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Hamilton-Jacobi Formulation of Siegel Superparticle

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Abstract

The Hamilton-Jacobi formalism of constrained systems is used to study Siegel superparticles moving in $R^{4|4}$ flat superspace. The equations of motion for a singular system are obtained as total differential equations in many variables. These equations of motion are in exact agreement with those obtained by Dirac's method.

Key Words: Hamilton-Jacobi formalism, Singular Lagrangian.
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1. Introduction

The Hamiltonian formulation of constrained systems was initiated by Dirac [1]. He obtained the equations of motion of a singular Lagrangian system by using the consistency conditions. He also showed that the number of degrees of freedom of the mechanical system can be reduced [2, 3]. An alternative Hamilton-Jacobi scheme for constrained systems was proposed by Güler [4,5]. Güler used the Hamilton-Jacobi partial differential equations to obtain the equations of motion as total differential equations. The Hamilton-Jacobi formalism applied only to a few number of physical examples as the electromagnetic field, relativistic particle in an external Electromagnetic field, the Young-Mills field and the Einstein gravitational field [6–9]. But a better understanding of this approach as a utility in the study of singular systems is still lacking, and such understanding can only be achieved through its application to other interesting physical systems. Here we apply this technique to the superparticle with spacetime supersymmetry (with bosonic and fermionic variables in a unique and compact notation), via the massless superparticle discovered by Siegel [10]. The advantage of the Hamilton-Jacobi formalism is that we have no difference between first and second class constraints and we do not need a gauge-fixing term because the gauge variables are separated in the processes of constructing an integrable system of total differential equations.

The work is organized as follows: In section 2, Hamilton-Jacobi Formulation is presented. The motion of Siegel Superparticle is analyzed by using Hamilton-Jacobi Formulation in section 3. In section 4 the conclusion is given.

2. Hamilton-Jacobi formalism of constrained systems

The system that is described by the Lagrangian $L(q_i, \dot{q}_i, t)$, $i = 1, ..., n$, is a constrained system if the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \qquad i, j = 1, \ldots, n, \tag{1}$$

has a rank $(n-p), p < n$. In this momenta p are dependent of each other. The generalized momenta P_i corresponding to the generalized coordinates q_i are defined as

$$P_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, \ldots, n-p, \tag{2}$$

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = n-p+1, \ldots, n. \tag{3}$$

Since the rank of the Hessian matrix is $(n-p)$, one may solve eqn. (2) for $\dot{q}_a$ as

$$\dot{q}_a = \omega_a(q_i, \dot{q}_\mu, P_b). \tag{4}$$

Substituting eqn. (4) into eqn. (3), we obtain relations in $q_i, P_a, \dot{q}_\nu$ and t in the form

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}_{|_{\dot{q}_a = \omega_a}} = -H_\mu(q_i, \dot{q}_\mu, \dot{q}_a = \omega_a, P_a, t). \tag{5}$$

The canonical Hamiltonian H_0 is defined as

$$H_0 = -L(q_i, \dot{q}_\mu, \dot{q}_a = \omega_a, t) + P_a \omega_a + \dot{q}_\mu P_{\mu|_{P_\nu = -H_\nu}}. \tag{6}$$

The set of Hamilton-Jacobi partial differential equations (HJPDE)
is expressed as

$$H'_\alpha(q_\beta; q_a; P_a = \frac{\partial s}{\partial q_a}; P_\mu = \frac{\partial s}{\partial q_\mu}) = 0, \qquad \alpha, \beta = 0, 1, \ldots, r, \tag{7}$$

where

$$H'_0 = P_0 + H_0; \tag{8}$$

$$H'_\mu = P_\mu + H_\mu. \tag{9}$$

with $q_0 = t$ and S being the action. The equations of motion are obtained as total differential equations in many variables such as,

$$dq_a = \frac{\partial H'_\alpha}{\partial P_a} dt_\alpha, \tag{10}$$

$$dP_r = -\frac{\partial H'_\alpha}{\partial q_r} dt_\alpha, \qquad r = 0, 1, \ldots, n, \tag{11}$$

$$dZ = (-H_\alpha + P_a \frac{\partial H'_\alpha}{\partial P_a}) dt_\alpha, \tag{12}$$

where $Z = S(t_\alpha, q_a)$. These equations are integrable if and only if [11, 12]

$$dH'_0 = 0, \tag{13}$$

$$dH'_\mu = 0, \qquad \mu = n-p+1, \ldots, n. \tag{14}$$

If conditions (13) and (14) are not satisfied identically, we consider them as new constraints and we examine their variations. Thus repeating this procedure, one may obtain a set of constraints such that all variations vanish.

