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The Fluctuation theory and Gravitational Galaxy Clustering

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Abstract

The description of gravitational galaxy clustering evolving through quasi-equilibrium thermodynamics is examined on the basis of thermodynamic fluctuations and statistical mechanical ensemble techniques. Second order fluctuation moments lead to an ordinary differential equation for $b(nT^{-3})$, ratio of gravitational correlation energy to (twice) the kinetic energy of peculiar velocities. The general solution of the differential equation gives a functional form of b in terms of a scale invariant function nT^{-3}, in conformity with the earlier results. A method for finding analytic expression for probability distribution function $f(N)$ of gravitational clustering in a expanding universe is also developed on the basis of ensemble theory. The results can provide a deeper understanding of gravitational clustering on the basis of statistical mechanics.

Key Words: Cosmology - gravitation - galaxies: Clustering - method: Analytical.

1. Introduction

The statistical studies of the distribution of galaxies reveal that they cluster together on all scales and can provide valuable insights into large scale structure of the universe. In recent years, different techniques have been developed to understand the complicated process of gravitational galaxy clustering, notable among which are the correlation functions and density distribution functions. In fact, the two-particle correlation function is one of the objective quantities which can be related analytically to the physical processes of clustering, with observations and computer simulations. Linking of two-point correlation function with gravitational thermodynamics of an infinite system involve the equations of state. These equations of state, in turn, determine the equilibrium fluctuations. By using these fluctuations, the probability distribution function $f(N)$ can be extracted. An attempt in this direction was first made by Saslaw and Hamilton [1] in developing distribution function $f(N)$ of galaxy clustering with the help of gravitational thermodynamic technique. There is fairly good agreement between gravitational thermodynamic results with N-body simulations [2–4] as well as with observed galaxy clustering [5–11]. A completely independent confirmation of Saslaw and Hamilton's work is provided by the Riemannian geometric approach to thermodynamics developed by Ruppeiner [12]. The geometric theory [12] has since been refined and extended to a wider class of interacting systems [13].

The applicability of thermodynamics to the cosmological many-body problem is based on the quasi-equilibrium approximation [14, 15]. At first sight, this may appear a little improper because the expanding universe in which clusters continually grow does not appear to be an equilibrium state. Moreover, gravitating systems do not have any non-singular equilibrium state, because of their long range, unshielded, attractive interaction. A simple physical assumption makes the thermodynamic description possible. This is that the local dynamic time scales $(G\rho)^{-1/2}$ in an over dense region is faster than the global gravitational time scale $(G\bar{\rho})^{-1/2}$, for the average density $\bar{\rho}$, and this difference makes it possible for gravitational clustering to evolve through a sequence of quasi-equilibrium states. In other words, local equilibria can arise faster than the cosmic expansion can disrupt them. N-body simulation analysis shows that the difference in time scales need not be large for this to happen. (Indeed, most of the general relativity cosmologies satisfy this criterion). Consequently, average macroscopic thermodynamic quantities such as temperature, pressure, density, chemical potential, internal energy etc. can be reasonably defined along with local fluctuations around these averages. At any given time, there would be thermodynamic equilibrium relations such as equation of state among these quantities. Even though the macroscopic quantities may themselves change over the longer timescale, they continue to satisfy these equilibrium relations to a good approximation at any given time. Thus, if the macroscopic timescale exceeds the microscopic timescale i.e. the condition of quasi-equilibrium holds, we can apply standard thermodynamic fluctuation theory to gravitating masses. Further, it has been shown [15] that the expansion of the universe effectively removes the mean long-range gravitational field from the thermodynamic description of the cosmological many-body problem. The thermodynamic equations of state depend only on the local fluctuations of the field. The analysis of Saslaw and Fang [15] shows that the condition of quasi-equilibrium to hold is a pre-requisite for a thermodynamical description. Computer N-body experiments confirm the applicability of the idea of quasi-equilibrium evolution to galaxy clustering [2–4]. However, a detailed description of the conditions and lengthscales for quasi-equilibrium evolution remains an important unsolved problem.

