

Non-homogeneous Bell-type Inequalities for Two- and Three-qubit States

Mingjun Shi,¹ Changliang Ren,¹ and Jiangfeng Du^{1,2,*}

¹*Department of Modern Physics, University of Science and Technology of China,
Hefei, Anhui 230026, People's Republic of China*

²*Hefei National Laboratory for Physical Sciences at Microscale,
Hefei, Anhui 230026, People's Republic of China*

A systematic approach is presented to construct non-homogeneous two- and three-qubit Bell-type inequalities. When projector-like terms are subtracted from homogeneous two-qubit CHSH polynomial, non-homogeneous inequalities are attained and the maximal quantum mechanical violation asymptotically equals a constant with the subtracted terms becoming sufficiently large. In the case of three-qubit system, it is found that most significant three-qubit inequalities presented in literature can be recovered in our framework. We also discuss the behavior of such inequalities in the loophole-free Bell test and obtain corresponding thresholds of detection efficiency.

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I. INTRODUCTION

Bell's original inequality reveals the conflict of quantum mechanics and local hidden variable (LHV) theory [1]. Since then, various forms of Bell-type inequalities have been derived. The most well-known is Clauser-Horne-Shimony-Holt (CHSH) inequality [2]. Mermin-Ardehali-Belinskii-Klyshko (MABK) generalized the result to the case of N qubits [3]. Subsequently, Werner and Żukowski (WWŻB) presented explicit description of N -qubit Bell-correlation inequalities for two dichotomic observables per site [4, 5]. It should be stressed that all these results only concern with the "full" correlation function, that is, the expectation value of the product of all N local observables. We call this class of Bell-type inequalities

*Electronic address: djf@ustc.edu.cn

as *homogeneous* inequalities.

Besides full correlations, one can take into account partial correlations, which involve the product of not N but $n < N$ observables. It is meaningful to construct such *non-homogeneous* Bell-type inequalities that incorporate partial correlations as well as full correlations. The main reason is as follows. Gisin proved that the bipartite pure entangled states violate CHSH inequality [6, 7]. The problem whether Gisin theorem can be generalized for an arbitrary n -partite pure entangled states remains open. In the three-qubit case there are generalized GHZ states that do not violate the MABK inequalities [8, 9]. More generally it has been shown that these states do not violate any Bell-type inequality for n -partite correlation functions (that is, homogeneous inequality) for experiments involving two dichotomic observables per site [9]. Subsequently some Bell-type inequalities, which bear the non-homogeneous form, were presented and can be violated by generalized GHZ states [10, 11]. Besides, non-homogeneous inequalities have appeared in the discussion of the non-locality of cluster states [12, 13]. However one can hardly make clear how and why these inequalities come out. In other words, there is no systematic formulation of such inequalities.

Motivated by this issue, we present in this paper a feasible method to construct non-homogeneous Bell-type inequalities. The key point is so simple that we can describe it very briefly in one sentence, that is, non-homogeneous inequalities are derived by subtracting some projector-like terms from the homogeneous ones.

We will first present the construction of two-qubit non-homogeneous inequalities. The newly obtained inequalities are weaker than CHSH inequality in that the quantum mechanical violation of the former is smaller than that of the latter. Nevertheless, they pave the way for the considering more general cases. Consequently we attain so many three-qubit non-homogeneous inequalities and some meaningful results. In Ref. [14], Pitowsky and Svozil have presented several optimal three-qubit Bell-type inequalities in the sense that they represent the best possible upper bounds for the conceivable classical probabilities. These inequalities are filtrated from tens of thousands of inequalities describing the facets of classical correlation polytope. Though these inequalities appear very intricate, they can be constructed systematically in our framework. Additionally the two Bell-type inequalities, which were presented in Ref. [11] to disclose the non-locality of generalized three-qubit GHZ states, are included in our results.

When these inequalities are utilized to display the non-locality experimentally, one has to consider the effect of non-ideal detector. Using the software Mathematica and MatLab, we numerically analyze the detection efficiency limit of the above-mentioned three-qubit Bell-type inequalities. Three detectors may have the same efficiency or not [15]. With respect to these two different cases, we attain the thresholds of detection efficiency. Some inequalities have the advantage that the efficiency of one detector can be arbitrarily low if that of the other two detectors satisfy certain conditions.

II. TWO-QUBIT NON-HOMOGENEOUS BELL-TYPE INEQUALITIES

We start by briefly recalling LHV model and Bell-type inequalities. Bell-type inequalities always refer to correlations between two or more sites. In the two-qubit case, each of two space-separated observers, A and B, gets a qubit and measures two ± 1 -valued local observables, denoted A_1, A_2 for observer A, and B_1, B_2 for observer B. The outcomes of measurement are labeled by $\mathbf{a}_i = \pm 1$ and $\mathbf{b}_j = \pm 1$ for $i, j = 1, 2$.

A typical joint probability can be expressed as $P(\mathbf{a}_2, \mathbf{b}_1 | A_2, B_1)$, where after the vertical bar we write the observables chosen at two sites, and before the bar the particular outcomes.

