

RENORMALIZED QUANTUM TOMOGRAPHY

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ABSTRACT. The core of quantum tomography is the possibility of writing a generally unbounded complex operator in form of an expansion over operators that are generally nonlinear functions of a generally continuous set of spectral densities—the so-called *quorum* of observables. The expansion is generally non unique, the non unicity allowing further optimization for given criteria. The mathematical problem of tomography is thus the classification of all such operator expansions for given (suitably closed) linear spaces of unbounded operators—e.g. Banach spaces of operators with an appropriate norm. Such problem is a difficult one, and remains still open, involving the theory of general basis in Banach spaces, a still unfinished chapter of analysis. In this paper we present new non-trivial operator expansions for the quorum of quadratures of the harmonic oscillator, and introduce a first very preliminary general framework to generate new expansions based on the Kolmogorov construction. The material presented in this paper is intended to be helpful for the solution of the general problem of quantum tomography in infinite dimensions, which corresponds to provide a coherent mathematical framework for operator expansions over functions of a continuous set of spectral densities.

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1. INTRODUCTION

The state of a physical system is the mathematical description that provides a complete information on the system. Its knowledge is equivalent to know the result of any possible measurement on the system. In classical mechanics it is always possible, at least in principle, to devise a procedure made of multiple measurements which fully recovers the state of a single system. In quantum mechanics, on the contrary, there is no way, not even in principle, to infer the quantum state of a single system without having some prior knowledge on it [1]. It is however possible to estimate the quantum state of a system when many identical copies are available prepared in the same state, so that a different measurement can be performed on each copy. Such a procedure is called *quantum tomography*.

The problem of finding a strategy for determining the state of a system from multiple copies dates back to 1957 by Fano [2], who called *quorum* a set of observables sufficient for a complete determination of the density matrix. However, quantum tomography entered the realm of experiments more recently, with the pioneering experiments by Raymer's group [3] in the domain of quantum optics. In quantum optics, in fact, using a balanced homodyne detector one has the unique opportunity of measuring all possible linear combinations of position and momentum—the so-called *quadratures*—of the harmonic oscillator representing a single mode of the radiation field.

The first technique to reconstruct the density matrix from homodyne measurements — so called *homodyne tomography* — originated from the observation by Vogel and Risken [4] that the collection of probability distributions achieved by homodyne detection is just the Radon transform of the Wigner function W . Therefore, similarly to classical imaging, one can obtain W by inverting the Radon transform, and then from W one can recover the matrix elements of the density operator. This original method, however, works well only in a semi-classical regime, whereas generally for small photon numbers it is affected by an unknown bias caused by the smoothing procedure needed for the analytical inversion of the Radon transform. The solution to the problem is to bypass the evaluation of the Wigner function, and to evaluate the matrix elements of the density operator by simply averaging suitable functions of homodyne data: this is the basis of the first unbiased tomographic technique presented in Ref. [5]. Clearly the state is perfectly recovered in principle only in the limit of infinitely many measurements: however, for finitely many measurements one can estimate the statistical error affecting each matrix element. For infinite dimensions there is the further problem that the propagation of statistical errors of the density matrix elements make them useless for estimating the ensemble average of some operators (e. g. unbounded), and a method for estimating the ensemble average is needed, which bypasses the evaluation of the density matrix itself, as was first suggested in Ref. [6]. For a brief historical excursus on quantum tomography, along with a review on the generalization to any number of radiation modes, arbitrary quantum systems, noise deconvolution, adaptive methods, and maximum-likelihood strategies the reader is addressed to Ref. [7].

The most comprehensive theoretical approach to quantum tomography uses the concept of *frame of observables*, i. e. a set of observables spanning the linear space of operators, from which one derives contextually the *quorum* of observables and the estimation rule. The ensemble average $\langle H \rangle$ of any arbitrary operator H on a Hilbert space \mathcal{H} is estimated using measurement outcomes of the quorum $\{O_l\}$ upon expanding H over a set of functions $f_n(O_l)$ of the observables $\{O_l\}$. What makes the general theory nontrivial in infinite dimensions is the crucial role of the non linear functions $f_n(O_l)$ in making the infinite expansion convergent. Let's denote by $P_j := f_n(O_l)$, $j = (n, l)$, such complete set of operators. Once you have the P_j , then the problem is reduced to the *linear* problem of expanding an operator as $H = \sum_j \langle Q_j^\dagger, H \rangle P_j$, for a suitable “dual” set of operators $\{Q_j\}$. (Notice that generally the index l is continuous, whence also the operator expansion.) The scalar product in the expansion is generally not simply the Frobenius's when we need to expand operators that are not Hilbert-Schmidt. The mathematical *theory of frames* [8, 9, 10, 11] is the perfect tool for establishing completeness of $\{P_j\}$ and for finding dual sets $\{Q_j\}$. In most practical situations the set $\{P_j\}$ is over-complete, and there are many alternate dual sets $\{Q_j\}$, the non unicity providing room for optimization. A general theory should also classify the operators H having bounded scalar product with $\{Q_l\}$ and expansion $H = \sum_j \langle Q_j^\dagger, H \rangle P_j$ converging in average for a given class of quantum states. In infinite dimensions—e. g. for homodyne tomography—the easy known approach