In thermodynamic description of gravitational galaxy clustering, the influence of the gravitational correlation energy is measured in terms of a function $b(n,T)$; ratio of gravitational correlation energy to (twice) the kinetic energy of peculiar velocities. Specifying the correct functional form of $b(n,T)$ is essential for understanding the process of galaxy clustering. Saslaw and Hamilton [1] assumed a functional form of $b(n,T)$, which is derived by Masood et al [16], from the basic results of thermodynamics. Further, this functional form of b qualitatively matches the Riemannian geometric equation of thermodynamic fluctuation theory [12].

The aim of present paper is to show as to how the functional form of $b(n,T)$ can be obtained by combining the results of fluctuations from thermodynamics and grand canonical ensemble of galaxy clustering. The distribution function $f(N)$ of galaxies, clustering under gravitation, is obtained from the ensemble theory of statistical mechanics. Both $b(n,T)$ and probability distribution function so derived are independent of earlier techniques [1, 15, 16]. In fact, the functional form of $b(n,T)$ is obtained directly from the general solution of an ordinary differential equation, which is developed from the second order moments of particles clustering gravitationally. A method, based on ensemble theory of statistical mechanics, is also developed to obtain analytical expression for the distribution function of galaxies.

The paper is organized as follows: In section 2, we derive mean square number fluctuations for general functional form of $b(nT^{-3})$ in thermodynamics and grand canonical ensemble of particles, clustering gravitationally. On comparison of the two results, we get first order differential equation relating b and nT^{-3}. The functional form of $b(nT^{-3})$ is derived from series solution of this differential equation. In section 3, we obtain canonical partition function of particles clustering gravitationally from the theory of fluctuation and contour integration. Also, analytical expression for probability distribution function of galaxy clustering is derived with the help of ensemble theory. Finally, results are concluded in section 4.

2. Fluctuations and Functional form of $b(n, T)$

Greene and Callen [17] proved that fluctuation moments computed with statistical mechanical ensembles are same as those computed with classical thermodynamic fluctuation theory, and our results are based on this principle. The fluctuations are intrinsic part of thermodynamic equilibrium. Although, fluctuations are normally small but for large fluctuations gravity becomes dominant. These large fluctuations are identified with the formation of well-defined and sometimes bound clusters of galaxies. Gravitational galaxy clustering in the expanding universe can be described by quasi-equilibrium thermodynamic theory, because time dependence of thermodynamics represented by macroscopic variables including $b(t)$, which changes over the Hubble time scale, are longer [18]. This time dependence is slow enough that, when locally relaxed regions respond to it, they retain their description in terms of thermodynamic variables (e.g. fluctuations remain relatively small). Thermodynamics can be applied to an ensemble of slowly evolving gravitational systems, much like the ordinary equilibrium thermodynamics [19]. It is possible to use the usual macroscopic thermodynamic variables U, S, V and N to describe the macroscopic state of these infinite statistically homogeneous gravitating systems. This is because their fluctuations over a grand canonical ensemble are sufficiently small that average values of these variables are well defined.

The pairwise gravitational interaction of particles (galaxies) as point masses leads to the specific form of equation of state in terms of thermodynamic quantities like internal energy U and Pressure P in terms of the clustering parameter b [20]:

$$U = \frac{3}{2}NT(1 - 2b) \tag{1}$$

and

$$P = \frac{NT}{V}(1 - b), \tag{2}$$

where b measures the influence of gravitational correlation energy W_c, and is related to the two-particle correlation function by the relation

$$b(n, T) \equiv -\frac{W_c}{2K} = \frac{2\pi G m^2 \bar{n}}{3T} \int_V \xi(n, T, r) \frac{\mathrm{d}V}{4\pi r}, \tag{3}$$

where $\bar{n} = \frac{\bar{N}}{V}$ is the average number density of galaxies. The kinetic energy K of peculiar motion is related to temperature T (in energy units with Boltzmann's Constant $k = 1$) by

$$K = \frac{3}{2}NT = \frac{1}{2}\sum_{i=1}^{N} m V_i^2. \tag{4}$$

The functional form of $b(n, T)$ depends on n and T. Saslaw and Hamilton [1] obtained a specific form of $b(nT^{-3})$ on the basis of certain hypothesis. However, Masood et al [16] have shown, explicitly, how it can be derived from the first and second laws of thermodynamics, which is equivalent to the entropy being a perfect differential. We derive analytical functional form of $b(nT^{-3})$ on the basis of second order fluctuations of particles, clustering gravitationally, in thermodynamics and grand canonical ensemble, without much physical constraints.