In the formalism of local hidden variable (LHV) theory, there exists a hidden variable λ which takes values in space Λ . With the presence of λ , the probability of measuring A_i and obtaining the outcome \mathbf{a}_i is represented by $P(\mathbf{a}_i | A_i, \lambda)$. Similarly for $P(\mathbf{b}_j | B_j, \lambda)$. Given λ , one can calculate the mean value of $\mathbf{a}_i(\lambda)$ from the probability $P(\mathbf{a}_i | A_i, \lambda)$,

$$\bar{\mathbf{a}}_i(\lambda) = P(+1 | A_i, \lambda) - P(-1 | A_i, \lambda). \quad (1)$$

Similarly for $\bar{\mathbf{b}}_i(\lambda)$.

Furthermore LHV theory requires that with the presence of λ the joint probability $P(\mathbf{a}_i, \mathbf{b}_j | A_i, B_j, \lambda)$ is factorisable, that is,

$$P(\mathbf{a}_i, \mathbf{b}_j | A_i, B_j, \lambda) = P(\mathbf{a}_i | A_i, \lambda) P(\mathbf{b}_j | B_j, \lambda), \quad (2)$$

where $P(\mathbf{a}_i | A_i, \lambda)$ is independent of observable B_j and its outcome \mathbf{b}_j , and $P(\mathbf{b}_j | B_j, \lambda)$ independent of A_i and \mathbf{a}_i .

Given the probability measure μ on Λ , using (2), one can compute the joint probability

$$P(\mathbf{a}_i, \mathbf{b}_j | A_i, B_j) = \int_{\Lambda} P(\mathbf{a}_i | A_i, \lambda) P(\mathbf{b}_j | B_j, \lambda) \mu(\lambda) d\lambda. \quad (3)$$

Consequently the expectation value of joint measurement is

$$\begin{aligned}
\langle A_i B_j \rangle_{\text{lhv}} &= \sum_{\mathbf{a}_i, \mathbf{b}_j} \mathbf{a}_i \mathbf{b}_j P(\mathbf{a}_i, \mathbf{b}_j | A_i, B_j) \\
&= \int_{\Lambda} \sum_{\mathbf{a}_i, \mathbf{b}_j} \mathbf{a}_i \mathbf{b}_j P(\mathbf{a}_i | A_i, \lambda) P(\mathbf{b}_j | B_j, \lambda) \mu(\lambda) d\lambda \\
&= \int_{\Lambda} \bar{\mathbf{a}}_i(\lambda) \bar{\mathbf{b}}_j(\lambda) \mu(\lambda) d\lambda.
\end{aligned} \tag{4}$$

In the context of hidden variable model, correlation function is defined as

$$\begin{aligned}
E(\lambda) &= \bar{\mathbf{a}}_1(\lambda) \bar{\mathbf{b}}_1(\lambda) - \bar{\mathbf{a}}_1(\lambda) \bar{\mathbf{b}}_2(\lambda) \\
&\quad - \bar{\mathbf{a}}_2(\lambda) \bar{\mathbf{b}}_1(\lambda) - \bar{\mathbf{a}}_2(\lambda) \bar{\mathbf{b}}_2(\lambda).
\end{aligned} \tag{5}$$

One can see that $|E(\lambda)| \leq 2$, and the inequality is saturated only if $\bar{\mathbf{a}}_i(\lambda)$ and $\bar{\mathbf{b}}_j(\lambda)$ take the extremal values of ± 1 . CHSH inequality is attained by averaging over hidden variable λ .

$$|\langle A_1 B_1 \rangle_{\text{lhv}} - \langle A_1 B_2 \rangle_{\text{lhv}} - \langle A_2 B_1 \rangle_{\text{lhv}} - \langle A_2 B_2 \rangle_{\text{lhv}}| \leq 2. \tag{6}$$

In the following discussion, it is convenient to adopt $|E(\lambda)| \leq 2$ to represent CHSH inequality, that is,

$$-2 \leq a_1 b_1 - a_1 b_2 - a_2 b_1 - a_2 b_2 \leq 2, \tag{7}$$

where we replace $\bar{\mathbf{a}}_i(\lambda)$ and $\bar{\mathbf{b}}_j(\lambda)$ with a_i and b_j for simplicity. We also make the following definitions.

$$E_1 = a_1 b_1 - a_1 b_2 - a_2 b_1 - a_2 b_2, \tag{8}$$

$$E_2 = -a_1 b_1 + a_1 b_2 - a_2 b_1 - a_2 b_2, \tag{9}$$

$$E_3 = -a_1 b_1 - a_1 b_2 + a_2 b_1 - a_2 b_2, \tag{10}$$

$$E_4 = -a_1 b_1 - a_1 b_2 - a_2 b_1 + a_2 b_2. \tag{11}$$

Each E_k is called (homogeneous) CHSH polynomial and has the same bound of 2, i.e., $|E_k| \leq 2$ for $k = 1, 2, 3, 4$.

A. Method

Now we start to construct non-homogeneous polynomial, denoted E' . We require that E' satisfies two conditions:

- (i) $E' \leq 2$. That means in the LHV upper bound of E' is 2.
- (ii) In the case of quantum mechanics, E' is replaced with the operator form \mathcal{E}' , and the expectation value given by $\langle \mathcal{E}' \rangle_{\text{qm}}$ can be larger than 2.

