Turk J Phys
30 (2006) , 379 – 390.

The Symmetric Teleparallel Gravity

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Received 14.04.2006

Abstract

We study symmetric teleparallel (STP) gravity model, in which only spacetime non-metricity is nonzero. First we obtain STP equivalent Einstein-Hilbert Lagrangian and give an approach for a generic solution in terms of only metric tensor. Then we obtain a spherically symmetric static solution to the Einstein's equation in STP space-time and discuss the singularities. Finally, we study a model given by a Lagrangian 4-form quadratic in non-metricity. Thus, we seek Schwarzschild-type solutions because of its observational success and obtain some sets of solutions. Finally, we discuss physical relevance of the solutions.

Key Words: Non-Riemannian geometry, Non-metricity, Teleparallel gravity.

1. Introduction

It can be thought that Einstein's general relativity (GR) is one of the biggest achievements of the last century. This work, first time, formulated a comprehensive theory containing gravity and matter that gave rise to a new understanding of universe Some deficiencies, however, appeared in Einstein's approach in last decades and people started to investigate whether GR was a unique and basic theory that explains exactly the gravitational interactions. These matters come from basically cosmology and the quantum field theory. In the former; the standard cosmology model based on GR and the standard model of particle physics is inadequate for explaining the universe at limit zones because of the existence of the bing bang singularity, flatness and horizon problems. On the other hand, if someone wants to achieve the quantum explanation of the space-time (or gravitation), it is realized that GR is a *classical* theory. Because of these realities and the absence of a definite quantum gravity theory, efforts of finding an alternative gravity theory are continued.

One of the most efficient approaches is the non-Riemannian formulation of gravity (see [1] and references therein), but little evidence for physical relevance of additional fields. In non-Riemannian gravity models metric, co-frame and full connection are considered as gauge potentials. The corresponding field strengths are the non-metricity $Q^a{}_b$, the torsion T^a and the curvature $R^a{}_b$. Because of the lack of experimental results for $Q^a{}_b$ and T^a, in general, the non-Riemannian gravity models are studied theoretically. Classification of the space-time and related theories are given summarily in table 1.

As seen from the table there is nearly no work on the symmetric teleparallel gravity (STPG). This work aims to fill this gap. Due to the fact that curvature and torsion vanish, it is usually asserted that this model is a gravitational field theory that is closest possible to flat space-time. Here we adhere to the following conventions: $\alpha,\ \beta,\ \cdots = \hat{0}, \hat{1}, \hat{2}, \hat{3}$ are holonomic or coordinate indices and $a,\ b,\ \cdots = 0, 1, 2, 3$

Table 1. Classification of space-times

Spacetime	Physical Theory	Literature
$Q^a{}_b = 0$, $T^a = 0$, $R^a{}_b = 0$ *Minkowski*	Special Relativity	A. Einstein (1905) and many people
$Q^a{}_b = 0$, $T^a = 0$, $R^a{}_b \neq 0$ *(Pseudo-)Riemannian*	General Relativity	A. Einstein (1916) and many people
$Q^a{}_b = 0$, $T^a \neq 0$, $R^a{}_b = 0$ *Weitzenböck*	Teleparallel Gravity	K. Hayashi & T. Nakano (1967) and many people [2]-[6]
$Q^a{}_b \neq 0$, $T^a = 0$, $R^a{}_b = 0$ *???????*	Symmetric Teleparallel Gravity	J..M. Nester & H.J.Yo (1999) and few people [7]-[9]
$Q^a{}_b \neq 0$, $T^a = 0$, $R^a{}_b \neq 0$ *Riemann-Weyl*	Einstein-Weyl	H. Weyl (1919) and some people [10]-[11]
$Q^a{}_b = 0$, $T^a \neq 0$, $R^a{}_b \neq 0$ *Riemann-Cartan*	Einstein-Cartan	A. Trautman (1972) and many people [12]-[13]
$Q^a{}_b \neq 0$, $T^a \neq 0$, $R^a{}_b = 0$ *???????*	???????	???????
$Q^a{}_b \neq 0$, $T^a \neq 0$, $R^a{}_b \neq 0$ *Non-Riemannian*	Einstein-Cartan-Weyl	A few [14]-[19]

are anholonomic or frame indices. Vierbein (tetrad) $h^a{}_\alpha$ and its inverse $h^\alpha{}_a$ (i.e. $h^a{}_\alpha h^\alpha{}_b = \delta^a_b$) give transformations between them. Abbreviations $e^{ab\cdots} = e^a \wedge e^b \wedge \cdots$ and $(ab) = \frac{1}{2}(a+b)$ and $[ab] = \frac{1}{2}(a-b)$ are used.

