Monogamy and entanglement in tripartite quantum states

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Abstract

We present an interesting monogamy equation for $(2 \otimes 2 \otimes n)$-dimensional pure states, by which a quantity is found to characterize the tripartite entanglement with the GHZ type and W type entanglements as a whole. In particular, we, for the first time, reveals that for any quantum state of a pair of qubits, the difference between the two remarkable entanglement measures, concurrence and negativity, characterizes the W type entanglement of tripartite pure states with the two-qubit state as reduced density.

Key words: entanglement measure, monogamy, tripartite entanglement

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1 Introduction

Entanglement or the nonseparability of quantum states of composite systems is an essential feature of quantum mechanics and plays crucial role in various applications in quantum information processing [1-4]. It is of a paramount importance and one of the main tasks of quantum entanglement theory to quantitively characterize the extent to which composite quantum systems are entangled by constructing a so-called entanglement measure (a mathematical function) that should not increase on averaging under local operations and classical communications (LOCC). However, only the entanglements of bipartite pure states and low-dimensional systems are well understood [5-7]. Needless to say, it is still a challenge to quantify entanglement for high-dimensional systems and multipartite mixed states, the quantification of entanglement is not well solved even for multipartite pure states [8]. One of the main reasons is that multipartite entanglement can be classified into many inequivalent classes [9-11]. Therefore, we have to concern which class the multipartite entanglement belongs to when we consider multipartite entanglement.

In recent years, several works have shown that multipartite entanglements of some kinds have close contact with the monogamy of entanglement, a key property of entanglement which, quite different from classical correlation, demonstrates that the degree to which either of two parties can be entangled with anything else seems to be constrained by the entanglement that may exist between the two quantum parties [12-14]. A remarkable example [15] is that the residual entanglement of the Coffman-Kundu-Wootters (CKW) inequality characterizes the GHZ type entanglement of tripartite pure states of qubits. The same residual entanglement can also be obtained from the monogamy inequality dual to CKW inequality based on concurrence of assistance (COA) [16] presented for tripartite systems of qubits by Gour et al [17]. Quite recently, we have found an interesting monogamy equation for $(2 \otimes 2 \otimes n)$-dimensional (or multiple qubits) quantum pure states which relates the bipartite concurrence, COA and GHZ type tripartite entanglement [18,19]. In this paper, we first present a new monogamy equation for $(2 \otimes 2 \otimes n)$-dimensional (or multiple qubits) quantum pure states. The distinct advantage is that the "residual quantity" deduced from the equation can characterize the tripartite entanglement with GHZ type and W type entanglements as a whole. Furthermore, it is invariant under local unitary transformations and does not increase under the local operations performed on the n- dimensional subsystem, hence it is an entanglement semi-monotone.

What is more, it is well-known to all that there are two remarkable entanglement measures for bipartite systems of qubits——— One is the concurrence [20] and the other is the negativity [21,22]. It has been shown that the negativity is not greater than the concurrence [23]. But what does the difference between the two entanglement measures imply? In this paper, based on the new monogamy equation we find a surprising fact that the tripartite W type entanglement of a $(2\otimes 2\otimes n)$-dimensional quantum pure state can be characterized by the difference between concurrence and negativity of the $(2\otimes 2)$ -dimensional reduced density matrix. This paper is organized as follows. We first introduce our interesting monogamy inequality for tripartite pure states; Then we reveals that the difference between concurrence and negativity can characterize the W type entanglement; The conclusion is drawn finally.