In other words, $E' \leq 2$ is a Bell-type inequality.

The key point of the method presented here is so transparent that at first sight it seems to be trivial: since every CHSH polynomial satisfies $E_k \leq 2$, it is evident that E_k minus some non-negative terms must be equal to or less than 2. These non-negative terms are chosen to be

$$P_{a_i}^\pm = \frac{1 \pm a_i}{2}, \quad P_{b_j}^\pm = \frac{1 \pm b_j}{2}. \quad (12)$$

A typical non-homogeneous polynomial is then given by

$$E'(r) = E_1 - rP_{a_2}^+P_{b_2}^+, \quad (13)$$

where r is any non-negative number.

Considering condition (i), we note that $E' \leq 2$ and E' reaches the maximal value of 2 at specific extremal values of a_i and b_j . For example, when $a_1 = 1$, $a_2 = 1$, $b_1 = 1$ and $b_2 = -1$, E' equals 2. So (13) satisfies condition (i).

We now show that condition (ii) can also be fulfilled. In quantum-mechanical case, a_i and b_j in (12) are replaced with spin observables, namely, $a_i \Rightarrow A_i$, $b_j \Rightarrow B_j$. Correspondingly, we have

$$P_{a_i}^\pm \Rightarrow \mathcal{P}_{a_i}^\pm = \frac{I \pm A_i}{2}, \quad P_{b_j}^\pm \Rightarrow \mathcal{P}_{b_j}^\pm = \frac{I \pm B_j}{2}; \quad (14)$$

$$E_1 \Rightarrow \mathcal{E}_1 = A_1B_1 - A_1B_2 - A_2B_1 - A_2B_2, \quad (15)$$

where I is 2×2 identity matrix.

Then the quantum mechanical form of (13) is

$$\mathcal{E}'(r) = \mathcal{E}_1 - r\mathcal{P}_{a_2}^+\mathcal{P}_{b_2}^+, \quad (16)$$

Any two-qubit entangled pure state can be expressed as the form of Schmidt decomposition (see, for example, [16]),

$$|\Psi\rangle = \cos \xi |00\rangle + \sin \xi |11\rangle, \quad \xi \in (0, \pi/2). \quad (17)$$

We let

$$\left. \begin{aligned} A_1 &= \sigma_x, & A_2 &= -\sigma_z, \\ B_1 &= \sigma_x \sin \theta + \sigma_z \cos \theta, \\ B_2 &= -\sigma_x \sin \theta + \sigma_z \cos \theta. \end{aligned} \right\} \quad (18)$$

The expectation value of \mathcal{E}' is given by

$$\begin{aligned} \langle \Psi | \mathcal{E}' | \Psi \rangle &= \left(2 + \frac{r}{2} \sin^2 \xi \right) \cos \theta \\ &+ 2 \sin 2\xi \sin \theta - \frac{r}{2} \sin^2 \xi. \end{aligned} \quad (19)$$

When $\theta = \arctan \frac{4 \sin 2\xi}{4+r \sin^2 \xi}$, $\langle \Psi | \mathcal{E}' | \Psi \rangle$ acquires the maximal value

$$\langle \Psi | \mathcal{E}' | \Psi \rangle_{\max} = \left[\left(2 + \frac{r}{2} \sin^2 \xi \right)^2 + 4 \sin^2 2\xi \right]^{1/2} - \frac{r}{2} \sin^2 \xi, \quad (20)$$

which is obviously larger than 2. Therefore (13) represents a series of non-homogeneous inequalities and can be used to detect non-locality.

The above procedure can be formulated in more general forms. Let's consider the following non-homogeneous polynomial.

$$E''(s, t) = E_4 - sP_{a_1}^+ P_{b_2}^+ - tP_{a_2}^+ P_{b_1}^+, \quad (21)$$

where s and t are non-negative numbers. This time we select E_4 to construct E'' . It is just for the later convenience and not necessary. Obviously we have $E'' \leq 2$ and the bound is attained for some extremal points, say, $a_1 = a_2 = 1$ and $b_1 = b_2 = -1$. In order to show the violation of $E'' \leq 2$ in quantum mechanics, we evaluate the expectation value $\langle \Psi | \mathcal{E}'' | \Psi \rangle$, where $|\Psi\rangle$ is given by (17) and \mathcal{E}'' is the quantum counterpart of E'' .

For $\xi \in [0, \pi/4]$, we choose the observables as

$$\left. \begin{aligned} A_1 &= \sigma_z, & A_2 &= -\sigma_x, \\ B_1 &= \sigma_x \sin \theta + \sigma_z \cos \theta, \\ B_2 &= -\sigma_x \sin \theta + \sigma_z \cos \theta. \end{aligned} \right\} \quad (22)$$

For this choice, we have

$$\begin{aligned} &\langle \Psi | \mathcal{E}'' | \Psi \rangle \\ &= -\frac{1}{4} [s + 8 + (s + t) \cos 2\xi] \cos \theta \\ &\quad + \frac{t + 8}{4} \sin 2\xi \sin \theta - \frac{1}{4} (s + t + s \cos 2\xi) \end{aligned} \quad (23)$$

The maximum of $\langle \Psi | \mathcal{E}'' | \Psi \rangle$ over all θ is given by

$$\begin{aligned}
f_1(s, t, \xi) &= \max_{\text{all } \theta} \langle \Psi | \mathcal{E}'' | \Psi \rangle \\
&= -\frac{1}{4}(s + t + s \cos 2\xi) \\
&\quad + \frac{1}{4} \left[(8 + s + (s + t) \cos 2\xi)^2 + (8 + t)^2 \sin^2 2\xi \right]^{1/2}.
\end{aligned} \tag{24}$$

It can be found that if s and t satisfy

$$32 - st + (32 - st + 8t) \cos 2\xi > 0, \quad \xi \in [0, \pi/4], \tag{25}$$

the function $f_1(s, t, \xi)$ will be larger than 2 (when $\xi = 0$, $f_1 = 2$).