2. Mathematical Preliminaries

The triple $\{M, g, \nabla\}$ denotes the space-time where M is a 4-dimensional differentiable manifold, g is a non-degenerate Lorentzian metric and ∇ is a linear connection. g can be written in terms of the co-frame 1-forms

$$g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta = \eta_{ab} e^a \otimes e^b \tag{1}$$

where $\{e^a\}$ orthonormal and $\{dx^\alpha\}$ co-ordinate co-frame 1$-$forms, and $\eta_{ab} = (-,+,+,+)$ orthonormal and $g_{\alpha\beta}$ co-ordinate components of the metric. Orthonormal co-frame is dual to orthonormal frame $e^b(X_a) = \imath_a e^b = \delta^b_a$. Similarly, $dx^\beta(\partial_\alpha) = \imath_\alpha dx^\beta = \delta^\beta_\alpha$. Here $\imath$ denotes the interior product operator mapping any p-form into $(p-1)$-form. Besides, we set space-time orientation by $\epsilon_{0123} = +1$ or $*1 = e^{0123}$ where $*$ is the Hodge star operator mapping any p-form into $(4-p)$-form. Finally, the connection is specified by a set of connection 1-forms $\{\Lambda^a{}_b\}$. In the gauge approach to gravity, η_{ab}, e^a, $\Lambda^a{}_b$ are interpreted as the generalized gauge potentials, and the corresponding field strengths; the non-metricity 1-forms, torsion 2-forms and curvature 2-forms are defined through the Cartan structure equations; table 2.

Table 2. Gauge potentials and field strengths

	Gauge Potential	Field Strength	
η_{ab}	*o.n. metric*	$Q_{ab} := -\frac{1}{2}D\eta_{ab} = \frac{1}{2}(\Lambda_{ab} + \Lambda_{ba})$	*Nonmetricity* 1$-$*form*
e^a	*o.n. co-frame*	$T^a := De^a = de^a + \Lambda^a{}_b \wedge e^b$	*Torsion* 2$-$*form*
$\Lambda^a{}_b$	*Full connection*	$R^a{}_b := D\Lambda^a{}_b := d\Lambda^a{}_b + \Lambda^a{}_c \wedge \Lambda^c{}_b$	*Curvature* 2$-$*form*

Here d is the exterior derivative mapping any p-form into $(p+1)$-form. These field strengths satisfy the Bianchi identities; table 3.

Table 3. Bianchi identities

$DQ_{ab} = \frac{1}{2}(R_{ab} + R_{ba})$	*Zeroth*
$DT^a = R^a{}_b \wedge e^b$	*First*
$DR^a{}_b = 0$	*Second*

We also need the identities *

$$\begin{aligned} D * e_a &= -Q \wedge *e_a + T^b \wedge *e_{ab} && (2) \\ D * e_{ab} &= -Q \wedge *e_{ab} + T^c \wedge *e_{abc} && (3) \\ D * e_{abc} &= -Q \wedge *e_{abc} + T^d \wedge *e_{abcd} && (4) \\ D * e_{abcd} &= -Q \wedge *e_{abcd} && (5) \end{aligned}$$

where with $Q = \Lambda^a{}_a = Q^a{}_a$ Weyl 1-form. The full connection 1-forms are decomposed uniquely as follows [14]-[16]:

$$\Lambda^a{}_b = \omega^a{}_b + K^a{}_b + q^a{}_b + Q^a{}_b \tag{6}$$

where $\omega^a{}_b$ are the Levi-Civita connection 1-forms

$$\omega^a{}_b \wedge e^b = -de^a \qquad \text{or} \qquad 2\omega_{ab} = -\imath_a(de_b) + \imath_b(de_a) + [\imath_a\imath_b(de_c)]e^c \ , \tag{7}$$

$K^a{}_b$ are the contortion 1-forms

$$K^a{}_b \wedge e^b = T^a \qquad \text{or} \qquad 2K_{ab} = \imath_a T_b - \imath_b T_a - (\imath_a\imath_b T_c)e^c \ , \tag{8}$$

and $q^a{}_b$ anti-symmetric tensor 1-forms

$$q_{ab} = -(\imath_a Q_{bc}) \wedge e^c + (\imath_b Q_{ac}) \wedge e^c \ . \tag{9}$$

In this decomposition the symmetric part

$$\Lambda_{(ab)} = Q_{ab} \tag{10}$$

while the anti-symmetric part

$$\Lambda_{[ab]} = \omega_{ab} + K_{ab} + q_{ab} \ . \tag{11}$$

In gravity models it is complicated to keep all the components of $Q^a{}_b$. Therefore, people sometimes deal only with certain irreducible parts of that. To obtain the irreducible decompositions of non-metricity invariant under the Lorentz group, firstly we write

$$Q_{ab} = \underbrace{\overline{Q}_{ab}}_{trace-free\ \ part} + \underbrace{\frac{1}{4}\eta_{ab}Q}_{trace\ \ part} \tag{12}$$

where Weyl 1-form $Q = Q^a{}_a$ and $\eta^{ab}\overline{Q}_{ab} = 0$. Now we sum up the components

$$Q_{ab} = {}^{(1)}Q_{ab} + {}^{(2)}Q_{ab} + {}^{(3)}Q_{ab} + {}^{(4)}Q_{ab} \tag{13}$$

*Since $Q^{ab} = \frac{1}{2}D\eta^{ab} \neq 0$ special attention in lowering and raising index in front of covariant exterior derivative.

in terms of

$$
\begin{aligned}
{}^{(2)}Q_{ab} &= \frac{1}{3} * (e_a \wedge \Omega_b + e_b \wedge \Omega_a) , && (14)\\
{}^{(3)}Q_{ab} &= \frac{2}{9}(\Lambda_a e_b + \Lambda_b e_a - \frac{1}{2}\eta_{ab}\Lambda) , && (15)\\
{}^{(4)}Q_{ab} &= \frac{1}{4}\eta_{ab} Q , && (16)\\
{}^{(1)}Q_{ab} &= Q_{ab} - {}^{(2)}Q_{ab} - {}^{(3)}Q_{ab} - {}^{(4)}Q_{ab} && (17)
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_b &:= \imath_a \overline{Q}^a{}_b \ , \qquad \Lambda := \Lambda_a e^a ,\\
\Theta_b &:= * (\overline{Q}_{ab} \wedge e^a) \ , \quad \Theta := e^b \wedge \Theta_b \ , \quad \Omega_a := \Theta_a - \frac{1}{3}\imath_a \Theta . \qquad (18)
\end{aligned}
$$