2 Monogamy and entanglement for tripartite pure states

Let us first briefly recall the concurrence and the negativity for the bipartite quantum state ϱ of qubits. The concurrence is defined as

$$C\left(\varrho\right) = \max\{0, \lambda_1 - \sum_{i>1}\lambda_i\}, \tag{1}$$

where λ_i are the square roots of the eigenvalues of $\varrho\tilde{\varrho}$ in decreasing order with $\tilde{\varrho} = (\sigma_y\otimes\sigma_y)\,\varrho^*\,(\sigma_y\otimes\sigma_y)$ and σ_y is the Pauli matrix, and the negativity is defined as [22]

$$N\left(\varrho\right) = \left\|\varrho^{T_\alpha}\right\|_1 - 1 \tag{2}$$

which corresponds to the doubled absolute value of the sum of the negative eigenvalues of ϱ^{T_α}, where ϱ^{T_α} denotes the partial transpose of ϱ and $\|\cdot\|$ is the trace norm of a matrix. It has been shown [23] that

$$C\left(\varrho\right) \geq N\left(\varrho\right) \tag{3}$$

where the " $=$ " always holds for pure ϱ and some given mixed states [24].

Now suppose that a tripartite $(2 \otimes 2 \otimes n)$- dimensional pure state $|\Psi\rangle_{ABC}$ is shared by three parties Alice, Bob and Charlie where the subsystem at Charlie's side is an auxilliary one. The $(2 \otimes 2)$- dimensional reduced density matrix by tracing over party C can be given by $\rho_{AB} = Tr_C\left(|\Psi\rangle_{ABC}\langle\Psi|\right)$. Based on GHJW theorem [25,26], any decomposition of ρ_{AB} can always be realized with the help of Charlie who can perform positive-operator-value-measurements [27] on his subsystem C. Let $\mathcal{E} = \{p_i, \left|\varphi_i^{AB}\right\rangle\}$ is a decomposition of ρ_{AB} such that

$$\rho_{AB} = \sum_i p_i \left|\varphi_i^{AB}\right\rangle\left\langle\varphi_i^{AB}\right|, \sum_i p_i = 1. \tag{4}$$

Oppositely to the mixed-state concurrence which is defined by the minimal average pure-state concurrence on $\mathcal{E}$, the COA is defined [28,29] as

$$C_a\left(|\Psi\rangle_{ABC}\right) = \max_{\mathcal{E}} \sum_i p_i C\left(\left|\varphi_i^{AB}\right\rangle\right) \tag{5}$$

$$= C_a\left(\rho_{AB}\right) = tr\sqrt{\sqrt{\rho_{AB}}\tilde{\rho}_{AB}\sqrt{\rho_{AB}}} = \sum_{i=1}^{4} \lambda_i, \tag{6}$$

where Charlie is to maximize the entanglement shared by Alice and Bob and the corresponding parameters are defined the same as eq. (1). Besides, it should be noted that the COA is a tripartite entanglement monotone instead of a bipartite one [30]. Quite recently we have found an interesting monogamy equation [1],

$$C_a^2(\rho_{AB}) - C^2(\rho_{AB}) = \tau^2(|\Psi\rangle_{ABC}), \tag{7}$$

where $\tau(|\Psi\rangle_{ABC})$ is a good entanglement measure for GHZ type entanglement. Since eq. (3) always holds for the states of two qubits, if one replaces the concurrence by the negativity, eq. (7) can lead to another monogamy equation as

$$C_a^2(\rho_{AB}) - N^2(\rho_{AB}) = \chi^2(|\Psi\rangle_{ABC}). \tag{8}$$

Now we claim

Theorem 1: For a $(2 \otimes 2 \otimes n)$- dimensional quantum pure state $|\Psi\rangle_{ABC}$, $\chi(|\Psi\rangle_{ABC})$ is an entanglement semi-monotone that is invariant under local unitary transformations and does not increase under the local operations performed on the n- dimensional subsystem. $\chi(|\Psi\rangle_{ABC})$ characterizes tripartite entanglement with GHZ type and W type entanglement as a whole.

