Weak Value and Correlation Function

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(Dated: January 27, 2009)

We show that there exists, in quantum theory, a close relationship between the weak value and the correlation function, which sheds new lights on the concept of the weak value.

In this report, we prove an equality which connects two fundamental concepts in quantum theory: the weak value and the correlation function. The weak value has been a topic of active researches [1, 2, 3, 4, 5, 6], and is related to the foundation of quantum mechanics [7]. On the other hand, the (symmetric) correlation function is defined as $\mathrm{Re}\langle\psi|\hat{B}(t_2)\hat{A}(t_1)|\psi\rangle$, where "Re" means "the real part", $|\psi\rangle$ is a state vector, and $\hat{A}(t_1)$ and $\hat{B}(t_2)$ are observables at time t_1 and t_2 in the Heisenberg picture. The correlation function is a useful quantity to characterize the dynamics of quantum systems, and has played an important role in quantum statistical mechanics such as the linear response theory [8]. For example, the correlation function is related to the susceptibility by the fluctuation-dissipation theorem.

To introduce the weak value, we consider two observables $\hat{A}$ and $\hat{B}$ of a quantum system. We denote the spectrum decompositions of them as $\hat{A} = \sum_a a|a\rangle\langle a|$ and $\hat{B} = \sum_b b|b\rangle\langle b|$, where a's (b's) are the eigenvalues of $\hat{A}$ ($\hat{B}$), and $|a\rangle$'s ($|b\rangle$'s) are the corresponding eigenvectors. Let $\hat{P}_a^A \equiv |a\rangle\langle a|$ and $\hat{P}_b^B \equiv |b\rangle\langle b|$ be the projection operators. We consider the unitary evolution of the system from time 0 to t_2, and suppose that the projection measurement of $\hat{B}$ is performed at time t_2, and the outcome b is obtained. Let $|\psi\rangle$ be the initial state, t_1 ($0 < t_1 < t_2$) be an intermediate time, and $\hat{U}_1$ ($\hat{U}_2$) be the unitary evolution from 0 to t_1 (t_1 to t_2). Then the definition of the (real) weak value of $\hat{A}$ at time t_1 with the post-selection $|b\rangle$ is given by

$$_b\langle A\rangle_{\mathrm{w}} \equiv \mathrm{Re}\frac{\langle b|\hat{U}_2\hat{A}\hat{U}_1|\psi\rangle}{\langle b|\hat{U}_2\hat{U}_1|\psi\rangle}. \tag{1}$$

To relate the weak value to the correlation function, we first define a quasi-probability distribution based on the correlation function:

$$\mathrm{Pr}_\psi(b,a) \equiv \mathrm{Re}\langle\psi|\hat{P}_b^B(t_2)\hat{P}_a^A(t_1)|\psi\rangle, \tag{2}$$

where $\hat{P}_b^B(t_2) \equiv \hat{U}_1^\dagger\hat{U}_2^\dagger\hat{P}_b^B\hat{U}_2\hat{U}_1$ and $\hat{P}_a^A(t_1) \equiv \hat{U}_1^\dagger\hat{P}_a^A\hat{U}_1$ are described in the Heisenberg picture. We stress that this quasi-probability distribution does not correspond to the probability distribution which we can ob-

tain by performing the successive projection measurements of $\hat{A}$ at t_1 and $\hat{B}$ at t_2. The quasi-probability $\mathrm{Pr}_\psi(b,a)$ satisfies that $\sum_a \mathrm{Pr}_\psi(b,a) = \langle\psi|\hat{P}_b^B(t_2)|\psi\rangle \equiv \mathrm{Pr}_\psi(b)$, $\sum_b \mathrm{Pr}_\psi(b,a) = \langle\psi|\hat{P}_a^A(t_1)|\psi\rangle \equiv \mathrm{Pr}_\psi(a)$, and $\sum_{a,b} \mathrm{Pr}_\psi(b,a) = 1$. We note that, in the special case that $\hat{A}(t_1)$ and $\hat{B}(t_2)$ are commuting, the quasi-probability $\mathrm{Pr}_\psi(b,a)$ reduces to the true probability.

We then calculate the quasi-probability under the condition of b as $\mathrm{Pr}_\psi(a|b) \equiv \mathrm{Pr}_\psi(b,a)/\mathrm{Pr}_\psi(b) = \mathrm{Re}\langle\psi|\hat{U}_1^\dagger\hat{U}_2^\dagger|b\rangle\langle b|\hat{U}_2\hat{P}_a^A\hat{U}_1|\psi\rangle/\langle\psi|\hat{U}_1^\dagger\hat{U}_2^\dagger|b\rangle\langle b|\hat{U}_2\hat{U}_1|\psi\rangle$, and obtain

$$\mathrm{Pr}_\psi(a|b) = {}_b\langle P_a^A\rangle_{\mathrm{w}}. \tag{3}$$

Averaging the eigenvalue a over the conditional quasi-probability, we have

$$_b\langle A\rangle_{\mathrm{w}} = \sum_a a\mathrm{Pr}_\psi(a|b), \tag{4}$$

which is the main result of this report. The left-hand side of Eq. (4) is the weak value, and the right-hand side is related to the correlation function via Eq. (2).

In conclusion, we have derived an equality which gives us a new interpretation of the weak value $_b\langle A\rangle_{\mathrm{w}}$: it is the average of $\hat{A}$ under the condition of b over the quasi-probability which is defined via the correlation function. While we have only considered the real weak value and the symmetric correlation function, we can straightforwardly generalize our result to the complex weak value and the complex correlation function by removing the notation "Re".

Acknowledgments

[1] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).

[2] N. W. M. Ritchie, J. G. Story, and Randall G. Hulet, Phys. Rev. Lett. **66**, 1107 (1991).

[3] H. M. Wiseman, Phys. Rev. A **65**, 032111 (2002).

[4] D. R. Solli, C. F. McCormick, and R. Y. Chiao, S. Popescu, J. M. Hickmann, Phys. Rev. Lett. **92**, 043601 (2004).

[5] Michael J. W. Hall, Phys. Rev. A **69**, 052113 (2004).

[6] G. J. Pryde, J. L. O'Brien, A. G. White, T. C. Ralph, and H. M. Wiseman, Phys. Rev. Lett. **94**, 220405 (2005).

[7] L. Vaidman, Found. Phys. **26**, 895 (1996).

[8] R Kubo, M Toda, and N Hashitsume "*Statistical Mechanics II, Nonequilibrium Statistical Mechanics*" (Springer, Berlin, 1985).