The components have properties

$$
\begin{aligned}
\eta_{ab}{}^{(1)}Q^{ab} = \eta_{ab}{}^{(2)}Q^{ab} = \eta_{ab}{}^{(3)}Q^{ab} = 0 ,\\
\imath_a{}^{(1)}Q^{ab} = \imath_a{}^{(2)}Q^{ab} = 0 ,\\
e_a \wedge {}^{(1)}Q^{ab} = 0 ,\\
\imath_{(a}{}^{(2)}Q_{bc)} = 0 . \qquad (19)
\end{aligned}
$$

Thus the components are orthogonal in the following sense

$$
{}^{(i)}Q^{ab} \wedge *{}^{(j)}Q_{ab} = \delta^{ij} N_{ij} \qquad \text{(no summation over } ij) \qquad (20)
$$

where δ^{ij} is the Kronecker symbol and N_{ij} any 4-form. Then

$$
\begin{aligned}
{}^{(1)}Q^{ab} \wedge *{}^{(1)}Q_{ab} &= Q^{ab} \wedge *Q_{ab} - {}^{(2)}Q^{ab} \wedge *{}^{(2)}Q_{ab} - {}^{(3)}Q^{ab} \wedge *{}^{(3)}Q_{ab}\\
&\quad - {}^{(4)}Q^{ab} \wedge *{}^{(4)}Q_{ab} , && (21)\\
{}^{(2)}Q^{ab} \wedge *{}^{(2)}Q_{ab} &= \frac{2}{3}(Q_{ac} \wedge e^a) \wedge *(Q^{bc} \wedge e_b) - \frac{2}{9}(\imath^a Q_{ac})(\imath_b Q^{bc}) * 1 - \frac{2}{9} Q \wedge *Q\\
&\quad + \frac{4}{9}(\imath_a Q)(\imath_b Q^{ab}) * 1 , && (22)\\
{}^{(3)}Q^{ab} \wedge *{}^{(3)}Q_{ab} &= \frac{4}{9}(\imath^a Q_{ac})(\imath_b Q^{bc}) * 1 + \frac{1}{36} Q \wedge *Q - \frac{2}{9}(\imath_a Q)(\imath_b Q^{ab}) * 1 , && (23)\\
{}^{(4)}Q^{ab} \wedge *{}^{(4)}Q_{ab} &= \frac{1}{4} Q \wedge *Q . && (24)
\end{aligned}
$$

3. Symmetric Teleparallel Gravity

STP space-time is defined as $Q^a{}_b \neq 0$, $T^a = 0$, $R^a{}_b = 0$. One generic solution to that is obtained in the coordinate frame; $\Lambda^\alpha{}_\beta = 0$, the so-called *natural or inertial gauge*:

$$
R^\alpha{}_\beta = d\Lambda^\alpha{}_\beta + \Lambda^\alpha{}_\gamma \wedge \Lambda^\gamma{}_\beta = 0 , \qquad (25)
$$

$$
T^\alpha = d(dx^\alpha) + \Lambda^\alpha{}_\beta \wedge dx^\beta = 0 , \qquad (26)
$$

$$
Q_{\alpha\beta} = -\frac{1}{2} D g_{\alpha\beta} = -\frac{1}{2} d g_{\alpha\beta} \neq 0 . \qquad (27)
$$

After a frame transformation via vierbein $e^a = h^a{}_\alpha dx^\alpha$ and $\Lambda^a{}_b = h^a{}_\alpha \Lambda^\alpha{}_\beta h^\beta{}_b + h^a{}_\alpha dh^\alpha{}_b$ we obtain the field strengths in orthonormal components

$$
R^a{}_b = h^a{}_\alpha R^\alpha{}_\beta h^\beta{}_b = 0 , \qquad (28)
$$

$$
T^a = h^a{}_\alpha T^\alpha = 0 , \qquad (29)
$$

$$
Q_{ab} = Q_{\alpha\beta} h^\alpha{}_a h^\beta{}_b \neq 0 . \qquad (30)
$$

3.1. The Einstein-Hilbert Lagrangian in STPG

The orthonormal teleparallel representation of the Einstein's theory is interesting and useful. Therefore, we derive the Einstein-Hilbert Lagrangian 4-form. Firstly we use the decomposition of the full connection (6), with $K^a{}_b = 0$,

$$\Lambda^a{}_b = \omega^a{}_b + \Omega^a{}_b \quad \text{where} \quad \Omega^a{}_b = Q^a{}_b + q^a{}_b \, . \tag{31}$$