Proof. It is obvious that

$$C_a^2(\rho_{AB}) - N^2(\rho_{AB}) \geq 0. \tag{9}$$

Furthermore, according to the definitions of COA and negativity, one can find

[1] The monogamy relation is different from the familiar monogamy first introduced in Ref. [9]. However, it does not violate the property of monogamy, which does show the limitation of entanglement shared between different parties. In this sense (at least a generalized sense), all that imply the similar limitations of entanglement are considered as monogamy.

that it is impossible to change $\chi(|\Psi\rangle_{ABC})$ by local unitary transformations on any subsystems. The most general local operations can be given in terms of Kraus operators denoted by M_k, $\sum_k M_k^\dagger M_k \leq I_C$. The state after the operation M_k can be represented by

$$|\psi\rangle_k = (I_{AB} \otimes M_k) |\Psi\rangle_{ABC} \langle\Psi| \left(I_{AB} \otimes M_k^\dagger\right) /p_k,$$

with I_{AB} being the $(2 \otimes 2)$-dimensional identity and $p_k = \mathrm{tr}|\psi\rangle_{kk} \langle\psi|$. Thus the average $\bar{\chi}(\cdot)$ can be given by

$$\begin{aligned}
&\bar{\chi} = \sum_k p_k \chi(|\psi\rangle_k) \\
&= \sum_k p_k \sqrt{C_a^2 (|\psi\rangle_k) - N^2 (|\psi\rangle_k)} \\
&= \sum_k \sqrt{p_k \left[C_a (|\psi\rangle_k) - N (|\psi\rangle_k)\right]} \\
&\quad \times \sqrt{p_k \left[C_a (|\psi\rangle_k) + N (|\psi\rangle_k)\right]} \\
&\leq \left\{\sum_k p_k \left[C_a (|\psi\rangle_k) - N (|\psi\rangle_k)\right]\right\}^{1/2} \\
&\quad \times \left\{\sum_k p_k \left[C_a (|\psi\rangle_k) + N (|\psi\rangle_k)\right]\right\}^{1/2} \\
&= \sqrt{\left[\sum_k p_k C_a (|\psi\rangle_k)\right]^2 - \left[\sum_k p_k N (|\psi\rangle_k)\right]^2} \\
&\leq \sqrt{C_a^2 (\rho_{AB}) - N^2 (\rho_{AB})} = \chi(|\Psi\rangle_{ABC}).
\end{aligned} \tag{10}$$

Here the first inequality follows from the Cauchy-Schwarz inequality

$$\left(\sum_i x_i^2\right)^{1/2} \left(\sum_j y_j^2\right)^{1/2} \geq \sum_k x_k y_k, x, y \geq 0. \tag{11}$$

The second inequality is derived from the definition of COA and the convexity of negativity. Eq. (10) shows that $\chi(|\Psi\rangle_{ABC})$ is an entanglement semi-monotone.

Now we show $\chi(|\Psi\rangle_{ABC})$ characterizes the tripartite entanglement of $|\Psi\rangle_{ABC}$ with the GHZ type and W type entanglement as a whole. We first show that $\chi(|\Psi\rangle_{ABC})$ vanishes for separable states and then show that $\chi(|\Psi\rangle_{ABC})$ has nonzero value for both GHZ type entanglement and W type entanglement. Without loss of generality, a separable tripartite pure state can be written as

$$|\Phi_1\rangle_{ABC} = |\psi\rangle_{AB} \otimes |\phi\rangle_C \tag{12}$$

or

$$|\Phi_2\rangle_{ABC} = |\phi\rangle_A \otimes |\psi\rangle_{BC} \tag{13}$$

where $|\phi\rangle$ represents a pure state of a single qubit and $|\psi\rangle$ denotes a general bipartite pure state which may be entangled or not. Thus a fully separable state can be included in either of eq. (12) and eq. (13). The reduced density matrix ρ_{AB1} of $|\Phi_1\rangle_{ABC}$ is pure, hence

$$C_a\left(\rho_{AB1}\right) = N\left(\rho_{AB1}\right) = C\left(\rho_{AB1}\right), \tag{14}$$

which shows $\chi(|\Phi_1\rangle_{ABC}) = 0$. The reduced density matrix ρ_{AB2} of $|\Phi_2\rangle_{ABC}$ is separable, hence $C_a\left(\rho_{AB1}\right) = N\left(\rho_{AB1}\right) = C\left(\rho_{AB1}\right) = 0$, which also shows $\chi(|\Phi_2\rangle_{ABC}) = 0$. Therefore, $\chi\left(\cdot\right)$ vanishes for separable states.

