Turk J Phys
31 (2007) , 191 – 196.

Time Dependent Entropy of Constant Force Motion

Özgür ÖZCAN[1], **Ethem AKTÜRK**[2], **Ramazan SEVER**[3]
[1] *Hacettepe University, Department of Physics Education, Beytepe 06800, Ankara-TURKEY*
e-mail: ozcano@hacettepe.edu.tr
[2] *Hacettepe University, Department of Physics Engineering, Beytepe 06800, Ankara-TURKEY*
e-mail: eakturk@hacettepe.edu.tr
[3] *Middle East Technical University, Department of Physics, ODTÜ, Ankara-TURKEY*
e-mail: sever@metu.edu.tr

Received 07.02.2007

Abstract

Time dependent entropy of constant force motion is investigated. Obtained is the joint entropy, also known as the Leipnik entropy. The main purpose of this work is to calculate the Leipnik entropy via a time dependent wave function which is obtained by the Feynman path integral method. For this case it is found the Leipnik entropy increases with time, and is the same behavior as in the free particle case.

Key Words: Path integral, joint entropy, constant force motion.

1. Introduction

Information entropy plays a major role in a stronger formulation of the uncertainty relations [1]. This formulation may be mathematically defined by using the Boltzmann-Shannon information entropy and the von Neumann entropy. Unlike in classical (Shannon) information theory, quantum (von Neumann) conditional entropies can be negative when considering quantum entangled systems, a fact related to quantum non-separability. The possibility that negative (virtual) information can be carried by entangled particles suggests a consistent interpretation of quantum informational processes [2].

In the literature for both open and closed quantum systems, different information-theoretical entropy measures have been discussed [3, 4, 5]. In contrast, joint entropy [6, 7] can also be used to characterize the loss of information related to evolving pure quantum states [8]. The joint entropy of physical systems were conjectured by Dunkel and Trigger [9], and named their systems MACS, maximal classical states. The Leipnik entropy of the simple harmonic oscillator was found not to monotonically increase with time [10]. In this work, we give a uniform description of the complete joint entropy information of systems in motion under a constant force.

This paper is organized as follows. In section II, we explain fundamental definitions needed for the calculation. In section III, we deal with calculation and results for constant force systems. Moreover, we obtain the analytical solution of Kernel, wave function in both coordinate and momentum space and its joint entropy. Finally, we present the conclusion in section IV.

2. Fundamental Definitions

We consider a classical system with $n = sN$ degrees of freedom, where N is the particle number and s is number of spatial dimensions. Apart from this, let us describe $g(x, p, t) = g(x_1, ..., x_n, p_1, ..., p_d, t)$. It is non-negative, time dependent phase space density function of system. The density function is assumed normalized to unity:

$$\int dx\, dp\, g(x,p,t) = 1. \tag{1}$$

The Gibbs-Shannon entropy is described by

$$S(t) = -\frac{1}{N!}\int dx\, dp\, g(x,p,t)\ln(h^d g(x,p,t)), \tag{2}$$

where $h = 2\pi\hbar$ is the Planck constant. Schrödinger wave equation with the Born interpretation [11] is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi. \tag{3}$$

The quantum probability densities are defined in position and momentum spaces as $|\psi(x,t)|^2$ and $|\tilde{\psi}(p,t)|^2$, where $|\tilde{\psi}(p,t)|^2$ is given as

$$\tilde{\psi}(p,t) = \int \frac{dx\, e^{-ipx/\hbar}}{(2\pi\hbar)^{d/2}}\psi(x,t). \tag{4}$$

Leipnik proposed the product function as

$$g_j(x,p,t) = |\psi(x,t)|^2|\tilde{\psi}(p,t)|^2 \geq 0. \tag{5}$$

Substituting equation (5) into equation (2), we get the joint entropy $S_j(t)$ for the pure state $\psi(x,t)$, or equivalently, can be written in the form [9]

$$S_j(t) = -\int dx|\psi(x,t)|^2\ln|\psi(x,t)|^2 - \int dp|\tilde{\psi}(p,t)|^2\ln|\tilde{\psi}(p,t)|^2 - \ln h^d. \tag{6}$$