For $\xi \in [\pi/4, \pi/2]$, we choose

$$\left. \begin{aligned}
A_1 &= -\sigma_z, & A_2 &= -\sigma_x, \\
B_1 &= \sigma_x \sin \theta + \sigma_z \cos \theta, \\
B_2 &= -\sigma_x \sin \theta + \sigma_z \cos \theta.
\end{aligned} \right\} \tag{26}$$

Similarly, we have

$$\begin{aligned}
f_2(s, t, \xi) &= \max_{\text{all } \theta} \langle \Psi | \mathcal{E}'' | \Psi \rangle \\
&= -\frac{1}{4}(s + t - s \cos 2\xi) \\
&\quad + \frac{1}{4} \left[(8 + s - (s + t) \cos 2\xi)^2 + (8 + t)^2 \sin^2 2\xi \right]^{1/2}.
\end{aligned} \tag{27}$$

If s and t satisfy

$$32 - st - (32 - st + 8t) \cos 2\xi > 0, \quad \xi \in [\pi/4, \pi/2], \tag{28}$$

then $f_2(s, t, \xi)$ is larger than 2 (when $\theta = \pi/2$, $f_2 = 2$).

It can be seen from (25) and (28) that both of f_1 and f_2 are larger than 2 if $st < 32$. So, under this condition, $\langle \mathcal{E}'' \rangle$ violate LHV upper bound.

Some remarks are needed here. In the above calculation, we do not intend to find the maximal violation of LHV bound. The choices of observable, given by (18), (22) and (26), do not cover all possible local measurements. They are so selected as to simplify the calculation. Therefore $\langle \Psi | \mathcal{E}' | \Psi \rangle_{\max}$ and $\langle \Psi | \mathcal{E}'' | \Psi \rangle_{\max}$ do not represent the maximal violation of LHV bound. In the same way, the condition of $st < 32$ is just required in the case considered. In the following subsection, we discuss the maximal violation of inequalities $E' \leq 2$ and $E'' \leq 2$.

B. Violations of inequality

Two typical forms of non-homogeneous inequality have been presented in (13) and (21). We now analyze the violation of them in quantum mechanics. The quantum bound can be computed by means of Min-Max principle [17], which states that for self-adjoint transformations the operator norm is bounded by the minimal and maximal eigenvalues. For the purpose of clarity, we choose the directions of local measurement to be orthogonal. In other words, A_1 and A_2 anti-commute. So do B_1 and B_2 . Furthermore, we let

$$A_1 = B_1 = \sigma_x, \quad A_2 = B_2 = \sigma_y. \quad (29)$$

The operator $\mathcal{E}'(r)$, which is quantum correspondent of $E'(r)$, is written as

$$\mathcal{E}'(r) = \begin{pmatrix} -1 - r & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}. \quad (30)$$

The eigenvalue of $\mathcal{E}'(r)$, denoted $\varepsilon(r)$, is determined by the equation

$$\varepsilon^4 + r\varepsilon^3 - (r + 8)\varepsilon^2 - 4r\varepsilon = 0. \quad (31)$$

Among the solutions of (31), the largest one must be between 2 and 3. To see this, note that the left-hand side of (31) is negative when $\varepsilon = 2$, while positive and monotone increasing when $\varepsilon \geq 3$.

The numerical result of the largest eigenvalue of $\mathcal{E}'(r)$ is plotted in Fig. 1. It can be seen that the largest eigenvalue asymptotically equals a constant (about 2.5) for sufficiently large r . The asymptotic value can be calculated from (31). In fact, when $r \rightarrow \infty$, the terms containing r should be finite. It follows that

$$\varepsilon^3 - \varepsilon^2 - 4\varepsilon = 0. \quad (32)$$

The solution $\frac{1+\sqrt{17}}{2} \approx 2.56$ is what we want.

Similar analysis shows that the largest eigenvalue of $\mathcal{E}''(s, t)$ is also between 2 and 3, and asymptotically equals 2.43 when $s, t \rightarrow \infty$.

