satisfying this condition. Many special-case protocols have been proposed in the literature [8, 9, 11, 12, 13]. Compared with which, ours is easier to implement experimentally. For example, in the $d = n$ case, the most complex operation in our protocol involves m qubits, while the others involve $(m+n)$ qubits.

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