By substituting that into $R^a{}_b(\Lambda)$ we decompose the non-Riemannian curvature as the follows:

$$\begin{aligned} R^a{}_b(\Lambda) &= d\Lambda^a{}_b + \Lambda^a{}_c \wedge \Lambda^c{}_b \\ &= R^a{}_b(\omega) + D(\omega)\Omega^a{}_b + \Omega^a{}_c \wedge \Omega^c{}_b \end{aligned} \tag{32}$$

where $R^a{}_b(\omega)$ is the Riemannian curvature 2-form and $D(\omega)$ is the covariant exterior derivative with respect to the Levi-Civita connection. To set $R^a{}_b(\Lambda) = 0$ for STP space-time yields the Einstein-Hilbert Lagrangian 4-form

$$\begin{aligned} L_{EH} &= R^a{}_b(\omega) \wedge *e_a{}^b \\ &= -[D(\omega)\Omega^a{}_b] \wedge *e_a{}^b - \Omega^a{}_c \wedge \Omega^c{}_b \wedge *e_a{}^b \end{aligned} \tag{33}$$

Here after using the equality

$$d(\Omega^a{}_b \wedge *e_a{}^b) = [D(\omega)\Omega^a{}_b] \wedge *e_a{}^b - \Omega^a{}_b \wedge [D(\omega) * e_a{}^b] \tag{34}$$

we discard the exact form and we notice that $D(\omega) * e_a{}^b = 0$ because T^a and $Q^a{}_b$ vanish for $\omega^a{}_b$ (see eq.(3)). Thus

$$\begin{aligned} L_{EH} &= \frac{1}{2\kappa}\Omega^a{}_c \wedge \Omega^c{}_b \wedge *e_a{}^b \\ &= \frac{1}{2\kappa}(Q^{ac} + q^{ac}) \wedge (Q_{cb} + q_{cb}) \wedge *e_a{}^b \\ &= \frac{1}{2\kappa}(Q^{ac} \wedge Q_{cb} + q^{ac} \wedge q_{cb}) \wedge *e_a{}^b \\ &= \frac{1}{2\kappa}[-Q_{ab} \wedge *Q^{ab} + 2(Q_{ac} \wedge e^a) \wedge *(Q^{bc} \wedge e_b) - Q \wedge *Q + 2(\imath_b Q)(\imath_a Q^{ab}) * 1] \end{aligned} \tag{35}$$

where κ is gravitational coupling constant. In two-dimension we note $q^{ac} \wedge q_{cb} \wedge *e_a{}^b = 0$.

3.2. A Symmetric Teleparallel Solution to the Einstein Equation

Now we give a brief outline of GR. GR is written in (pseudo-) Riemannian spacetime in which torsion and non-metricity are both zero, i.e., connection is Levi-Civita. Einstein equation can be written in the following form

$$G_a := -\frac{1}{2}R^{bc}(\omega) \wedge * e_{abc} = \kappa\tau_a \tag{36}$$

or alternatively

$$* \, G_a := (\mathrm{Ric})_a - \frac{1}{2}\mathcal{R}e_a = \kappa * \tau_a \tag{37}$$

where G_a is Einstein tensor 3-form, $R^{ab}(\omega)$ is Riemannian curvature 2-form, $(\mathrm{Ric})^a = \imath^b R^a{}_b(\omega)$ is Ricci curvature 1-form, $\mathcal{R} = \imath_a(\mathrm{Ric})^a$ is scalar curvature, τ_a is the energy-momentum 3-form. For the symmetric teleparallel equivalent of Einstein equation we use the the decomposition of the non-Riemannian curvature 2-form (32) and set $R^a{}_b(\Lambda) = 0$. Thus we obtain the symmetric teleparallel equivalent of (36)

$$G_a := \frac{1}{2}[D(\omega)q^{bc} + q^b{}_k \wedge q^{kc} + Q^b{}_k \wedge Q^{kc}] \wedge *e_{abc} = \kappa\tau_a \ . \tag{38}$$

We now proceed the attempt for finding a solution to the STPG model. As usual in the study of exact solutions, we have two steps. The first one is to choose the convenient local coordinates and make corresponding ansatz for the dynamical fields. The second step concerns providing the invariants of the resulting geometry. While the choice of an ansatz helps to solve the field equations easily, the invariant description provides the correct understanding of the physical contents of a solution.

Since metric and connection are independent quantities in non-Riemannian spacetimes, we have to predict separately appropriate candidates for them. Therefore we first write a line element in order to determine the metric. We naturally start dealing with the case of spherical symmetry for realistic simplicity,

$$g = -f^2 dt^2 + g^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{39}$$

where $f = f(r)$ and $g = g(r)$. A convenient choice for a tetrad reads

$$e^0 = f dt, \quad e^1 = g dr, \quad e^2 = r d\theta, \quad e^3 = r\sin\theta d\varphi \ . \tag{40}$$

In addition, for the non-Riemannian connection we choose

$$\begin{aligned}
\Lambda_{12} &= -\Lambda_{21} = -\frac{1}{r}e^2, \quad \Lambda_{13} = -\Lambda_{31} = -\frac{1}{r}e^3, \quad \Lambda_{23} = -\Lambda_{32} = -\frac{\cot\theta}{r}e^3, \\
\Lambda_{00} &= \frac{f'}{fg}e^1, \quad \Lambda_{11} = \frac{1}{r}(1-\frac{1}{g})e^1, \quad \Lambda_{22} = \frac{1}{r}(1-\frac{1}{g})e^1, \\
\Lambda_{33} &= \frac{1}{r}(1-\frac{1}{g})e^1, \quad \text{others} = 0
\end{aligned} \tag{41}$$

where prime denotes derivative with respect to r. These gauge configurations (40) and (41) satisfy the constraint equations $R^a{}_b(\Lambda) = 0$, $\quad T^a(\Lambda) = 0$. One can certainly perform a locally Lorentz transformation

$$e^a \to L^a{}_b e^b \quad , \quad \Lambda^a{}_b \to L^a{}_c \Lambda^c{}_d L^{-1}{}^d{}_b + L^a{}_c dL^{-1}{}^c{}_b \tag{42}$$

which yields the Minkowski gauge $\Lambda^a{}_b = 0$. This may mean that we propose a set of connection components in a special frame and coordinate which seems contrary to the spirit of relativity theory. However in physically natural situations we can choose a reference and coordinate system at our best convenience.

