

Regular and chaotic Bose-Einstein condensate in an accelerated Wannier-Stark lattice

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We investigate a Bose-Einstein condensate held in a quasi-one-dimensional Wannier-Stark lattice which is a combination of linear potential with an accelerated optical lattice. It is demonstrated that the system can be reduced to a periodically driven Gross-Pitaevskii one, in which we find the first exact analytical solution and the regular and chaotic numerical solutions with accelerated atomic flow densities. The results suggest an experimental scheme for generating and controlling the accelerating regular and chaotic matter-waves.

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I. INTRODUCTION

Recently, experimental investigation on the Bose-Einstein condensates (BECs) in a Wannier-Stark (WS) system have attracted much attention [1]. In some experiments, a BEC of rubidium atoms was created, which then was loaded into the optical lattice potential of a standing laser light [2] and was adopted to observe the collective tunneling effects [1]. The optical potentials have been used in many theoretical and experimental studies of quantum dynamics, for example, the coherent pulse output from BECs in WS system [3], the observation of Bloch oscillations both with single atoms and with a BEC in an accelerated standing wave [2, 4], the studies of WS ladders for the accelerated optical potential [5]. An atomic-scale analog of the kicked rotor is realized by placing laser-cooled atoms in a pulsed standing wave [6]. Very recently, M. Glück et al. investigated the properties of a coherent superposition of WS resonances [3] and gave the lifetime of WS states [7]. Chaotic behaviors have also been found in the WS systems without acceleration [6, 8, 9, 10]. It is a natural motivation for us to demonstrate the chaotic and regular features for the BEC in an accelerated WS lattice.

On the other hand, it is well-known that the BEC governed by a Gross-Pitaevskii equation (GPE) without external potential is an integrable system and the integrability could be easily broken by external potentials of different forms [11]. So previously, only few analytical works concern exact solutions of the system, where one-dimensional (1D) stationary systems with some simple potentials are treated, such as the infinite or finite square-wells [12, 13, 14, 15], the step-potentials [16], δ or δ comb potentials [17, 18, 19], linear ramp potential [20, 21] and optical lattice potentials [22, 23]. Under some rigorous conditions on the interaction intensities or external potentials, several exact nonstationary-state solutions were constructed [24, 25], including the exact soliton solutions [26, 27, 28, 29, 30]. It is worth noting that the balance between nonlinearity and dispersion was found in the seminal work of soliton [31], and the new balances between the atom-atom interaction and the external potentials are demonstrated recently [25, 32, 33].

By using the balance conditions, although some exact solutions have been constructed for the GPE with periodic potential, however, any exact solution in the nonintegrable WS system with a combination of the linear and periodic potentials has not been reported yet.

The aim of this paper is to present the first exact analytical solution with balance condition and to illustrate the regular and chaotic numerical solutions of the accelerated WS system. The corresponding atomic flow densities accelerated by the constant force are demonstrated. Based on the relations between the system parameters and the solution behaviors, we suggest an experimental method for controlling the regular and chaotic states by applying the accelerated optical potential and adjusting the system parameters.

II. SIMPLIFICATION OF THE WANNIER-STARK SYSTEM

The mean-field theory is a successful one for describing the BECs. In this theory, the dynamical behaviors are governed by the GPE [34, 35], which provides us a nonlinear macroscopic quantum system. Let us consider a BEC trapped in one-dimensional tilted optical lattice potential

$$V(x', t') = V_0 \cos(2k_L \xi') + Fx', \quad \xi' = x' + \frac{1}{2}at'^2; \quad (1)$$

here x' and t' are spatial and time coordinates, $V_0 \cos(2k_L \xi')$ is the accelerating optical potential [5, 8] with strength V_0 , wave vector k_L and acceleration a , and $F = ma$ is a constant force [8] with m being the atomic mass. Due to this force, a “tilted” potential is produced that leads the atoms to accelerate in x direction with linearly increasing flow density and makes the atoms tunnel out of the optical traps. The corresponding dimensionless GPE reads as

$$i \frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + [V_0 \cos(2\xi) + \alpha x + g_{1d}|\psi|^2]\psi, \quad (2)$$
$$\xi = k_L \xi' = x + \alpha t^2, \quad \alpha = \frac{1}{2}k_L a \hbar^2 / E_r^2,$$

where the dimensionless spatial and time coordinates are $x = k_L x'$ and $t = E_r t' / \hbar$. The wave function ψ has been normalized in units of the radical $\sqrt{k_L}$, the potential depth V_0 is normalized by the recoil energy $E_r = \hbar^2 k_L^2 / (2m)$. The constant force is rescaled by the unit $k_L E_r$, so all variables and parameters in equation (3) are dimensionless. In such units, the interatomic interaction intensity related to the s -wave scattering length a_s is in the form $g_{1d} = 4a_s / (k_L l_r^2)$ with $l_r = \sqrt{\hbar / (m\omega_r)}$ being the radial length of harmonic oscillator.