Since GHZ type and W type entanglement are distinguished by sLOCC operations, which shows that quantum states belonging to the same type can be converted to each other by invertible local operations[9], it is enough to only consider the standard states corresponding to every type entanglement.

Generically, GHZ type entangled state can be written as [9]

$$|\psi_{GHZ}\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\theta} |111\rangle , \tag{15}$$

where $\lambda_i > 0$, $\sum\limits_i \lambda_i^2 = 1$, $\theta \in [0, \pi]$. W type entangled state can be given by [9]

$$|\psi_W\rangle = \tilde{\lambda}_0 |001\rangle + \tilde{\lambda}_1 |010\rangle + \tilde{\lambda}_2 |100\rangle + \tilde{\lambda}_3 |000\rangle , \tag{16}$$

where $\tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_2 > 0$, $\tilde{\lambda}_3 = \sqrt{1 - \sum_{i=0}^{2} \tilde{\lambda}_i^2}$. It is easy to see that

$$C_a (|\psi_{GHZ}\rangle) = 2\lambda_0\lambda_1 \neq 0, N(\rho'_{AB}) = 0 \tag{17}$$

with $\rho'_{AB} = \mathrm{tr}_C |\psi_{GHZ}\rangle \langle\psi_{GHZ}|$ and

$$C_a (|\psi_W\rangle) = C(\varrho'_{AB}) = 2\tilde{\lambda}_1\tilde{\lambda}_2, \tag{18}$$

$$N(\varrho'_{AB}) = \sqrt{\tilde{\lambda}_0^4 + 4\tilde{\lambda}_1^2\tilde{\lambda}_2^2} - \tilde{\lambda}_0^2, \tag{19}$$

with $\varrho'_{AB} = \mathrm{tr}_C |\psi_W\rangle \langle\psi_W|$. Eq. (17) shows that $\chi(\cdot)$ has nonzero value for GHZ state. Eq. (18) and eq. (19) lead to

$$C_a (|\psi_W\rangle) = C(\varrho'_{AB}) > N(\varrho'_{AB}), \tag{20}$$

which shows that $\chi(\cdot)$ has also nonzero value for W state.

The previous paragraph has shown that $\chi(\cdot)$ has also nonzero value for tripartite quantum pure state with local rank (2,2,2) [11]. The local rank is defined as the rank of the reduced density matrix traced out for all except one party. It has been shown that $(2 \otimes 2 \otimes n)$- dimensional quantum pure states can be divided into 9 classes in terms of different local ranks. The standard states of the classes corresponding to high local ranks can be given by

$$|\Phi\rangle_{223} = |000\rangle + |011\rangle + |112\rangle\,, \tag{21}$$

$$|\Phi'\rangle_{223} = |000\rangle + \frac{1}{\sqrt{2}}\left(|011\rangle + |101\rangle + |112\rangle\right), \tag{22}$$

$$|\Phi\rangle_{224} = |000\rangle + |011\rangle + |102\rangle + |113\rangle\,, \tag{23}$$

where we omit the normalized constant and the subscripts denote the local rank. For example, '223' denotes the local rank is (2,2,3) and so on. A quantum pure state with high local rank can always be converted into these standard states with corresponding local rank based on stochastic LOCC operations. One can also verify that $\chi(\cdot)$ does not vanish for the standard states with high local ranks. That is to say, $\chi(\cdot)$ vanishes for separable states including partially entangled and fully separable states and has nonzero value for tripartite entangled states, hence $\chi(\cdot)$ characterizes the tripartite entanglement.