We find time dependent wave function by means of the Feynman path integral, which has form [12]

$$\begin{aligned} K(x'',t'';x',t') &= \int_{x'=x(t')}^{x''=x(t'')} Dx(t)e^{\frac{i}{\hbar}S[x(t)]} \\ &= \int_{x'}^{x''} Dx(t)e^{\frac{i}{\hbar}\int_{t'}^{t''}L[x,\dot{x},t]dt}. \end{aligned} \tag{7}$$

The Feynman kernel can be related to the time dependent Schrödinger's wave function:

$$K(x'',t'';x',t') = \sum_{n=0}^{\infty}\psi_n^*(x',t')\psi_n(x'',t''). \tag{8}$$

The propagator in semiclassical approximation reads

$$K(x'',t'';x',t') = \left[\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial x'\partial x''}S_{cl}(x'',t'';x',t')\right]^{1/2}e^{\frac{i}{\hbar}S_{cl}(x'',t'';x',t')}. \tag{9}$$

The prefactor is often referred to as the Van Vleck-Pauli-Morette determinant [13, 14]. The function $F(x'',t'';x',t')$ is given by

$$F(x'',t'';x',t') = \left[\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial x'\partial x''}S_{cl}(x'',t'';x',t')\right]^{1/2}. \tag{10}$$

3. Calculation and Results

3.1. Constant Force

The Lagrangian for the present case is

$$L(x,\dot{x},t)=\frac{1}{2}m\dot{x}^2+fx \tag{11}$$

The classical path obeys

$$m\ddot{x}_{cl}=f. \tag{12}$$

The solution of the above equation is

$$x_{cl}(\tau)=x_0+\Big(\frac{x-x_0}{t-t_0}-\frac{1}{2}\frac{f}{m}(t-t_0)\Big)\tau+\frac{1}{2}\frac{f}{m}\tau^2. \tag{13}$$

One obtains for classical action integral along the classical path

$$S(x_{cl}(\tau))=\frac{1}{2}m\frac{(x-x_0)^2}{t-t_0}+\frac{1}{2}(x+x_0)f(t-t_0)-\frac{1}{24}\frac{f^2}{m}(t-t_0)^3. \tag{14}$$

And finally, for the kernel we have

$$K(x'',x';T)=\Big[\frac{m}{2\pi i\hbar T}\Big]^{1/2}\exp\left[\frac{im}{2\hbar}\frac{(x-x_0)^2}{T}+\frac{i}{2\hbar}(x+x_0)fT-\frac{i}{24\hbar}\frac{f^2}{m}T^3\right]. \tag{15}$$

The dependent wave function at time $t>0$ is

$$\begin{aligned}\Psi(x,t) \quad &= \quad \Big[\frac{1-i\frac{\hbar t}{m\sigma^2}}{1+i\frac{\hbar t}{m\sigma^2}}\Big]^{1/4}\Big[\frac{1}{\pi\sigma^2(1+\frac{\hbar^2t^2}{m^2\sigma^4})}\Big]^{1/4}\exp\Big[-\frac{(x-\frac{p_0}{m}t-\frac{ft^2}{2m})^2}{2\sigma^2(1+\frac{\hbar^2t^2}{m^2\sigma^4})}\times \\ &\times \quad (1-i\frac{\hbar t}{m\sigma^2})\Big]\exp\Big[\frac{i}{\hbar}(p_0+ft)x-\frac{i}{\hbar}\int_0^t d\tau\frac{(p_0+f\tau)^2}{2m}\Big]\end{aligned} \tag{16}$$