3. Hamilton-Jacobi Formulation of Siegel Superparticle

Consider the Siegel Superparticle action [10, 13]

$$S = \int \{ \frac{1}{2e}(\dot{x}^m + i\theta\sigma^m\dot{\bar{\theta}} - i\dot{\theta}\sigma^m\bar{\theta} + i\psi\sigma^m\bar{\rho} - i\rho\sigma^m\bar{\psi})^2 - \rho^\alpha\dot{\theta}_\alpha - \bar{\rho}_{\dot{\alpha}}\dot{\bar{\theta}}^{\dot{\alpha}} \} d\tau. \tag{15}$$

The Lagrangian is

$$L = \frac{1}{2e}(\dot{x}^m + i\theta\sigma^m\dot{\bar{\theta}} - i\dot{\theta}\sigma^m\bar{\theta} + i\psi\sigma^m\bar{\rho} - i\rho\sigma^m\bar{\psi})^2 - \rho^\alpha\dot{\theta}_\alpha - \bar{\rho}_{\dot{\alpha}}\dot{\bar{\theta}}^{\dot{\alpha}}. \tag{16}$$

The singularity of the the Lagrangian follows from the fact that the rank of the Hessian matrix A_{ij} is one.

The canonical momenta defined in eqns. (2, 3) take the forms

$$P_m = \frac{\partial L}{\partial \dot{x}^m} = \frac{1}{e}(\dot{x}^m + i\theta\sigma^m\dot{\bar{\theta}} - i\dot{\theta}\sigma^m\bar{\theta} + i\psi\sigma^m\bar{\rho} - i\rho\sigma^m\bar{\psi}), \tag{17}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = -iP_m\sigma^m\bar{\theta} - \rho^\alpha = -H_\theta, \tag{18}$$

$$P_{\bar{\theta}} = \frac{\partial L}{\partial \dot{\bar{\theta}}} = i\theta\sigma^m P_m - \rho_{\dot{\alpha}} = -H_{\bar{\theta}}, \tag{19}$$

$$P_\psi = \frac{\partial L}{\partial \dot{\psi}} = 0 = -H_\psi, \tag{20}$$

$$P_{\bar{\psi}} = \frac{\partial L}{\partial \dot{\bar{\psi}}} = 0 = -H_{\bar{\psi}}, \tag{21}$$

$$P_\rho = \frac{\partial L}{\partial \dot{\rho}} = 0 = -H_\rho, \tag{22}$$

$$P_{\bar{\rho}} = \frac{\partial L}{\partial \dot{\bar{\rho}}} = 0 = -H_{\bar{\rho}}, \tag{23}$$

and

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_e. \tag{24}$$

Since the rank of the Hessian matrix is one, we can solve eqn. (17) for $\dot{x}^m$ in terms of P_m and other coordinates:

$$\dot{x}^m = eP_m - i\theta\sigma^m\dot{\bar{\theta}} + i\dot{\theta}\sigma^m\bar{\theta} - i\psi\sigma^m\bar{\rho} + i\rho\sigma^m\bar{\psi} \equiv \omega_m. \tag{25}$$

The canonical Hamiltonian H_0 is obtained as

$$H_0 = \frac{1}{2}eP^2 - i\psi\sigma^m\bar{\rho}P_m + i\rho\sigma^m\bar{\psi}P_m. \tag{26}$$

The set of HJPDE's are:

$$H_0' = P_0 + \frac{1}{2}eP^2 - i\psi\sigma^m\bar{\rho}P_m + i\rho\sigma^m\bar{\psi}P_m, \tag{27}$$

$$H'_{\theta} = P_{\theta} + P_m \sigma^m \bar{\theta} + \rho^{\alpha}, \tag{28}$$

$$H'_{\bar{\theta}} = P_{\bar{\theta}} - i\theta\sigma^m P_m + \rho_{\dot{\alpha}}, \tag{29}$$

$$H'_{\psi} = P_{\psi}, \tag{30}$$

$$H'_{\bar{\psi}} = P_{\bar{\psi}}, \tag{31}$$

$$H'_{\rho} = P_{\rho}, \tag{32}$$

$$H'_{\bar{\rho}} = P_{\bar{\rho}}, \tag{33}$$

$$H'_{e} = P_{e}. \tag{34}$$

Therefore, the total differential equations for the characteristics equations (10, 11) read as

$$dx^m = (eP_m - i\psi\sigma^m \bar{\rho} + i\rho\sigma^m \bar{\psi})dt + i\sigma^m \bar{\theta} d\theta - i\theta\sigma^m d\bar{\theta}, \tag{35}$$