In order to get a simple description and better understanding of BEC dynamics, we let the wave function be in the form

$$\psi(x, t) = u(\xi, t) \exp \left[-i \left(\alpha x t + \frac{1}{3} \alpha^2 t^3 \right) \right], \quad (3)$$

where undetermined function $u(\xi, t)$ may be real or complex, which is normalized to the total number of atoms, $\int u^2(\xi, t) dx = N$. Given Eq. (2), we perform the calculations

$$\begin{aligned} i \frac{\partial \psi}{\partial t} &= i \left(\frac{\partial u}{\partial t} + 2\alpha t \frac{\partial u}{\partial \xi} \right) e^{-i(\alpha x t + \frac{1}{3} \alpha^2 t^3)} + (\alpha x + \alpha^2 t^2) \psi, \\ \frac{\partial^2 \psi}{\partial x^2} &= \left(\frac{\partial^2 u}{\partial \xi^2} - i 2\alpha t \frac{\partial u}{\partial \xi} \right) e^{-i(\alpha x t + \frac{1}{3} \alpha^2 t^3)} - \alpha^2 t^2 \psi. \end{aligned} \quad (4)$$

Substituting Eq. (4) into Eq. (2) yields

$$i \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial \xi^2} + [V_0 \cos(2\xi) + g_{1d} |u|^2] u \quad (5)$$

in which the linear potential is removed and its effect is included in the parameter α . When the optical potential is switched off, $V_0 = 0$, Eq. (5) becomes a standard nonlinear Schrödinger equation (NLSE), whose single-soliton and multi-soliton solutions are well-known for us. It is worth noting that in the transformation $x \rightarrow x + n\pi$ for $n = 0, 1, 2, \dots$, Eq. (5) and its solution $u(\xi, t)$ are kept such that Eq. (3) gives n solutions

$$\psi_n(x, t) = \psi(x + n\pi, t) = \psi(x, t) e^{-i\alpha n \pi t}. \quad (6)$$

The solutions with different n possess the different phases and the same amplitude.

III. EXACT PERIODIC WAVE WITH ACCELERATED FLOW

We are interested in the exact analytical solution and regular and chaotic numerical solutions of Eq. (5). Noticing that ξ and t in Eq. (5) are two independent variables, we can rewrite the function $u(\xi, t)$ in the separation form of variables

$$u(\xi, t) = \phi(\xi) e^{-i\mu t}, \quad (7)$$

and transform Eq. (5) to the ordinary differential equation

$$\mu \phi = -\frac{d^2 \phi}{d\xi^2} + [V_0 \cos(2\xi) + g_{1d} |\phi|^2] \phi, \quad (8)$$

where μ is a constant adjusted by the normalization condition and can be called the chemical potential. By using the balance technique, we establish the balance condition [25, 33]

$$g_{1d} |\phi|^2 + V_0 \cos(2\xi) = \mu - 1, \quad (9)$$

and then reduce Eq. (7) to the linear Schrödinger equation

$$\frac{d^2 \phi}{d\xi^2} = -\phi. \quad (10)$$

The exact solution of Eq. (8) must obey Eqs. (9) and (10) simultaneously. General solution of the complex equation (10) can be written as

$$\phi = (A + iC) \cos \xi + (B + iD) \sin \xi, \quad (11)$$

where A, B, C and D are real constants which are determined partly by the balance condition (9). From Eq. (11) we construct the quadratic norm

$$\begin{aligned} |\phi|^2 &= A^2 + C^2 + (B^2 + D^2 - A^2 - C^2) \sin^2 \xi \\ &\quad + (AB + CD) \sin(2\xi). \end{aligned} \quad (12)$$

Comparing Eq. (9) with Eq. (12) and noticing $\cos(2\xi) = 1 - 2 \sin^2 \xi$ produce the algebraical equations

$$\begin{aligned} g_{1d}(A^2 + C^2) &= \mu - V_0 - 1, \quad AB + CD = 0, \\ g_{1d}(B^2 + D^2 - A^2 - C^2) &= V_0, \end{aligned} \quad (13)$$

which denote a group of indefinite equations with infinite numbers of solutions.