$$\begin{aligned}\Psi(x,t)=\left[\frac{1-i\frac{\hbar t}{m\sigma^2}}{1+i\frac{\hbar t}{m\sigma^2}}\right]^{1/4}\left[\frac{1}{\pi\sigma^2\left(1+\frac{\hbar^2t^2}{m^2\sigma^4}\right)}\right]^{1/4}&\times\exp\left[-\frac{\left(x-\frac{p_0}{m}t-\frac{ft^2}{2m}\right)^2}{2\sigma^2\left(1+\frac{\hbar^2t^2}{m^2\sigma^4}\right)}\left(1-i\frac{\hbar t}{m\sigma^2}\right)\right] \\ &\times\exp\left[\frac{i}{\hbar}(p_0+ft)x-\frac{i}{\hbar}\int_0^t d\tau\frac{(p_0+f\tau)^2}{2m}\right],\end{aligned} \tag{17}$$

where σ is width of Gaussian curve. The corresponding probability distribution is

$$|\Psi(x,t)|^2=\left[\frac{1}{\pi\sigma^2(1+\frac{\hbar^2t^2}{m^2\sigma^4})}\right]^{1/2}\exp\left[-\frac{\left(x-\frac{p_0t}{m}-\frac{ft^2}{2m}\right)^2}{\sigma^2\left(1+\frac{\hbar^2t^2}{m^2\sigma^4}\right)}\right], \tag{18}$$

or

$$|\Psi(x,t)|^2=\left[\frac{1}{\pi\sigma^2(1+\frac{\hbar^2t^2}{m^2\sigma^4})}\right]^{1/2}\exp\left[-\frac{\left(x-\frac{p_0t}{m}-\frac{ft^2}{2m}\right)^2}{\sigma^2\left(1+\frac{\hbar^2t^2}{m^2\sigma^4}\right)}\right]. \tag{19}$$

The probability density in coordinate space is shown Figure 1. The probability density in momentum space can be written easily as

$$|\Psi(p,t)|^2 = \left[\frac{\sigma^2}{\pi\hbar^2}\right]^{1/2} \exp\left[\frac{-\sigma^2}{\hbar^2}(p+(p_0+ft))^2\right] \tag{20}$$

The time dependent joint entropy can be obtained from equation 6 as

$$S_j(t) = \ln\left(\frac{e}{2}\right)\sqrt{1+\frac{\hbar^2 t^2}{m^2\sigma^4}}. \tag{21}$$

The joint entropy of this system is shown Figure 2 and Figure 3. It is important that equation 20 is in agreement with following general inequality for the joint entropy:

$$S_j(t) \geq \ln\left(\frac{e}{2}\right) \tag{22}$$

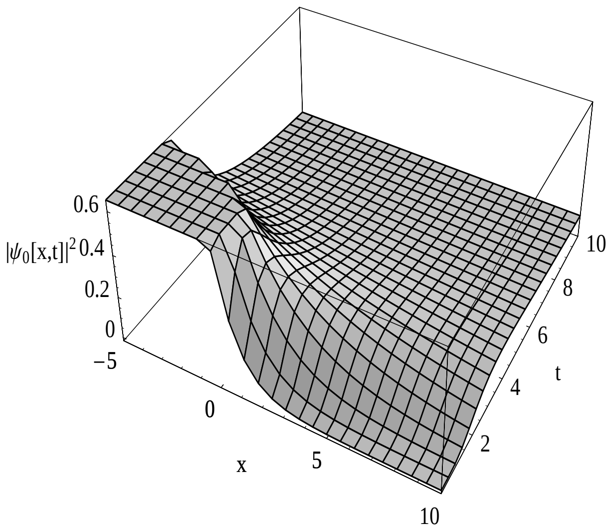

Figure 1. $|\Psi(x,t)|^2$ versus time and coordinate

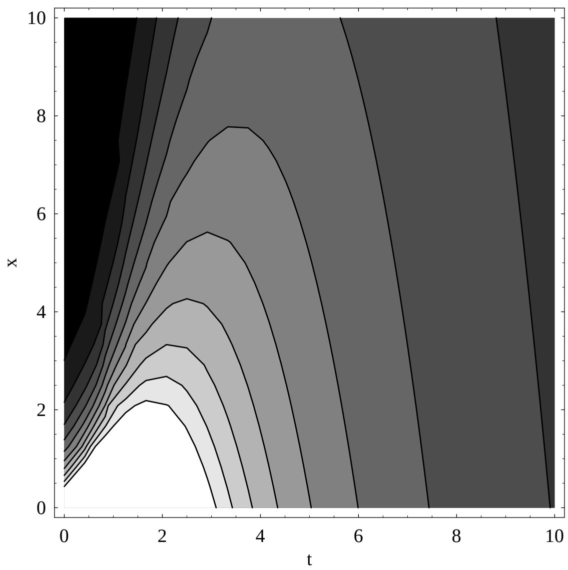

Figure 2. The counter graph of $|\Psi(x,t)|^2$

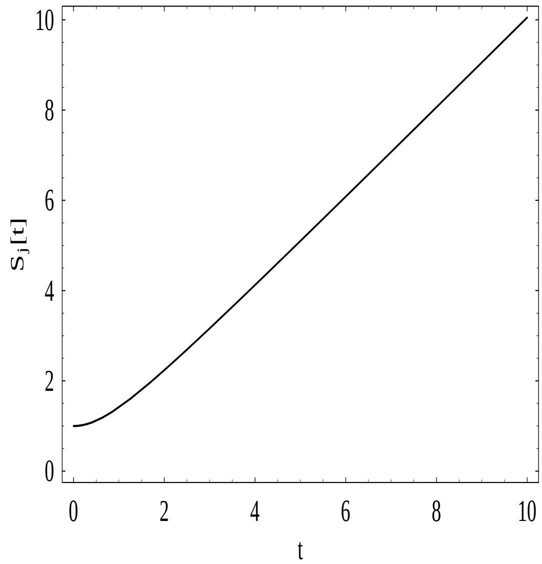

Figure 3. The joint entropy of constant forces motion versus time

originally derived by Leipnik for arbitrary one-dimensional one-particle wave functions.

4. Conclusion

We have investigated the joint entrophy for a motion going with a constant force. We have obtained the time dependent wave function by means of Feynman Path integral technique. We have found that the joint entropy increases with time and the results are in harmony with prior studies. Joint entropy has same the behavior as the free particle case. This result indicates that the information entropy increases with time.

Acknowledgements

This research was partially supported by the Scientific and Technological Research Council of Turkey.

References

[1] E. Aydiner , C. Orta and R.Sever, *quant-ph/0602203*

[2] N. J. Cerf and C. Adami, *Phys. Rev. Lett.*, **79**, (1997), 5194-5197.

[3] W. H. Zurek, *Phys. Today*, **44**, (1991), 36.

[4] R. Omnes, *Rev. Mod. Phys.*, **64**, (1992), 339.

[5] C. Anastopoulos, *Ann. Phys.*, **303**, (2003), 275.

[6] R. Leipnik, *Inf. Control.*, **2**, (1959), 64.

[7] V. V.Dodonov, *J.Opt. B: Quantum Semiclassical Opt*, **4**, (2002), 98.

[8] S. A. Trigger, Bull. Lebedev Phys. Inst., **9**, (2004), 44.

[9] J. Dunkel and S. A. Trigger, *Phys. Rev.A*, **71**, (2005), 052102.

[10] P. Garbaczewski, *Phys. Rev. A*, **72**, (2005), 056101.

[11] M. Born, *Z. Phys.*, **40**, (1926), 167.

[12] R. P. Feynman, A. R. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw-Hill, USA 1965).

[13] D. C. Khandekar, S.V. Lawande, K.V. Bhagwat, *Path-Integral Methods and Their Applications*, (World Scientific, Singapore 1993).

[14] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, (World Scientific, 3rd Edition, Singapore 2004).