In a word, the inequalities $E'(r) \leq 2$ and $E''(s, t) \leq 2$ can be violated by quantum mechanics for arbitrary non-negative numbers of (r, s, t) . The maximal violation corresponds

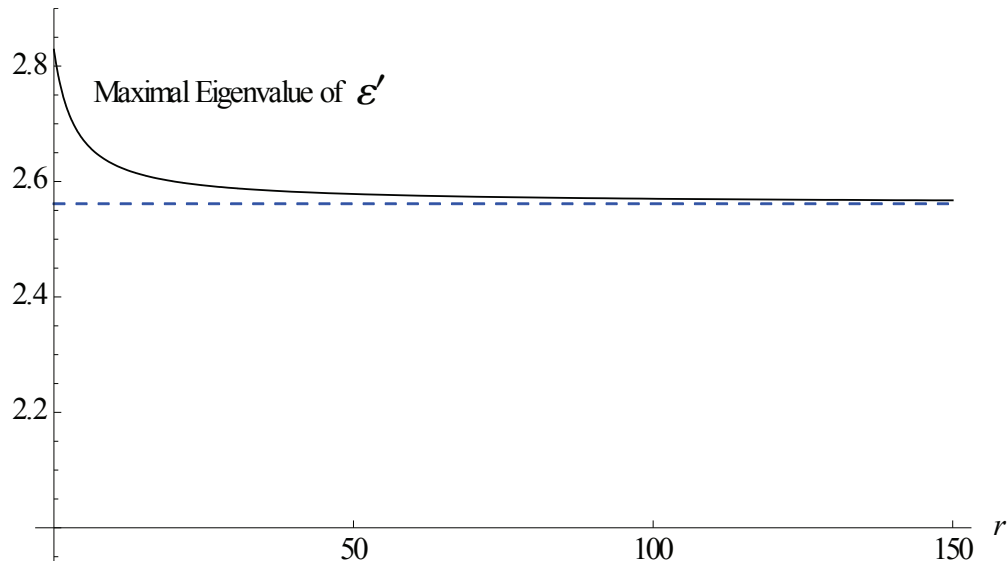


FIG. 1: (Color online) The maximal eigenvalue of $\mathcal{E}'(r)$. When r tends to infinity, the asymptotic value is $\frac{1+\sqrt{17}}{2} \approx 2.56$.

to the case of $r = s = t = 0$. With r , s and t sufficiently large, the violation is close to a constant.

III. THREE-QUBIT NON-HOMOGENEOUS INEQUALITIES

Based on the results obtained previously, we will in this section construct three-qubit non-homogeneous inequalities.

A. General results

We now introduce the third qubit C. Two observables C_1 and C_2 are also ± 1 -valued. Just like Eq. (1), we define

$$c_i = \bar{c}_i(\lambda) = P(+1|C_i, \lambda) - P(-1|C_i, \lambda). \quad (33)$$

The general form of three-qubit non-homogeneous polynomial can be expressed as

$$F(a, b)c_1 + G(a, b)c_2 + H(a, b), \quad (34)$$

where F , G and H are polynomials associated with qubit A and B. We will construct three-qubit inequalities by the following steps. Firstly, we set the upper bound of (34) to be an arbitrary number, for example,

$$F(a, b)c_1 + G(a, b)c_2 + H(a, b) \leq 2 + u, \quad (35)$$

where we let $u \geq 0$ simply for the technical simplicity. Secondly we let c_1 and c_2 take extremal values of ± 1 . (35) will reduce to four inequalities with the upper bound of $2 + u$. We let three of them be the known two-qubit Bell-type inequalities (homogeneous or non-homogeneous). For example, we make the following choice.

$$F + G + H = E'' + u \leq 2 + u, \quad (36)$$

$$F - G + H = E' + u \leq 2 + u, \quad (37)$$

$$-F - G + H = \frac{2+u}{2}(-E_4) \leq 2 + u. \quad (38)$$

So F , G and H can be expressed in terms of $E'(r)$, $E''(s, t)$ and E_4 .

$$F = \frac{2+u}{4}E_4 + \frac{E'}{2} + \frac{u}{2}, \quad (39)$$

$$G = \frac{E'' - E'}{2}, \quad (40)$$

$$H = -\frac{2+u}{4}E_4 + \frac{E''}{2} + \frac{u}{2}. \quad (41)$$

Finally, there remains a reduced inequality, namely, $-F + G + H \leq 2 + u$. We use it to determine the possible values of r , s and t . From (39), (40) and (41), it follows that

$$-F + G + H = -\frac{2+u}{2}E_4 + E'' - E'. \quad (42)$$

The requirement that the upper bound of (42) is $2 + u$ leads to the following restrictions on r , s and t .

$$r \leq 4 + 2u, \quad (43a)$$

$$r - s \leq 2u, \quad (43b)$$

$$r - t \leq 2u, \quad (43c)$$

$$r - s - t \leq 0. \quad (43d)$$

In a word, we require that four reduced two-qubit polynomials, namely $\pm F \pm G + H$, have the same upper bound of $2 + u$. We point out that the choice given by (36)–(38) is not unique. What is more, some reduced inequality are not necessarily required to be Bell-type inequalities, as will be seen in the next subsection.