We deduce from equations (40)-(41)

$$\begin{aligned}
\omega_{01} &= -\frac{f'}{fg}e^0, \quad \omega_{12} = -\frac{1}{rg}e^2, \quad \omega_{13} = -\frac{1}{rg}e^3, \quad \omega_{23} = -\frac{\cot\theta}{r}e^3 \ , \\
Q_{00} &= \frac{f'}{fg}e^1, \quad Q_{11} = \frac{1}{r}(1-\frac{1}{g})e^1, \quad Q_{22} = \frac{1}{r}(1-\frac{1}{g})e^1, \quad Q_{33} = \frac{1}{r}(1-\frac{1}{g})e^1 \ , \\
q_{01} &= \frac{f'}{fg}e^0, \quad q_{12} = \frac{1}{r}(\frac{1}{g}-1)e^2, \quad q_{13} = \frac{1}{r}(\frac{1}{g}-1)e^3, \quad \text{others} = 0 \ .
\end{aligned} \tag{43}$$

When we put (43) into (38) we obtain, with $\tau_a = 0$

$$\left(dq^{bc} + 2\omega^b{}_f \wedge q^{fc} + q^b{}_f \wedge q^{fc}\right) \wedge *e_{abc} = 0 \tag{44}$$

whose components read explicitly

$$Zeroth\ \ component \qquad \left[\frac{2(g^{-1})'}{rg} - \frac{g^2-1}{r^2g^2}\right] e^{123} = 0 \tag{45}$$

$$First\ \ component \qquad -\left[\frac{2f'}{rfg^2} - \frac{g^2-1}{r^2g^2}\right] e^{023} = 0 \tag{46}$$

$$Second\ \ component \qquad \left[\frac{(f'g^{-1})'}{fg} + \frac{f'}{rfg^2} + \frac{(g^{-1})'}{rg}\right] e^{013} = 0 \tag{47}$$

$$Third\ \ component \qquad -\left[\frac{(f'g^{-1})'}{FG} + \frac{f'}{rfg^2} + \frac{(g^{-1})'}{rg}\right] e^{012} = 0\ . \tag{48}$$

Then from (45) and (46)

$$g(r) = 1/f(r) \tag{49}$$

and from (47) and (48)

$$f^2(r) = 1 - \frac{C}{r} \tag{50}$$

where C is a constant.

In order to have a correct understanding of the resulting solution, we need to construct invariants of the *Riemannian* curvature and the non-metricity. Although the total curvature is identically zero in the teleparallel gravity, the Riemannian curvature of the Levi-Civita connection is nontrivial:

$$\begin{gathered} R^{01}(\omega) = \frac{(f'g^{-1})'}{fg}e^{10} \quad , \quad R^{02}(\omega) = \frac{f'}{rfg^2}e^{20} \quad , \quad R^{03}(\omega) = \frac{f'}{rfg^2}e^{30} \quad , \\ R^{12}(\omega) = \frac{(g^{-1})'}{rg}e^{21} \quad , \quad R^{13}(\omega) = \frac{(g^{-1})'}{rg}e^{31} \quad , \quad R^{23}(\omega) = \frac{1}{r^2}(1-\frac{1}{g^2})e^{32}\ . \end{gathered} \tag{51}$$

Thus the quadratic invariant of the Riemannian curvature reads

$$\begin{aligned} R_{ab}(\omega) \wedge *R^{ab}(\omega) &= \left\{2\left[\frac{(f'g^{-1})'}{fg}\right]^2 + 4\left(\frac{f'}{rfg^2}\right)^2 + 4\left[\frac{(g^{-1})'}{rg}\right]^2 + 2\left[\frac{1}{r^2}\left(1-\frac{1}{g^2}\right)\right]^2\right\} *1 \\ &= \frac{6C^2}{r^6} *1 \end{aligned} \tag{52}$$

and the spacetime geometry is naturally characterized by the quadratic invariant of the nonmetricity

$$\begin{aligned} Q_{ab} \wedge *Q^{ab} &= \left\{\left(\frac{f'}{fg}\right)^2 + 3\left[\frac{1}{r}\left(1-\frac{1}{g}\right)\right]^2\right\} *1 \\ &= \left\{\frac{C^2}{4r^3(r-C)} - \frac{3C}{r^3} + \frac{6}{r^2}\left[1-\left(1-\frac{C}{r}\right)^{1/2}\right]\right\} *1\ . \end{aligned} \tag{53}$$

These two quadratic invariants provide the sufficient tools for understanding the contents of the classical solutions. Important observation is that the Riemannian curvature invariant (52) is singular at $r=0$, but regular at the zero ($r=C$) of the metric function $f(r)$, which means that we have a horizon here. The resulting geometry then describes the well known Schwarzschild black hole at $r=0$ with the horizon at $r=C$. Since we are dealing with symmetric teleparallel gravity, it is necessary also to analyze the behavior of nonmetricity As seen from (53), the nonmetricity invariant diverges not only at the origin $r=0$, but also at the Schwarzschild horizon $r=C$ The horizon is a regular surface from the viewpoint of the Riemannian geometry, but it is singular from the viewpoint of symmetric teleparallel gravity.