The grand canonical ensemble is a powerful tool of statistical mechanical problems of complex systems. In this ensemble, the average density $\bar{n}$ and temperature T are fixed, but the density and energy fluctuate among members of the ensemble. The chemical potential $\mu(n, T)$ is a measure of exchange of particles in grand canonical ensemble and is related to entropy S at equilibrium by a standard relation:

$$(\mu/T) = \frac{U}{NT} - \frac{S}{N} + \frac{\psi}{N}, \tag{5}$$

where ψ is the grand canonical potential given by

$$\psi = \frac{PV}{T} = \bar{N}(1-b). \tag{6}$$

Also, the entropy of the system is

$$\mathrm{d}\left(\frac{S}{N}\right) = \frac{1}{T}\,\mathrm{d}\left(\frac{U}{N}\right) + \frac{P}{T}\,\mathrm{d}\left(\frac{V}{N}\right). \tag{7}$$

We can get general expression for the chemical potential (μ/T) for any functional form of $b(n,T)$ by combining equations (1), (2), (5), (6) and (7) as

$$(\mu/T) = \ln(nT^{-3/2}) - b - \int n^{-1} b \,\mathrm{d}n. \tag{8}$$

The combination of first and second laws of thermodynamics with entropy being a perfect differential immediately gives $b = b(nT^{-3})$, which is equivalent to the scale invariance of gravitational galaxy clustering in a homogeneous expanding universe. Further, Ruppiener [12] also shows that necessary and sufficient condition for Maxwell relation between pressure and entropy is satisfied when $b(n,T) = b(nT^{-3})$. This property can also be observed from the invariance of grand canonical partition function [15]. For convenience, we put

$$x = \beta n T^{-3}, \tag{9}$$

where β is a positive constant. For positive correlations and pressure, we see that the possible values of b lie between zero and one, and $x \to 0$ corresponds to ideal gas limit.

Let the general functional form of x in terms of b be given by the series

$$x = f(b) = \sum_{\lambda=0}^{\infty} a_\lambda b^{k+\lambda}, \qquad a_0 \neq 0, \tag{10}$$

where k is a constant and a_λ are coefficients. The values of k and a_λ can determine the functional form of $b(x)$. Substitution of equation (10) in equation (8) gives a general expression for the chemical potential

$$\left(\frac{\mu}{T}\right) = \ln\left[\frac{f(b)T^{3/2}}{\beta}\right] - b - \int \frac{b f'(b)}{f(b)}\,\mathrm{d}b. \tag{11}$$

Mean square fluctuations of number counts (density) in any volume can be calculated from thermodynamic fluctuations [21] via

$$\langle(\Delta N)^2\rangle = \frac{T\bar{N}}{V}\left(\frac{\partial N}{\partial P}\right)_{T,V}. \tag{12}$$

Substitution of equation (2) in equation (12) gives the thermodynamic fluctuation for any functional form of $b(x)$ as

$$\langle(\Delta N)^2\rangle = \frac{\bar{N}}{(1 - b - x\frac{\mathrm{d}b}{\mathrm{d}x})}. \tag{13}$$

Also, the mean square fluctuations in number of particles for an infinite system, in grand canonical ensemble, is

$$\langle(\Delta N)^2\rangle = \left.\frac{\partial^2\psi}{\partial^2(\mu/T)}\right|_{T,V}. \tag{14}$$

Using equation (10) and (11), we obtain

$$\left.\frac{\partial(\mu/T)}{\partial b}\right|_{T,V} = \frac{\Big[(1-b)f'(b) - f(b)\Big]}{f(b)},$$

hence

$$\langle(\Delta N)^2\rangle = \bar{N}\frac{f'(b)}{(1-b)f'(b) - f(b)}. \tag{15}$$