B. Specific inequalities

We present some examples to illustrate the application of our method.

$$1. \quad u = 0, r = s = t = 0$$

It follows that

$$E' = E_1, \quad E'' = E_4, \quad (44)$$

$$F = \frac{1}{2}(E_4 + E_1), \quad G = \frac{1}{2}(E_4 - E_1), \quad H = 0. \quad (45)$$

Then we have

$$-a_1b_1c_2 - a_1b_2c_1 - a_2b_1c_1 + a_2b_2c_2 \leq 2. \quad (46)$$

It is the well-known three-qubit MABK inequality. The largest eigenvalue of the quantum correspondent is 4, which means the violation factor is $\frac{4}{2} = 2$.

$$2. \quad u = 2, r = 8, s = t = 4$$

In this case, conditions (43) are satisfied and saturated. Direct calculation gives

$$\begin{aligned} & -a_1 - a_2 - b_1 - b_2 + a_1b_1 - a_2b_2 - a_1c_2 \\ & - 2a_2c_1 + a_2c_2 - b_1c_2 - 2b_2c_1 + b_2c_2 \\ & - a_1b_1c_1 - 2a_1b_1c_2 - 3a_1b_2c_1 - a_1b_2c_2 \\ & - 3a_2b_1c_1 - a_2b_1c_2 - a_2b_2c_1 + 4a_2b_2c_2 \leq 8. \end{aligned} \quad (47)$$

In Ref. [14], Pitowsky and Svozil enumerate some new Bell-type inequalities. In their notation, one of the inequalities (Eq. (5) in [14]) is

$$\begin{aligned} & -P(A_1) - 2P(B_1) - 2P(C_1) + 2P(A_1, B_1) \\ & + 2P(A_1, C_1) + P(A_1, B_2) + P(A_1, C_2) + P(A_2, B_1) \\ & + P(A_2, C_1) - P(A_2, B_2) - P(A_2, C_2) + 2P(B_1, C_1) \\ & + 2P(B_2, C_1) + 2P(B_1, C_2) - 2P(B_2, C_2) \\ & - P(A_1, B_1, C_1) - 2P(A_2, B_1, C_1) - 3P(A_1, B_2, C_1) \\ & - 3P(A_1, B_1, C_2) - P(A_2, B_2, C_1) - P(A_2, B_1, C_2) \\ & - P(A_1, B_2, C_2) + 4P(A_2, B_2, C_2) \leq 0. \end{aligned} \quad (48)$$

Here $P(A_1)$ denotes the probability that observer A measures observable A_1 and obtains the outcome $+1$, that is, in our notation, $P(\mathbf{a}_1 = +1|A_1)$. We know that

$$\begin{aligned}
P(\mathbf{a}_1 = +1|A_1) &= \int_{\Lambda} P(\mathbf{a}_1 = +1|A_1, \lambda)\mu(\lambda) d\lambda \\
&= \int_{\Lambda} \frac{1 + \bar{a}_1(\lambda)}{2} \mu(\lambda) d\lambda \\
&= \int_{\Lambda} \frac{1 + a_1}{2} \mu(\lambda) d\lambda.
\end{aligned} \tag{49}$$

Similarly $P(A_1, B_1)$ refers to $P(\mathbf{a}_1 = +1, \mathbf{b}_1 = +1|A_1, B_1)$. And we have

$$\begin{aligned}
&P(\mathbf{a}_1 = +1, \mathbf{b}_1 = +1|A_1, B_1) \\
&= \int_{\Lambda} P(\mathbf{a}_1 = +1, \mathbf{b}_1 = +1|A_1, B_1, \lambda)\mu(\lambda) d\lambda \\
&= \int_{\Lambda} P(\mathbf{a}_1 = +1|A_1, \lambda)P(\mathbf{b}_1 = +1|B_1, \lambda)\mu(\lambda) d\lambda \\
&= \int_{\Lambda} \frac{1 + a_1}{2} \frac{1 + b_1}{2} \mu(\lambda) d\lambda.
\end{aligned} \tag{50}$$

Then the inequality (48) can be rewritten in terms of a_i, b_j, c_k .

We find that, under permutation $b \rightarrow a, a \rightarrow c$ and $c \rightarrow b$, Pitowsky's inequality (48) can be transformed to (47). Furthermore, the other inequalities presented in [14] can be recovered in our framework.

In quantum mechanics, the largest eigenvalue of the operator corresponding to the left-hand side of (47) is about 12.87 (numerically), which means the violation factor is about $\frac{12.87}{8} \approx 1.61$.

We also evaluate various violation factors for different values of (u, r, s, t) . Numerical results reveal such phenomena: (i) When $u \leq 2$, the optimal inequality, which can give rise to the maximal violation of LHV bound, corresponds to the case of $r = 4u$ and $s = t = 2u$, that is, conditions (43b), (43c) and (43d) are saturated. (ii) when $u \geq 2$, the optimal inequality is attained for $r = 4 + 2u$ and $s = t = 2 + u$, that is, conditions (43a) and (43d) are saturated. In Fig. 2, we plot the maximal violation factor for different values of u . It can be seen that the violation factor tends to be a constant (about 1.27) for large u .