4. Lagrange Formulation of STPG

We formulate STPG in terms of a Lagrangian 4-form

$$\mathcal{L} = L + \lambda_a \wedge T^a + R^a{}_b \wedge \rho_a{}^b \tag{54}$$

where $\rho_a{}^b$ and λ_a are the Lagrange multiplier 2-forms giving the constraints

$$R^a{}_b = 0 \quad , \qquad T^a = 0 \; . \tag{55}$$

$\mathcal{L}$ changes by a closed form under the transformations

$$\lambda_a \quad \rightarrow \quad \lambda_a + D\mu_a \; , \tag{56}$$
$$\rho_a{}^b \quad \rightarrow \quad \rho_a{}^b + D\xi_a{}^b - \mu_a \wedge e^b \tag{57}$$

of the Lagrange multiplier fields. Here μ_a and $\xi_a{}^b$ are arbitrary 1-forms. To show this invariance we use the Bianchi identities and discard exact forms Consequently the field equations derived from the Lagrangian 4-form (54) will determine the Lagrange multipliers only up to above transformations. The gravitational field equations are derived from (54) by independent variations with respect to the connection $\{\Lambda^a{}_b\}$ and the ortohonormal co-frame $\{e^a\}$ 1-forms, respectively:

$$\lambda_a \wedge e^b + D\rho_a{}^b \quad = \quad -\Sigma_a{}^b \; , \tag{58}$$
$$D\lambda_a \quad = \quad -\tau_a \tag{59}$$

where $\Sigma_a{}^b = \frac{\partial L}{\partial \Lambda^a{}_b}$ and $\tau_a = \frac{\partial L}{\partial e^a}$. In principle the first field equation (58) is used to solve for the Lagrange multipliers λ_a and $\rho_a{}^b$ and the second field equation (59) governs the dynamics of the gravitational fields. Here the first equation, however, has 64 and the second one has 16 independent components, thus giving the total number of independent equations 80. On the other hand, there are totally 120 unknowns: 24 for λ_a plus 96 for $\rho_a{}^b$. But we note that the left-hand side of (58) is invariant under the transformations (56)-(57) and consequently it is sufficient to determine the gauge invariant piece of the Lagrange multipliers, namely $\lambda_a \wedge e^b + D\rho_a{}^b$, in terms of $\Sigma_a{}^b$. One can consult Ref.[20] for further discussions on gauge symmetries of Lagrange multipliers. It is important to notice $D\lambda_a$ rather than the Lagrange multipliers themselves couple to the second field equations (59). As a result we must calculate $D\lambda_a$ directly and we can manage that by taking the set of covariant exterior derivative of (58):

$$D\lambda_a \wedge e^b = -D\Sigma_a{}^b \; . \tag{60}$$

Here we used the constraints

$$De^b \quad = \quad T^b = 0, \tag{61}$$
$$D^2\rho_a{}^b \quad = \quad D(D\rho_a{}^b) = R^b{}_c \wedge \rho_a{}^c - R^c{}_a \wedge \rho_c{}^b = 0 \tag{62}$$

where the covariant exterior derivative of a $(1,1)$-type tensor is

$$D\rho_a{}^b = d\rho_a{}^b + \Lambda^b{}_c \wedge \rho_a{}^c - \Lambda^c{}_a \wedge \rho_c{}^b \; . \tag{63}$$

The result (60) is unique because $D\lambda_a \rightarrow D\lambda_a$ under (56). Thus we arrive at the field equation

$$D\Sigma_a{}^b - \tau_a \wedge e^b = 0 \; . \tag{64}$$

Now we write down the following Lagrangian 4-form which is the most general quadratic expression in the non-metricity tensor [21]:

$$L = \frac{1}{2\kappa}\left[k_0 R^a{}_b \wedge *e_a{}^b + \sum_{I=1}^{4} k_I {}^{(I)}Q_{ab} \wedge *{}^{(I)}Q^{ab} + k_5 \left({}^{(3)}Q_{ab} \wedge e^b\right) \wedge * \left({}^{(4)}Q^{ac} \wedge e_c\right)\right] \; . \tag{65}$$

Here $k_0, k_1, k_2, k_3, k_4, k_5$ are dimensionless coupling constants and $\kappa = \frac{8\pi G}{c^3}$, with G the Newton's gravitational constant. Inserting (21)-(24) into (65) we find

$$
\begin{aligned}
L = \frac{1}{2\kappa} \big[& k_0 R^a{}_b \wedge *e_a{}^b + c_1 Q_{ab} \wedge *Q^{ab} + c_2 (Q_{ac} \wedge e^a) \wedge *(Q^{bc} \wedge e_b) \\
& + c_3 (\imath_a Q^{ac})(\imath^b Q_{bc}) * 1 + c_4 Q \wedge *Q + c_5 (\imath_a Q)(\imath_b Q^{ab}) * 1 \big]
\end{aligned} \tag{66}
$$

where the new coefficients are the following combinations of the original coupling constants:

$$
\begin{aligned}
c_1 &= k_1 , \\
c_2 &= -\frac{2}{3}k_1 + \frac{2}{3}k_2 \; , \\
c_3 &= -\frac{2}{9}k_1 - \frac{2}{9}k_2 + \frac{4}{9}k_3 \; , \\
c_4 &= -\frac{1}{18}k_1 - \frac{2}{9}k_2 + \frac{1}{36}k_3 + \frac{1}{4}k_4 + \frac{1}{16}k_5 \; , \\
c_5 &= -\frac{2}{9}k_1 + \frac{4}{9}k_2 - \frac{2}{9}k_3 - \frac{1}{4}k_5 \; .
\end{aligned} \tag{67}
$$