works only for Hilbert-Schmidt operators, whence including trace-class ones—e. g. for estimating the matrix elements of the density operator over an orthonormal basis. For unbounded operators, however, such operator expansion becomes an infinite sum of unbounded terms. On the other hand, converging (and even finite) alternate expansions are known to exist for various unbounded operators [12]. As we will see in this paper, the mechanism allowing "renormalization" of the expansion relies on the existence of *null-estimators*—namely operator-valued functions that have zero mean over the quorum—their existence being related to a group of symmetries of the quorum. The notion of null-estimator was first introduced in Ref. [13], in the context of homodyne tomography, where the quorum is made of the quadratures of the field mode—the quantum analogue of the Radon transform. Here the symmetry group of the quorum is the group $U(1)$ of rotations of the quadrature phase. The existence of null-estimators leads to infinitely many alternate expansions of the same operator over the quorum, allowing cancellations of the infinities in the expansion—a kind of "renormalization" procedure.

The problem of classifying all operator expansions for a given quorum in infinite dimensions for given spaces of unbounded operators is a difficult one, and remains still open. It involves the theory of *frames* or even more general notions of basis in Banach spaces [8, 9, 10, 11], a still unfinished chapter of analysis. In this paper we present new nontrivial operator expansions for the quorum of quadratures of the harmonic oscillator, and introduce a first preliminary general framework to generate and classify new expansions, based on the Kolmogorov construction. We hope that the material presented here will open the way to the solution of the problem of quantum tomography in infinite dimensions, leading to a general mathematical framework for operator expansions over functions of a set of spectral densities.

2. QUANTUM TOMOGRAPHY AND QUORUM OF OBSERVABLES

The general idea of quantum tomography is that there is a set of observables $\{X_\xi\}_{\xi \in \mathfrak{X}}$ on the Hilbert space of the system \mathbb{H} —called "quorum"—by which one can estimate any desired ensemble average by measuring the observables of the quorum, each at the time, in a scheme of a repeated measurements. The observables of the quorum must be independent each other, namely they are not commuting:

$$[X_{\xi'}, X_{\xi''}] = 0 \iff \xi' = \xi'' \quad (1)$$

Generally the set \mathfrak{X} parameterizing the quorum is infinite, and most commonly, is a continuum. In these cases, since clearly one can measure only a finite number of observables, these are randomly picked out according to a given probability measure on \mathfrak{X} , which, therefore, must be a probability space. In the following, for simplicity, we will also assume a probability density over \mathfrak{X} and denote it with the symbol $d\mu(\xi)$. It follows that the ensemble average of a (generally complex) operator is written in the form of double expectation

$$\langle X \rangle = \int_{\mathfrak{X}} d\mu(\xi) \langle f_\xi(X_\xi | X) \rangle, \quad (2)$$

where the generally nonlinear function $f_\xi(x|X)$ of the variable x has an analytic form which depends on the particular operator X . We will call the function $f_\xi(x|X)$ the *tomographic estimator* for X with quorum $\{X_\xi\}_{\xi \in \mathfrak{X}}$. If we want to achieve the estimation of $\langle X \rangle = \text{Tr}[\rho X]$ —the expectation being supposedly bounded on the state ρ —by averaging the estimator $f_\xi(x|X)$ over both quorum and measurement outcomes with a bounded variance, we need to have the function $f_\xi(x|X)$ square-summable over x and ξ , more precisely

$$\int_{\mathfrak{X}} d\mu(\xi) \int_{\mathcal{X}_\xi} \langle dE_\xi(x) | f_\xi(x|X) \rangle^2 < \infty, \quad (3)$$

where \mathcal{X}_ξ denotes the spectrum of X_ξ , and $dE_\xi(x)$ its spectral measure. In the following, for simplicity, we will consider the spectrum $\mathcal{X}_\xi \equiv \mathcal{X}$ independent on ξ . Clearly, the above square-summability will depend again on the state ρ and on the operator X .