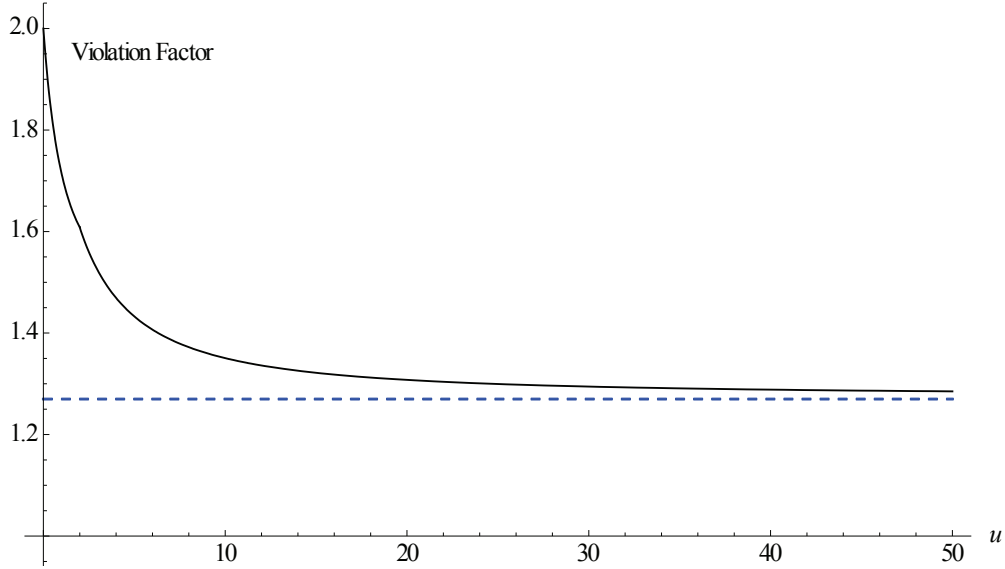


FIG. 2: (Color online) The maximal violation factor for different u . When u sufficiently large, the asymptotic value is about 1.27.

3. Another case

Let's recall the steps by which the three-qubit non-homogeneous inequalities are established. The key point is to let three reduced polynomials be related to three two-qubit Bell-type inequalities, which is demonstrated in Eqns. (36), (37) and (38). This requirement can be relaxed. For example, we consider the following procedure.

$$F + G + H = 2 - P_{a_1}^+ P_{b_2}^+ - P_{a_2}^+ P_{b_1}^+ \leq 2, \quad (51)$$

$$-F + G + H = E_4 \leq 2, \quad (52)$$

$$-F - G + H = 2 - P_{a_1}^- P_{b_1}^- - P_{a_2}^- P_{b_2}^- \leq 2. \quad (53)$$

The LHV upper bounds of above polynomials are the same. But only (52) is a real Bell-type inequality. From these equations, we get the following inequality.

$$\begin{aligned} & -a_1 b_1 - a_1 b_2 - a_2 b_1 - a_2 b_2 - a_1 c_1 \\ & -a_1 c_2 - a_2 c_1 - a_2 c_2 - b_1 c_1 - b_1 c_2 \\ & -b_2 c_1 - b_2 c_2 + a_1 b_1 c_1 - a_1 b_2 c_2 \\ & -a_2 b_1 c_2 - a_2 b_2 c_2 + 2a_2 b_2 c_2 \leq 4 \end{aligned} \quad (54)$$

This inequality was originally presented in Ref. [11] and said to be violated by all three-qubit pure entangled states (see Eq. (6) in [11]).

IV. THRESHOLDS OF DETECTION EFFICIENCY

In the actual experiment performed on Bell-type test, there are mainly two kinds of loophole to overcome, that is, locality loophole and detection loophole [18]. Locality loophole has been closed in several Bell experiments [19]. However, detection loophole is difficult to overcome so far [20]. In order to perform a loophole-free Bell test, one must make clear the threshold detection efficiency. Many useful results have been obtained for different Bell inequalities in two-qubit system [15, 21]. In this section we discuss the thresholds of detection efficiency for six three-qubit inequalities. Four of them come from Ref. [14] and are labeled from PI-(2) to PI-(5) where the number denotes the equation number in the reference. The other two inequalities, labeled CI-(2) and CI-(6), are respectively Eq. (2) and Eq. (6) in Ref. [11].

We take inequality (48), i.e., PI-(5), as an illustrative example. As said before, $P(A_1)$ is the probability of finding the outcome $\mathbf{a}_1 = +1$, and $P(A_1, B_1)$ is a joint probability of finding the outcomes $\mathbf{a}_1 = +1$ and $\mathbf{b}_1 = +1$. Other probabilities have the similar meaning and all of them are defined in ideal situation.

Now we assume that the detection efficiencies at site A, B and C are η_1 , η_2 and η_3 respectively. Then $P(A_1)$ should be multiplied by a factor η_1 . The joint probability $P(A_1, B_1)$ is replaced by $\eta_1\eta_2P(A_1, B_1)$, and $P(A_1, B_1, C_1)$ by $\eta_1\eta_2\eta_3P(A_1, B_1, C_1)$. Similarly for other probabilities. Thus we get the modified form of PI-(5), which is utilized to deal with non-ideal cases and should be satisfied by LHV model.