The grand canonical ensemble denotes the probability density of the system in a certain microstate compatible with the given macrostate. Also, it is well known that thermodynamics is nominally macroscopic, while statistical mechanics is nominally microscopic. From the principle that fluctuation moments are identical when computed with statistical ensembles and with thermodynamic fluctuation [17], we infer that the mean square fluctuations from thermodynamics, and grand canonical ensemble for any functional form of $b(x)$ must be same. Hence equating the two results, from thermodynamics and the grand canonical ensemble-derived in equations (13) and (15), gives a first order differential equation between x and b:

$$x\frac{\mathrm{d}b}{\mathrm{d}x} = \frac{f(b)}{f'(b)} = \frac{a_0 b^k + a_1 b^{k+1} + a_2 b^{k+2} + \cdots}{a_0 k b^{k-1} + a_1(k+1)b^k + a_2(k+2)b^{k+1} + \cdots}. \tag{16}$$

There is a critical point [22] at which the mean square fluctuation $\langle(\Delta N)^2\rangle \to \infty$ in equation (13), is given by

$$x\frac{\mathrm{d}b}{\mathrm{d}x} = (1-b). \tag{17}$$

Then combining equations (16) and (17), we have

$$1 - b = \frac{a_0 b^k + a_1 b^{k+1} + a_2 b^{k+2} + a_3 b^{k+3}\cdots}{a_0 k b^{k-1} + a_1(k+1)b^k + a_2(k+2)b^{k+1} + a_3(k+3)b^{k+2}\cdots}. \tag{18}$$

To determine a_λ and k, equate like powers of b on both sides of equation (18), which gives $a_0 k = 0$; and since $a_0 \neq 0$, $k = 0$, which is physically also true. This condition implies that as $x \to 0$, $b \to 0$ which corresponds to the ideal gas situation. After some algebra, we obtain a relation between various coefficients given by

$$a_{\lambda+1}(k+\lambda+2) - a_\lambda(k+\lambda+1) = a_\lambda, \tag{19}$$

which gives

$$a_1 = a_2 = a_3 = \cdots = a_n = a_0. \tag{20}$$

Substituting equation (20) in equation (10) and $k = 0$, we have

$$\begin{aligned} x &= \sum_{\lambda=0}^{\infty} a_\lambda b^\lambda \\ &= a_0(b + b^2 + b^3 + \ldots\ldots.. + b^n + \ldots..) \\ &= a_0 b(1-b)^{-1}, \end{aligned}$$

or

$$b = \frac{x}{1+x} = \frac{\beta n T^{-3}}{1 + \beta n T^{-3}}. \tag{21}$$

Here, a_0 is chosen to be unity. The above solution confirms earlier functional form of b [1, 15, 23]. Here, we have derived it independently, from more fundamental principles, without much physical constraints.

3. Galaxy Distribution Function

The grand canonical ensemble is a model for an infinite system in a thermostat. In such an ensemble, the volume of subregions are much smaller than the total volume. Secondly, grand canonical ensemble provides a powerful tool for statistical mechanical problems for systems of interacting particles. In our analysis, we consider an ensemble of systems, all replicas of the actual system of interest which can exchange both energy and particles. This describes a reasonable way to sample the universe.

The observed simple objective clustering statistics is the distribution function $f(N)$ which gives the probability of finding any number of galaxies in a volume of arbitrary size and shape. One way to obtain probability distribution function is to describe it by canonical ensemble which can be obtained from grand partition function Z_G defined by the relation [24]

$$Z_G(T, V, z) = \sum_{N=0}^{\infty} z^N Z(T, V, N). \tag{22}$$

This equation shows the grand canonical partition function as a weighted sum of all canonical partition functions Z. The weighting factor z, called the activity, is related to chemical potential of the system (again in units of Boltzmann's constant $k = 1$) by

$$z = \mathrm{e}^{\mu/T}. \tag{23}$$

Formally, in equation (23), one can allow for complex values of the auxiliary variable z. Then equation (22) is nothing but a complex Taylor expansion of analytic function $Z_G(z)$. From function theory, the expansion co-efficient $Z(T, V, N)$ can be determined quite easily, from which one gets the relation

$$Z(T, V, N) = \frac{1}{2\pi i} \oint \frac{Z_G(T, V, z)}{z^{N+1}}\, \mathrm{d}z. \tag{24}$$