The entanglement monogamy of eq. (7) and eq. (8) is embodied respectively in limited GHZ type entanglement and the general tripartite entanglement (W type entanglement is implied) by bipartite entanglement. That is to say, when the maximal and minimal entanglement shared by two parties are close enough, the two parties can not entangle with a third party. Since $\chi(\cdot)$ in eq. (8) characterizes the tripartite entanglement and $\tau^2(\cdot)$ in eq. (7) characterizes the GHZ type entanglement, it is a natural conjecture that $\chi(\cdot) - \tau^2(\cdot)$ should characterizes the W type entanglement.

Theorem 2.-For a $(2 \otimes 2 \otimes n)$- dimensional quantum pure state $|\Psi\rangle_{ABC}$,

$\varpi(|\Psi\rangle_{ABC})$ is defined as

$$\varpi(|\Psi\rangle_{ABC}) = \varpi(\rho_{AB}) = C^2(\rho_{AB}) - N^2(\rho_{AB}), \tag{24}$$

which is invariant under local unitary transformations and characterizes the W type entanglement, where $\rho_{AB} = Tr_C\left(|\Psi\rangle_{ABC}\langle\Psi|\right)$.

Proof. Analogously to Theorem 1, it is obvious that $\varpi(|\Psi\rangle_{ABC})$ is invariant under local unitary transformations. One can find that $\varpi(\cdot)$ is zero for separable states given in eq. (12) and eq. (13). A simple calculation can also show that the concurrence and the negativity both vanish for the reduced density matrix of $|\psi_{GHZ}\rangle$. However, for the $|\psi_W\rangle$ one can find that $\varpi(\rho_{AB})$ is always nonzero in terms of eq. (20). It is interesting that $\varpi(\cdot)$ vanishes for the standard states with high local ranks given by eqs. (21-23). That is to say, $\varpi(\cdot)$ only characterizes the W type entanglement with local rank $(2,2,2)$, i.e., W type entanglement, even the standard states with high local ranks can be converted to the W type entangled states with local rank $(2,2,2)$. In fact, it is not strange. It has been shown that $\tau(\cdot)$ quantify GHZ type entanglement by considering GHZ type entanglement with local rank $(2,2,2)$ as minimal unit, hence the entanglement states with high local ranks has been quantified as GHZ type entanglement. Thus the contributions of the standard states with high local ranks has been subtracted from the total tripartite entanglement $\chi(\cdot)$. The remaining is only the W type entanglement with local rank $(2,2,2)$.

Remark.-For a tripartite $(2\otimes 2\otimes n)$ -dimensional pure state $|\Psi\rangle_{ABC}$, let

$$\eta = C(\rho_{AB}) - N(\rho_{AB}), \tag{25}$$

with $\rho_{AB} = tr_C |\Psi\rangle_{ABC} \langle\Psi|$, then η characterizes W type entanglement.

Proof. The proof is straightforward in terms of theorem 2. One might wonder why the characterization of W type entanglement including Theorem 2 is the substraction of two different entanglement measures. In fact, concurrence of bipartite reduced density can also well distinguish W type from GHZ type entanglement. But concurrence per se can not distinguish states with W type entanglement from some separable states.

Before the end, we would like to emphasize that, even though $\chi(\cdot)$, $\varpi(\cdot)$ and $\eta(\cdot)$ are not entanglement monotones, but invariant under local unitary transformations (or only an entanglement semi-monotone), in many cases they can be safely used because it was shown in Ref. [8] that it is not necessary for an entanglement measure to be always an entanglement monotone. For example, when Alice prepares a $(2\otimes 2\otimes n)$- dimensional quantum pure state, and only sends the n-dimensional qudit to Bob via a quantum channel, Alice and Bob can safely employ $\chi(\cdot)$ to study the evolution of the tripartite entanglement. In addition, we present theorem 2 and the Remark is in order to reveal the nature of the difference between the concurrence and the negativity instead of only to present a W type entanglement measure. Of course, $\varpi(\cdot)$ and η are both invariant under local unitary transformations, which can also be employed to measure W type entanglement in some field. Both eq. (8) and eq. (24), as well as eq. (25), can be generalized to mixed states by extending the involved entanglement measures of pure states to mixed states in terms of convex roof construction. But even though the generalized equations can provide the monogamy relationship of entanglement, the corresponding $\chi(\cdot)$ or $\varpi(\cdot)$ generalized for mixed states can not exactly characterize tripartite entanglement as for pure states.