In quantum-mechanical formalism, the probability $P(A_i)$ is replaced by $\langle \Psi | \mathcal{P}_{a_i}^+ \otimes I \otimes I | \Psi \rangle = \langle \Psi | \frac{I+A_i}{2} \otimes I \otimes I | \Psi \rangle$ for some three-qubit state $|\Psi\rangle$, and the joint probability $P(A_1, B_1)$ by $\langle \Psi | \mathcal{P}_{a_1}^+ \otimes \mathcal{P}_{b_1}^+ \otimes I | \Psi \rangle = \langle \Psi | \frac{I+A_1}{2} \otimes \frac{I+B_1}{2} \otimes I | \Psi \rangle$. Similarly for other probabilities. Thus we obtain the quantum mechanical expression of the modified PI-(5), that is,

$$\langle \Psi | \mathcal{J} | \Psi \rangle \leq 0 \quad (55)$$

where \mathcal{J} contains such operators as $\eta_1\mathcal{P}_{a_1}^+$, $\eta_1\eta_2\mathcal{P}_{a_1}^+\mathcal{P}_{b_1}^+$, etc. Once we find $\langle \Psi | \mathcal{J} | \Psi \rangle > 0$, a loophole-free experiment can be performed. In other words, we hope to find that the maximal

eigenvalue of the operator \mathcal{J} is positive. Before we proceed to numerical calculation, it is necessary to lower the number of the parameters in \mathcal{J} . Since we are to find the eigenvalues of \mathcal{J} , local observables can be set to be in xy -plane. Without loss of generality, we let

$$\begin{aligned} A_1 &= B_1 = C_1 = \sigma_x, \\ A_2 &= \sigma_x \cos \theta_A + \sigma_y \sin \theta_A, \\ B_2 &= \sigma_x \cos \theta_B + \sigma_y \sin \theta_B, \\ C_2 &= \sigma_x \cos \theta_C + \sigma_y \sin \theta_C. \end{aligned}$$

Thus the 8×8 matrix \mathcal{J} contains six parameters: $\eta_1, \eta_2, \eta_3, \theta_A, \theta_B, \theta_C$. Then under the condition that the maximal eigenvalue of \mathcal{J} is positive, we find the threshold of detection efficiency numerically by means of Mathematica and MatLab. In the following we discuss two cases and give corresponding results.

a. Symmetric system — In this case, all three detection efficiencies are the same, i.e., $\eta_1 = \eta_2 = \eta_3 = \eta$. we obtain the threshold efficiency $\eta^{th} = 66.8\%$ on which detection loophole can be closed.

b. Asymmetric system — In a asymmetric system, three articles may be detected with different probabilities. We obtain the following results.

- (i). When one of three detectors is perfect, say, $\eta_1 = 1$ and the other two are imperfect but have the same efficiency, that is, $\eta_2 = \eta_3 = \eta$, the threshold efficiency is found to be $\eta^{th} = 50\%$.
- (ii). Two detectors are perfect, e.g., $\eta_1 = \eta_2 = 1$, and the third is not. In this case, no matter how the efficiency of the third detector is low, the inequality can be always violated by appropriate choice of detection orientation.
- (iii). Assuming that $\eta_1 = 1$, just as (i), we consider such a problem: With η_3 arriving at its lower bound, how about the value taken by η_2 ? Calculation results show that the lower bound of η_3 can be zero while η_2 must be larger than 50%. Obviously (ii) is the specific case of this result.

Inequalities PI-(2) and PI-(4) have similar behavior, whereas inequalities PI-(3), CI-(2) and CI-(6) do not. For example, for CI-(2) the lower bound of η_3 is 81.9% and can not be zero, and moreover when $\eta_3 = 81.9\%$ the other two detector must be perfect. All results are

listed in Table I. Obviously inequality PI-(5) is the most optimal one among them in the sense that this inequality can endure very low detection efficiency.

TABLE I: Threshold Detection Efficiency

	symmetric	asymmetric		
	η	η_1	η_2	η_3
PI-(5)	66.8%	1	50%	50%
		1	1	0
PI-(4)	71.5%	1	60%	60%
		1	1	0
PI-(3)	87.6%	1	77.2%	77.2%
		1	1	34%
PI-(2)	87%	1	66.8%	66.8%
		1	1	0
CI-(2)	91.5%	1	87.4%	87.4%
		1	1	76.4%
CI-(6)	93.6%	1	90.5%	90.5%
		1	1	81.9%

V. CONCLUSION

In conclusion, we propose a systematic approach of constructing non-homogeneous Bell-type inequalities for two- and three-qubit system. In the two-qubit case, non-homogeneous inequality is attained by subtracting positive projector-like terms from CHSH polynomial. We find that when the subtracted terms are sufficiently “large” the maximal quantum mechanical violation asymptotically tends to be a constant.

Three-qubit non-homogeneous inequalities are attained by direct generalization of two-qubit ones. Most of significant three-qubit inequalities presented in literature are recovered in our framework. The method presented in this paper can be generalized to construct Bell-type inequalities for multipartite system. The benefit of non-homogeneous inequalities lies in considering not only full correlations but also partial ones, which are needed in discussing the non-locality of multipartite system. We conjecture that our method may be used to

categorize Bell-inequalities with various and complicated forms.

Additionally, we analyze numerically the detection efficiency thresholds of the previously mentioned three-qubit Bell-type inequalities when they are employed to display non-locality in quantum states. Under different situations, that is, three detectors may have the same or distinct efficiency, we obtain the thresholds of detection efficiency respectively. For some inequalities, we find that the efficiency of one detector can be arbitrarily low as long as the other detectors satisfy certain conditions. Numerical results will help us find the most optimal one which can endure very low detection efficiency.

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