The existence of multiple solutions of Eq. (13) implies that phase of the exact solution

$$\arctan \frac{(C \cos \xi + D \sin \xi)}{(A \cos \xi + B \sin \xi)}, \quad (14)$$

has some arbitrariness. However, quadratic norm of the solution (7) is determined uniquely by Eq. (9), $|\phi|^2 = [\mu - 1 - V_0 \cos(2\xi)] / g_{1d}$. This means that under the exact state the atomic density profile shapes the periodic wave-packets which propagate with acceleration a . Therefore, we can achieve the accelerated transport of BEC, through the considered exact solution. Application of the normalization integral yields the average number of condensed atoms per well [22]

$$N' = (\pi)^{-1} \int_0^\pi |\phi(x)|^2 dx = (\mu - 1) / g_{1d}, \quad (15)$$

which determines the chemical potential as $\mu = g_{1d} N' + 1$. Given the chemical potential, the exact atomic density reads

$$|\phi|^2 = R^2 = N' - \frac{V_0}{g_{1d}} \cos(2\xi). \quad (16)$$

The result is accurately adjusted by the experimental parameters N' , V_0 , g_{1d} , and wave vector k_L and acceleration a implied in ξ .

Clearly, Eq. (8) is a periodically driven GPE of the spatiotemporal evolution in which the well-known Smale-horseshoe chaos exists for a certain parameter region [36]. Writing the solution ϕ in the form of $\phi = R(\xi) \exp[i\theta(\xi)]$ and inserting it into Eq. (8) lead to two couple equations

$$\frac{d^2 R}{d\xi^2} = R \left(\frac{d\theta}{d\xi} \right)^2 + g_{1d} R^3 + [V_0 \cos(2\xi) - \mu] R, \quad (17)$$

$$\frac{d^2 \theta}{d\xi^2} + 2\theta_\xi \frac{R_\xi}{R} = 0. \quad (18)$$

The square of the modulus $|\psi|^2 = |\phi|^2 = R^2$ denotes atomic number density and the total phase reads

$$\Theta(x, t) = \theta(\xi) - (\mu + \alpha n \pi) t - (\alpha x t + \frac{1}{3} \alpha^2 t^3). \quad (19)$$

The both are associated with the velocity field v and flow density J , through the formulas $v = \hbar \Theta_x / m$ and $J = v R^2$. Integrating Eq. (18) yields the part phase

$$\theta = \int \frac{J_0}{R^2} d\xi. \quad (20)$$

So the velocity field and flow density become

$$\begin{aligned} v(\xi, t) &= \frac{\hbar k_L}{m} (\theta_\xi - \alpha t) = \frac{\hbar k_L}{m} \left[\frac{J_0}{R^2(\xi)} - \alpha t \right], \\ J(\xi, t) &= \frac{\hbar k_L}{m} [J_0 - \alpha R^2(\xi) t], \end{aligned} \quad (21)$$

where $J_0 = \theta_\xi(\xi_0) R^2(\xi_0)$ denotes an integration constant determined by the flow density at the point $\xi_0 = x_0 + \alpha t_0^2$ for the initial time t_0 and boundary coordinate x_0 . Substituting Eq. (16) into Eq. (21) produces the exact flow velocity and flow density. The linear term of time in flow velocity implies the BEC superfluid being accelerated.

IV. REGULAR AND CHAOTIC NUMERICAL SOLUTIONS

In general, the balance condition (9) cannot be satisfied such that we have to solve Eq. (17) for the modulus R . Applying $\theta_\xi = J_0/R^2$ of Eq. (20) to Eq. (17), we arrive at the decoupled equation

$$\frac{d^2 R}{d\xi^2} = \frac{J_0^2}{R^3} + g_{1d} R^3 + [V_0 \cos(2\xi) - \mu] R. \quad (22)$$

This is a real equation on the accelerated reference frame, which can be reduced to the parametrically driven Duffing equation [37, 38] for the case $J_0 = 0$. The Smale-horseshoe chaos in $J_0 = 0$ case has been widely investigated. Generally, $J_0 \neq 0$, the system becomes more complicated and the chaotic property may be kept. Under balance condition (9) we can prove directly that Eq. (16) is an exact special solution of Eq. (22) for a particular integral constant. When the periodic driving is weak

enough, the chaotic perturbed solution has been constructed for the system without linear potential [36, 39]. For most of parameters and initial data Eq. (22) is not analytically solvable, which necessitates the numerical solutions.