Here, one integrates on a sphere around the point $z = 0$ in the interior of the convergence radius of equation (22). The probability $P_{i,N}$ of finding a system of the grand canonical ensemble for a particle number N and at phase space point i is

$$P_{i,N} = \frac{\exp(N\mu/T - U_i/T)}{Z_G} \tag{25}$$

Here, U_i corresponds to the phase cell i and Z_G is given by

$$\begin{aligned} Z_G &= \sum_{i,N} \exp(N\mu/T - U_i/T) \\ &= \int \exp\left[-\left(\sum_i \frac{P_i^2}{2m_i} + \frac{1}{2}\sum_{i \neq j} \frac{G m_i m_j}{r_{ij}}\right)\frac{1}{T}\right] \mathrm{d}^{3N}p\, \mathrm{d}^{3N}r. \end{aligned} \tag{26}$$

We define the canonical partition function Z as

$$Z = \sum_i \mathrm{e}^{-U_i/T}. \tag{27}$$

The probability of finding N particles (galaxies) in grand canonical ensemble is the summation of equation (25) over all phase space cell i:

$$\begin{aligned} f(N) &= \sum_i P_{i,N} = \frac{\sum_i \mathrm{e}^{N\mu/T} \mathrm{e}^{-U_i/T}}{Z_G} \\ &= \frac{\mathrm{e}^{N\mu/T} Z}{Z_G}, \end{aligned} \tag{28}$$

and grand canonical potential ψ is given by

$$\begin{aligned}\psi(1/T, V, \mu/T) &= \ln Z_G \\ &= \frac{\bar{P}V}{T} = \bar{N}(1-b)\end{aligned} \tag{29}$$

Using equation (29) in (28) gives distribution function as

$$f(N) = \mathrm{e}^{-\bar{N}(1-b)}\mathrm{e}^{N\mu/T}Z. \tag{30}$$

This is the basic equation for finding the distribution function for galaxies in the grand canonical ensemble in which only the canonical partition function Z is unknown. To calculate Z we make use of equations (23), (24) and (29) to obtain

$$Z(T, V, N) = \frac{1}{2\pi i}\int \frac{\mathrm{e}^{\psi}}{\mathrm{e}^{N\mu/T}}\mathrm{d}(\mu/T). \tag{31}$$

Now, using equation (21) in grand canonical potential ψ; and chemical potential, equation (8) gives:

$$\psi = \alpha b \tag{32}$$

and

$$\mathrm{e}^{-\mu/T} = \frac{\beta T^{-3/2}\mathrm{e}^{b}}{b}. \tag{33}$$

where

$$\alpha = \frac{V}{\beta T^{-3}}. \tag{34}$$

Differentiation of equation (33) gives

$$d(\mu/T) = \frac{(1-b)}{b}\,\mathrm{d}b. \tag{35}$$

Substitution of equation (32) to (35) in (31) gives canonical partition function Z as

$$Z = \frac{(\beta T^{-3/2})^N}{2\pi i}\left[\oint \frac{\mathrm{e}^{(\alpha+N)b}}{b^{N+1}}\,\mathrm{d}b - \oint \frac{\mathrm{e}^{(\alpha+N)b}}{b^{N}}\,\mathrm{d}b\right]. \tag{36}$$

To evaluate these integrals we must use contour integration [22]. The first integral has pole at $b = 0$ of order $N + 1$, while the second integral has pole at $b = 0$ of order N. Integration of equation (36) by Cauchy's residue theorem gives

$$\oint \frac{\mathrm{e}^{(\alpha+N)b}}{b^{N+1}}\,\mathrm{d}b = \frac{2\pi i(\alpha+N)^N}{N!}. \tag{37}$$

Similarly,

$$\oint \frac{\mathrm{e}^{(\alpha+N)b}}{b^{N}}\,\mathrm{d}b = \frac{2\pi i(\alpha+N)^{N-1}}{(N-1)!}. \tag{38}$$

With the help of equations (37) and (38), the canonical partition function Z, given by equation (36), takes the form:

$$Z = (\beta T^{-3/2})^N \frac{V}{N!(\beta T^{-3})}\left(\frac{V}{\beta T^{-3}} + N\right)^{N-1} \tag{39}$$