$$dP_m = 0, \tag{36}$$

$$dP_{\theta} = (i\sigma^m P_m)d\bar{\theta}, \tag{37}$$

$$dP_{\bar{\theta}} = (-iP_m\sigma^m)d\theta, \tag{38}$$

$$dP_{\psi} = (iP_m\sigma^m \bar{\rho})dt, \tag{39}$$

$$dP_{\bar{\psi}} = (-i\rho\sigma^m P_m)dt, \tag{40}$$

$$dP_{\rho} = (-i\sigma^m \bar{\psi} P_m)dt - d\theta, \tag{41}$$

$$dP_{\bar{\rho}} = (i\psi\sigma^m P_m)dt - d\bar{\theta}, \tag{42}$$

$$dP_e = (\frac{1}{2}P^2)dt. \tag{43}$$

To check whether the set of eqns. (35) to (43) are integrable or not, let us consider the total variations of the set of HJPDE's. The variation of

$$dH'_0 = 0, \tag{44}$$

$$dH'_{\theta} = 0, \tag{45}$$

$$dH'_{\bar{\theta}} = 0. \tag{46}$$

are identically zero, whereas the variation of

$$dH'_{\psi} = (iP_m\sigma^m\bar{\rho})dt = H''_{\psi}dt, \tag{47}$$

$$dH'_{\bar{\psi}} = (-i\rho\sigma^m P_m)dt = H''_{\bar{\psi}}dt, \tag{48}$$

$$dH'_{\rho} = (-i\sigma^m\bar{\psi}P_m)dt - d\theta = H''_{\rho}dt, \tag{49}$$

$$dH'_{\bar{\rho}} = (i\psi\sigma^m P_m)dt - d\bar{\theta} = H''_{\bar{\rho}}dt, \tag{50}$$

$$dH'_{e} = (\frac{1}{2}P^2)dt = H''_{e}dt \tag{51}$$

are not. Therefore we obtain the following set of additional constraints:

$$H''_{\psi} = iP_m\sigma^m\bar{\rho}, \tag{52}$$

$$H''_{\bar{\psi}} = -i\rho\sigma^m P_m, \tag{53}$$

$$H''_{\rho} = -i\sigma^m\bar{\psi}P_m, \tag{54}$$

$$H''_{\bar{\rho}} = i\psi\sigma^m P_m, \tag{55}$$

$$H''_{e} = \frac{1}{2}P^2. \tag{56}$$

One notices that the total differential of H''_{ψ}, $H''_{\bar{\psi}}$, H''_{ρ}, $H''_{\bar{\rho}}$ and H''_{e} vanish identically, i.e.

$$dH''_{\psi} = 0, \tag{57}$$

$$dH''_{\bar{\psi}} = 0, \tag{58}$$

$$dH''_{\rho} = 0, \tag{59}$$

$$dH''_{\bar{\rho}} = 0, \tag{60}$$

$$dH''_{e} = 0. \tag{61}$$

Thus the equations of motion (35–43) and the new constraints (52–56) represent an integrable system.

4. Conclusion

In this paper we studied Hamilton-Jacobi formulation of constrained systems. In this formalism we have no first or second class constraints as in the Dirac's classification at the classical level; but the constraints arises naturally in the set of Hamilton-Jacobi Equations. In this approach, the equations of motion are obtained as total differential equations in many variables. If the system is integrable, then we can obtain the canonical Hamiltonian in terms of the canonical physical variables without using any gauge fixing condition. Hamilton-Jacobi Formulation was applied to Siegel superparticle moving in $R^{4|4}$ flat superspace. The equations of motion are obtained as total differential equations in many variables; and as the integrability conditions dH'_{α}'s are satisfied, then the system is integrable. Our result are in agreement with those given in Dirac's method [13].

References

[1] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, N.Y. 1964).

[2] K. Sundermeyer, *Constrained Dynamics*, Lect. Notes Phys.,Vol. 169 (Springer, Verlag. 1982).

[3] D. M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints*(Springer, Verlag. 1990).

[4] Y. Güler, *Nuovo Cimento*, **B107**, (1992), 1389.

[5] Y. Güler, *Nuovo Cimento*, **B107**, (1992), 1143.

[6] S. Muslih, N. I. Farahat, and M. Heles, *Nuovo Cimento*, **B119**, (2004), 531.

[7] S. I. Muslih, *Nuovo Cimento*, **B117**, (2002), 383.

[8] N. I. Farahat, and Y. Güler, *Nuovo Cimento*, **B111**, (1996), 513.

[9] S. I. Muslih, *General Relativity and Gravitation*, **34**, (2002), 1059.

[10] W. Siegel, *Phys. Lett.*, **B128**, (1983), 397.

[11] S. I. Muslih, *il Nuovo Cimento*, **B188**, (2003), 505.

[12] S. I. Muslih, *Modern Physics Letters A*, **19**,(2004), 863.

[13] A. V. Galajinsky, and D. M. Gitman, *Nucl. Phys. B,* **536**, (1998), 435.