At the initial and boundary point ξ_0 , the values of the velocity field and number density can't be determined exactly in experiment due to the fluctuation of the atomic thermal cloud, we can randomly choose some possible values to perform the numerical calculations. In order to explore the analytically insolvable system (22), we numerically solve it by using the MATHEMATICA code

```
T = π; e[ {Rnew-, vnew-} ] := {R[T], v[T]}/.Flatten
[NDSolve[ {R'[ξ] == v[ξ], v'[ξ] == g1dR[ξ]^3 - μR[ξ]
+ J0^2/R[ξ]^3 + V0Cos[2ξ] * R[ξ], R[0] == Rnew,
v[0] == vnew}, {R, v}, {ξ, 0, T}]]; Do[pic_i = ListPlot
[Drop[Nestlist[e, {Random[Real, {-0.5, 0.5}],
Random[Real, {-0.5, 0.5}]}], 5100], 100], {i, 1, 10}]
```

to make 10 groups of orbits on the Poincaré section of the equivalent phase space (R, R_ξ) for the random initial conditions $\{R(\xi_0) \in [-0.5, 0.5], R_\xi(\xi_0) \in [-0.5, 0.5]\}$ and different parameter sets. Each of the groups contains 10 orbits, which corresponds to one of the following cases:

Case 1. For the parameter set $g_{1d} = -1$, $\mu = -0.5$, $J_0 = 0.01$, $V_0 = 0.05$ all the 10 orbits are regular, whose 3 typical profiles are shown in Fig. 1.

Case 2. By increasing the strength of lattice potential to $V_0 = 0.2$ and keeping the other parameters as in case 1, we observe 3 regular orbits of Fig. 1 and 7 chaotic orbits, whose 2 typical profiles are shown in Fig. 2.

Case 3. After further increasing the strength of lattice potential to $V_0 = 0.5$, all the orbits become chaotic.

Case 4. By increasing the flow density to $J_0 = 0.16$ and keeping the other parameters as in case 3, we find that all the orbits become regular.

Case 5. By increasing the strength to $V_0 = 3$ and keeping the other parameters as in case 4, we observe 4 regular and 6 chaotic orbits.

Case 6. After further increasing the strength to $V_0 = 5$, all the orbits become chaotic.

Case 7. For the parameter set $g_{1d} = -1$, $\mu = 0.5$, $J_0 = 0.16$, $V_0 = 0.5$ with positive chemical potential μ all the chaotic orbits in case 3 are transformed to regular ones.

Case 8. By increasing the strength to $V_0 = 2$ and keeping the other parameters as in case 7, we observe 2 regular and 8 chaotic orbits.

Case 9. After further increasing the strength to $V_0 = 4$ and keeping the other parameters as in case 8, all the orbits become chaotic.

Case 10. For the different parameter sets with positive interaction g_{1d} we make many regular orbits and no chaotic orbit is found.

The above results imply that for the attractive interaction and negative chemical potential increasing the lattice strength can strengthen chaoticity of the system.

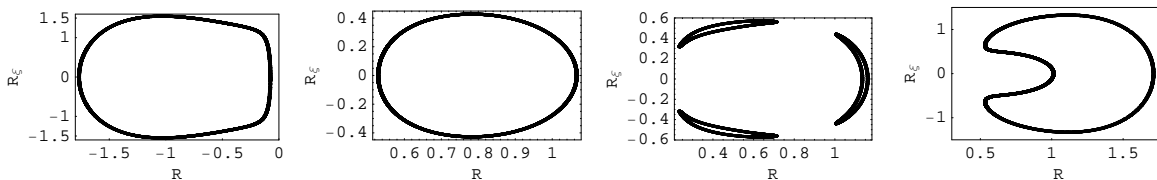


FIG. 1: Typical regular orbits on the Poincaré sections of the dimensionless ‘phase space’ (R , R_ξ) for ten different initial conditions and/or parameter sets. Here the phase orbits evolve in a finite region and display the regular state distributions.