We obtain the variational field equations from (66)

$$
\Sigma_a{}^b = \sum_{i=0}^{5} c_i \, {}^i\Sigma_a{}^b \quad , \qquad \tau_a = \sum_{i=0}^{5} c_i \, {}^i\tau_a \tag{68}
$$

where

$$
\begin{aligned}
{}^0\Sigma_a{}^b &= 2Q^{bc} \wedge *e_{ac} - Q \wedge *e_a{}^b + T_c \wedge *e_a{}^{bc} && (69) \\
{}^1\Sigma_a{}^b &= *(Q_a{}^b + Q^b{}_a) && (70) \\
{}^2\Sigma_a{}^b &= e_a \wedge *(Q^{bc} \wedge e_c) + e^b \wedge *(Q_{ac} \wedge e^c) && (71) \\
{}^3\Sigma_a{}^b &= \imath^c Q_{ac} * e^b + \imath_c Q^{bc} * e_a && (72) \\
{}^4\Sigma_a{}^b &= 2\delta_a^b * Q && (73) \\
{}^5\Sigma_a{}^b &= \frac{1}{2}(\imath^b Q) * e_a + \frac{1}{2}(\imath_a Q) * e^b + \delta_a^b (\imath_c Q^{cd}) * e_d && (74) \\
{}^0\tau_a &= R^b{}_c \wedge *e_{ab}{}^c && (75) \\
{}^1\tau_a &= -(\imath_a Q^{bc}) \wedge *Q_{bc} - Q^{bc} \wedge (\imath_a * Q_{bc}) && (76) \\
{}^2\tau_a &= -Q_{ab} \wedge *(Q^{bc} \wedge e_c) - (\imath_a Q^{cd}) e_c \wedge *(Q_{bd} \wedge e^b) + (Q_{bd} \wedge e^b) \wedge *(Q^{cd} \wedge e_{ca}) && (77) \\
{}^3\tau_a &= -2(\imath_a Q^{bd})(\imath^c Q_{cd}) * e_b + (\imath_b Q^{bd})(\imath^c Q_{cd}) * e_a && (78) \\
{}^4\tau_a &= -(\imath_a Q) * Q - Q \wedge (\imath_a * Q) && (79) \\
{}^5\tau_a &= (\imath_b Q)(\imath_c Q^{bc}) * e_a - (\imath_a Q)(\imath_c Q^{bc}) * e_b - (\imath_b Q)(\imath_a Q^{bc}) * e_c \; . && (80)
\end{aligned}
$$

One can consult Ref.[9] for the details of variations. Since ${}^0\tau_a = R^b{}_c \wedge *e_{ab}{}^c = 0$ and $D^0\Sigma_a{}^b = D^2 * e_a{}^b \sim R_a{}^b = 0$ we drop the Einstein-Hilbert term: $k_0 = 0$. The case that $k_0 \neq 0$ and others$= 0$ was discussed in the previous subsection.

4.1. Spherical symmetric solution to the model

Under the configuration (40)-(41) the only nontrivial field equation comes from the trace of (64):

$$
d\Sigma^a{}_a + e^a \wedge \tau_a = 0 \; . \tag{81}
$$

Symmetric and antisymmetric parts of the field equation (64) give trivially zero. From (81) we obtain

$$-\frac{\ell_1}{g}\left(\frac{f'}{fg}\right)' - \frac{\ell_2}{g}\left(\frac{1-g}{rg}\right)' + \ell_3\left(\frac{f'}{fg}\right)^2 + \ell_4\left(\frac{f'}{fg}\right)\left(\frac{1-g}{rg}\right) - \frac{2\ell_1}{rg}\left(\frac{f'}{fg}\right)$$
$$-\frac{2\ell_2}{rg}\left(\frac{1-g}{rg}\right) + \ell_5\left(\frac{1-g}{rg}\right)^2 = 0 \qquad (82)$$

where

$$\ell_1 = 2c_1 + 2c_2 + 8c_4 + c_5, \qquad (83)$$
$$\ell_2 = 6c_1 + 4c_2 + 2c_3 + 24c_4 + 7c_5, \qquad (84)$$
$$\ell_3 = -6c_4 - c_5, \qquad (85)$$
$$\ell_4 = -6c_1 - 4c_2 - 2c_3 - 12c_4 - 5c_5, \qquad (86)$$
$$\ell_5 = 6c_1 + 4c_2 + 2c_3 + 18c_4 + 6c_5\ . \qquad (87)$$