Using equation (21) in (39) gives (with $\bar{n}V = \bar{N}$):

$$Z(T,V,N) = \frac{1}{N!}(VT^{3/2})^N \left[1 + \frac{Nb}{\bar{N}(1-b)}\right]^{N-1} \tag{40}$$

The special case $N = 0$ corresponds to the situation that there is no galaxy in volume V. In that case, the canonical partition function $Z = 1$ is given by equation (40). Substituting $Z = 1$ and $N = 0$ in equation (30) immediately gives the probability distribution of a void in a randomly positioned volume V:

$$f(0) = \mathrm{e}^{-\bar{n}V(1-b)}. \tag{41}$$

It is also possible to develop formal generalization of probability of voids $f(0)$ to evaluate all other probabilities for understanding galaxy clustering [25, 26]. Also from equation (36), we can get the canonical partition function

$$Z = \frac{V^N T^{(3/2)N}}{N!}$$

if $b = 0$ (no gravity). Substitution of chemical potential (μ/T) from equation (33) and canonical partition function Z from equation (40) in equation (30) simply gives the distribution function for gravitational clustering of galaxies:

$$f(N) = \frac{\bar{N}(1-b)}{N!}\left[\bar{N}(1-b) + Nb\right]^{N-1} \mathrm{e}^{-\{\bar{N}(1-b)+Nb\}}, \tag{42}$$

where $\bar{N} = \bar{n}V$ is the average number expected in a volume V with an average density $\bar{n}$. The value of b ranges from 0 to 1. When there are no correlations, $b = 0$, and equation (42) gives standard result for a Poisson distribution. As clustering becomes important, b increases. The upper limit $b = 1$ would represent a state of virial equilibrium on all levels of clustering, an idealized state which is not fully realized because of the expansion.

Equation (42) has been derived earlier by Saslaw and Hamilton [1] with the help of generating function technique. They have derived only first few terms of $f(N)$ and generalized it by induction method. Here, we have shown as to how the distribution function can be obtained directly, independent of the earlier methods, by simple contour integration.

The distribution function (equation 42) fits well with what has been observed for the Zwicky Catalog [9], the UGC and ESO catalog [10] and IRAS Catalog [5]. Recently, Sivakoff and Saslaw [11] used wide range samples in the 2MASS survey (Two Micron All Sky Survey) to determine the spatial distribution function of galaxies. This catalog has 10–100 times as many galaxies available than earlier catalogs and hence is a better representation of the distribution of galaxies. The agreement, in general, is again very good. The slight disagreement may be due to several factors which include initial conditions when galaxies started clustering, segregation of more massive galaxies into smaller volumes or some other physical effects. A theoretical explanation for such disagreements has been a subject of interest for Ahmad et al [27, 28]

4. Conclusion

We derive various results associated with gravitational clustering of galaxies, evolving through quasi-equilibrium thermodynamics, by using fluctuation theory of ensembles. The evolution of galaxy clustering in a quasi-equilibrium manner, in the expanding universe, depends on the functional form of $b(nT^{-3})$. The specific combination nT^{-3} reveals the scale invariance of gravitational clustering in an expanding homogeneous universe. However, functional form of $b(nT^{-3})$ is restrained by the physical constraints on it [1, 14–16]. Here, we find its functional form independent of the earlier methods, with no constraints on $b(nT^{-3})$. Its derivation from the second order density fluctuations of thermodynamics and grand canonical ensemble, with no physical constraints on b, clearly provides a deeper insight of gravitational clustering, on the basis of ensemble theory.

We also propose a more basic derivation of the galaxy distribution function $f(N)$, first derived by Saslaw and Hamilton [1] on the basis of quasi-equilibrium gravitational thermodynamics. They derived only the first few terms and generalized the result by induction method. Here, we developed an independent method for evaluation of distribution function for gravitational clustering on the basis of ensemble theory of statistical mechanics. The distribution function emerges directly from the calculations, thereby making the theory more fundamental. Since distribution functions have long been associated with an understanding of the gravitational galaxy clustering, our study establishes a deep connection between ensemble theory and gravitational galaxy clustering.

Finally, we conclude that there is a considerable potential for broader understanding of gravitational galaxy clustering on the basis of statistical mechanics.

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