3 Conclusion and discussion

We have presented an interesting monogamy equation of entanglement by which a quantity $\chi(\cdot)$ can be found to characterize tripartite $(2\otimes 2\otimes n)$-dimensional quantum pure states with the GHZ type and W type entanglements as a whole. In particular, we find that the W type entanglement of $(2\otimes 2\otimes n)$-dimensional pure states can be characterized by the difference between the two remarkable entanglement measures, the concurrence and the negativity of the $(2\otimes 2)$ -dimensional reduced density matrix of the tripartite pure states. Finally, we have to mention that $\chi(\cdot)$ is an entanglement semi-monotone and $\varpi(\cdot)$ and η are both invariant under local unitary transformations [31]. However, they show the interesting relations between different entanglement measures and reveal some valuable implication after all. It is our forthcoming work to seek for the corresponding entanglement monotones.

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References

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[2] M. Zukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, Phys. Rev. Lett. **71**,1993, 4287.

[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. **70**,1993, 1895.

[4] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 1992, 2881.

[5] C. H. Bennet, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**,1996, 3824.

[6] W. K. Wootters, Quant. inf. Comp. **1**, 2001, 27.

[7] P. Horodecki and R. Horodecki, Quant. inf. Comp. **1**, 2001, 45.

[8] M. B. Plenio and S. Virmani, Quant. inf. Comp. **7**, 2007, 1.

[9] W. Dür, G. Vidal and J. I. Cirac, Phys. Rev. A **62**, 2000, 062314.

[10] A. Miyake, Phys. Rev. A **67**, 2003, 012108.

[11] A. Miyake and F. Verstraete, Phys. Rev. A **69**, 2004, 012101.

[12] B. M. Terhal, IBM J. Res. Dev. **48**, 2004, 71.

[13] M. Koashi and A. Winter, Phys. Rev. A **69**, 2004, 022309 and the references therein.

[14] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 2006, 220503.

[15] V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A **61**, 2000, 052306.

[16] T. Laustsen, F. Verstraete and S. J. Van Enk, Quant. Info. Comp. **4**, 2003, 64.

[17] G. Gour, D. A. Meyer and B. C. Sanders, Phys. Rev. A **72**, 2005, 042329.

[18] Chang-shui Yu and He-shan Song, Phys. Rev. A **76**, 2007, 022324.

[19] Chang-shui Yu and He-shan Song, Phys. Rev. A **77**, 2008, 032329.

[20] W. K. Wootters, Phys. Rev. Lett. **80**, 1998, 2245.

[21] A. Peres, Phys. Rev. Lett. **77**, 1996, 1413.

[22] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 2002, 032314.

[23] K. Audenaert, F. Verstraete, T. De Bie and B. De Moor, quant-ph/0012074.

[24] Frank Verstraete, Koenraad Audenaert, Jeroen Dehaene and Bart De Moor, J. Phys. A **34**, 2001, 10327.

[25] N. Gisin, Helv. Phys. Acta **62**, 1989, 363.

[26] L. P. Hughston, R. Josza and W. K. Wootters, Phys. Lett. A **183**, 1993, 14.

[27] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[28] G. Gour and R. W. Spekkens, Phys. Rev. A **73**, 2006, 062331.

[29] G. Gour, Phys. Rev. A **72**, 2005, 042318.

[30] G. Gour, Phys. Rev. A **71**, 2005, 012318.

[31] G. Vidal, J. Mod. Opt **47**, 2000, 355.