Mathematically, this equation has infinitely many solutions because there are two functions and only one equation. We give two classes of solutions. At first, let $f'/fg = u$, then (82) takes the form, if $\ell_3 \neq 0$,

$$-\left(\ell_1 u + \ell_2\frac{1-g}{rg}\right)' - \frac{2}{r}\left(\ell_1 u + \ell_2\frac{1-g}{rg}\right) + \frac{\ell_3 g}{\ell_1^2}\left(\ell_1^2 u^2 + \frac{\ell_1^2\ell_4}{\ell_3}u\frac{1-g}{rg} + \frac{\ell_1^2\ell_5}{\ell_3}\frac{(1-g)^2}{r^2g^2}\right) = 0\ . \qquad (88)$$

When let $\ell_4 = 2\ell_2\ell_3/\ell_1$ and $\ell_5 = 2\ell_3\ell_2^2/\ell_1^2$ if we define

$$z = \ell_1 u + \ell_2\frac{1-g}{rg} \qquad (89)$$

the equation becomes

$$-\frac{(r^2z)'}{(r^2z)^2} + \frac{\ell_3}{\ell_1^2}\frac{g}{r^2} = 0\ . \qquad (90)$$

This means that given g we obtain f. At the second class; $\ell_3 = 0$, $\ell_4 = \alpha\ell_1$, $\ell_5 = \alpha\ell_2$ with $\alpha \neq 0$ parameter, (82) turns out to be

$$-\frac{(r^2z)'}{r^2z} + \alpha\frac{1-g}{r} = 0\ . \qquad (91)$$

Again to specify g yields f. Physically, however, due to the observational success of the Schwarzschild solution of general relativity, we investigate solutions with $g = 1/f$. Then (82) becomes

$$-\ell_1 ff'' + \ell_3(f')^2 - (2\ell_1 + \ell_2 - \ell_4)\frac{ff'}{r} + (\ell_5 - \ell_2)\frac{f^2}{r^2} - \ell_4\frac{f'}{r} + (\ell_2 - 2\ell_5)\frac{f}{r^2} + \ell_5\frac{1}{r^2} = 0\ . \qquad (92)$$

We can not find an analytical exact solution of this nonlinear second order differential equation. Therefore, we treat the linear sector of the equation

$$r^2\left(f^2\right)'' + (2 + \frac{\ell_2}{\ell_1})r\left(f^2\right)' + \frac{\ell_2}{\ell_1}f^2 = \frac{\ell_2}{\ell_1} \qquad (93)$$

by choosing our parameters as follows;

$$\ell_3 = -\ell_1 \quad , \quad \ell_4 = 0 \quad , \quad \ell_5 = \frac{\ell_2}{2}\ . \qquad (94)$$

Here some special cases deserve attention.

1. For $\ell_2 = \ell_1$, we obtain the solution

$$f^2 = 1 + \frac{C_1}{r} + D_1 \frac{\ln r}{r} \tag{95}$$

which is asymptotically flat; $\lim_{r\to\infty} f = 1$. Here C_1 and D_1 are integration constants

2. For $\ell_2 \neq \ell_1$, the solution is found as

$$f^2 = 1 + \frac{C_2}{r} + \frac{D_2}{r^{\ell_2/\ell_1}} \tag{96}$$

where C_2 and D_2 are integration constants.

   (a) For $\ell_2 = 0$, we obtain a Schwarzschild-type solution with $D_2 = 0$ for asymptotically flatness and we identify the other constant with a spherically symmetric mass centered at the origin; $C_2 = -2M$.

   (b) For $\ell_2 = -2\ell_1$, we obtain a Schwarzschild-de Sitter-type solution. We again identify C_2 with mass $C_2 = -2M$ and D_2 with cosmological constant $D_2 = -\frac{1}{3}\Lambda$. The Λ term corresponds to a repulsive central force of magnitude $\frac{1}{3}\Lambda r$, which is independent of the central mass.

   (c) For $\ell_2 = 2\ell_1$, we obtain a Reissner-Nordström-type solution. We again identify C_2 with mass $C_2 = -2M$ while D_2 with a new kind of gravitational charge. We hope that besides ordinary matter that interacts gravitationally through its mass, the dark matter in the Universe may interact gravitationally through both its mass and this new gravitational charge.

5. Conclusion

In this paper we investigated the symmetric teleparallel gravity. After giving the irreducible decompositions of non-metricity under the Lorentz group we identified STPG theories and gave an approach for the generic solution: *natural or inertial gauge.* Then we obtained symmetric teleparallel equivalence of the Einstein-Hilbert Lagrangian and found a spherically symmetric static solution to Einstein's equation in STP geometry. We analyzed the singularity structure of the space-time according to that solution. The singularities need to be clarified in the non-Riemannian space-times. Finally, we studied the Lagrange formulation of the general STPG by considering a 5-parameter symmetric teleparallel Lagrangian without *a priori* restricting the coupling constants c_1, c_2, c_3, c_4, c_5. We obtained sets of solutions in which the Schwarzschild-type, Schwarzschild-de Sitter-type and Reissner-Nordström-type solutions are some physically interesting ones. Consequently, we suggest that in addition ordinary matter that interacts gravitationally through its mass, the dark matter in the Universe may interact gravitationally through both its mass and a new kind of gravitational charge [22]-[23]. The latter coupling is analogous to the coupling of electric charge to electromagnetic field where the analogue of the Maxwell field is the non-metricity field strength. That is, such "charges" may provide a source for the non-metricity. The novel gravitational interactions may have a significant influence on the structure of black holes For example, we may speculate that this unknown gravitational charge may have repulsive nature.

Acknowledgment

The author is grateful to Prof. Dr. Metin GÜRSES for his fruitful comments.

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