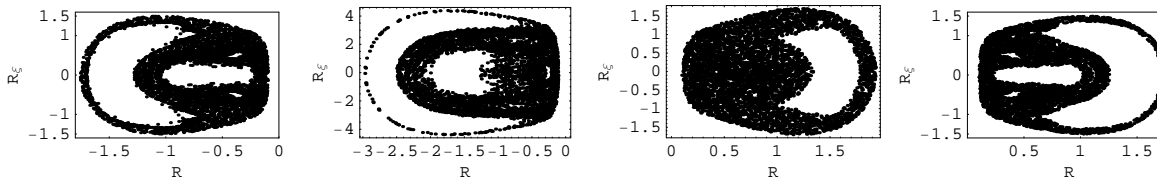


FIG. 2: Typical chaotic orbits on the Poincaré sections of the dimensionless ‘phase space’ (R , R_ξ) for ten different initial conditions and/or parameter sets. Here the phase orbits evolve in a finite region and exhibit the confused state distributions.

Conversely, increasing the flow density can weaken the chaoticity. On the other hand, after changing the chemical potential from negative to positive, the chaoticity is weakened effectively. Finally, for the positive interaction g_{1d} we have not found the chaotic orbits. The 4 typical regular orbits and 4 typical chaotic orbits among the $10 \times 10 = 100$ orbits are shown in Fig. 1 and Fig. 2 respectively. They are associated with different initial conditions and/or parameter sets. As the examples of densities of atomic number we plot their spatiotemporal evolutions for the parameters of Case 2 and two fixed initial conditions as in Fig. 3. Figure 3(a) corresponds to the first closed orbit of Fig. 1 and describes a quasiperiodic evolution thereby. Figure 3(b) is associated with the first chaotic orbit of Fig. 2, which possesses obvious aperiodicity. From Eq. (21) the corresponding flow densities can be easily illustrated. They will increase linearly in time that leads the condensed atoms to tunnel out of the optical traps. These results reveal the relations between the system parameters and the solution behaviors, and suggest a method for controlling the regular and chaotic states.

V. CONCLUSIONS

In summary, we have investigated a BEC interacting with an accelerated WS potential. Using the mean-field method and the macroscopic one-body wave function, we seek the exact analytical solution and the regular and chaotic numerical solutions of the system. It is demonstrated that after the linear potential being removed by a function transformation, the governing equation becomes a periodically driven GPE on the accelerated reference frame. With the help of the balance condition, we establish the first exact analytical solution of the WS system, which is accurately controlled by the experimental pa-

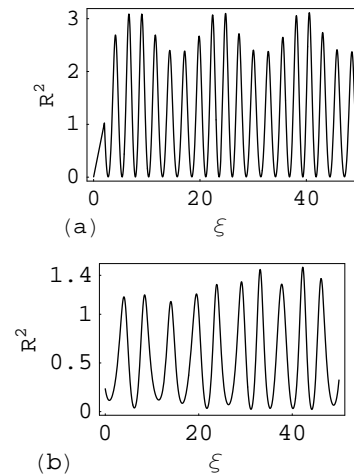


FIG. 3: Spatiotemporal evolutions of the densities of atomic number for the parameters of Case 2 and the initial conditions (a) $R^2(0) = -0.07793183579$, $R_\xi^2(0) = -0.199975080313$, (b) $R^2(0) = -0.48451144892$, $R_\xi^2(0) = 0.31792019031$.

rameters. Further writing the solution of GPE in the exponential form, we obtain the equation of modulus, which contains the parametrically driven Duffing equation. The well-known Smale-horseshoe chaos and quasiperiodic orbits on the Poincaré sections of the dimensionless ‘phase space’ are shown numerically for different initial conditions and parameter regions of different chaoticity. The accelerated atomic flow densities are demonstrated for both the regular and chaotic states.

It is well known that chaos could emerge in the processes of BEC collapsing and may play a destructive role for the BEC system. Therefore, predicting and controlling chaos are quite important for the creation and application of BEC. Our analytical and numerical results have supplied a method for controlling the regular and chaotic

states, through the application of accelerated optical potential and the adjustments of system parameters.

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