We first want to notice two main features of estimators:

- (1) The estimator $f_\xi(x|X)$ is generally not unique, namely there can be many different estimators for the same operator X . This is equivalent to the existence of *null estimators*, namely functions $n_\xi(x)$ such that

$$\int_{\mathfrak{X}} d\mu(\xi) \int_{\mathcal{X}} dE_\xi(x) n_\xi(x) = 0. \quad (4)$$

Accordingly, the estimators are grouped into equivalence classes, each class corresponding to an operator X . For such equivalence we will use the notation \simeq , i. e. we will write $f \simeq g$ or $f - g \simeq 0$ to denote that the two estimators are equivalent, namely they differ by a null estimator.

- (2) For fixed x and ξ the estimator $f_\xi(x|X)$ must be a linear functional of X , namely

$$f_\xi(x|aX + bY) = af_\xi(x|X) + bf_\xi(x|Y), \quad f_\xi(x|X^\dagger) = f_\xi(x|X)^*. \quad (5)$$

Example 1 (Homodyne tomography). [7] *The quorum is given by $\{X_\phi\}_{[0,\pi]}$ with X_ϕ denoting the quadrature $X_\phi \doteq \frac{1}{2}(a^\dagger e^{i\phi} + ae^{-i\phi})$, a, a^\dagger being the annihilation and creation operators of the harmonic oscillator with commutator $[a, a^\dagger] = 1$. Estimators for the dyads $|n\rangle\langle m|$ made with the orthonormal basis $\{|n\rangle\}$, $n = 0, \dots, \infty$ are*

$$\begin{aligned} f_\phi(x|n)\langle m| &= \int_{-\infty}^{+\infty} \frac{dk|k|}{4} e^{\frac{1-2\eta}{8\eta}k^2 - ikx} \langle n+d|e^{ikX_\phi}|n\rangle \\ &= e^{id(\phi + \frac{\pi}{2})} \sqrt{\frac{n!}{(n+d)!}} \int_{-\infty}^{+\infty} dk|k| e^{\frac{1-2\eta}{2\eta}k^2 - i2kx} k^d L_n^d(k^2), \end{aligned} \quad (6)$$

where $L_n^d(x)$ denotes the generalized Laguerre polynomials. For the unbounded operators a and a^\dagger one can check that the following are unbiased estimators

$$\begin{aligned} f_\phi(x|a) &= 2e^{i\phi}x, \\ f_\phi(x|a^\dagger a) &= 2x^2 - \frac{1}{2}. \end{aligned} \quad (7)$$

The problem of quantum tomography is to establish the general rule for estimation, namely

Definition 1 (Estimation rule). *Given the quorum $\{X_\xi\}_{\xi \in \mathfrak{X}}$ find the correspondence:*

$$f_\xi(x|X) \iff X, \quad \text{for every operator } X \text{ on } \mathbb{H}, \quad (8)$$

where we possibly mean to find the whole equivalence class of estimators $f_\xi(X_\xi|X)$.

Before solving this task, first one needs to know that the set of observables $\{X_\xi\}_{\xi \in \mathfrak{X}}$ is actually a quorum. The easiest thing to do, however, is to derive both the quorum and the estimation rule contextually, starting from a spanning-set of observables—shortly *observable spanning-set*—namely a set of observables $\{F_\omega\}_{\omega \in \mathfrak{D}}$ in terms of which we can linearly expand operators as follows

$$X = \int_{\mathfrak{D}} d\omega c_\omega(X) F_\omega. \quad (9)$$

Notice that the notion of operator spanning-set used here generalizes the notion of frames for Banach spaces to unbounded operators (see also the following), and is generally not strictly a frame according to the definition of Refs. [8, 9]. In the following throughout the paper we will always assume probability distributions admitting densities. Generally, the set \mathfrak{D} is unbounded, and the measure $d\omega$ is not normalizable, whence, as such, the expansion (9) cannot be used for quantum tomography. However, generally this feature is related to the redundancy of the observable spanning-set, which includes many observables F_ω that are just different functions of the same observable. Then, collecting the observables of the spanning-set into functional equivalence classes \mathfrak{R}_ξ , each corresponding to an observable of the quorum $\{X_\xi\}_{\xi \in \mathfrak{X}}$, one can relabel the observable spanning-set as $F_{\kappa, \xi} \doteq f_\kappa(X_\xi)$ with $\kappa \in \mathfrak{R}_\xi$, writing

$$X = \int_{\mathfrak{X}} d\mu(\xi) \int_{\mathfrak{R}_\xi} d\nu(\kappa) c_{\kappa, \xi}(X) f_\kappa(X_\xi) = \int_{\mathfrak{X}} d\mu(\xi) f_\xi(X_\xi|X), \quad (10)$$

where the function $f_\xi(x|X)$ is the integral over the observables equivalent to X_ξ , namely

$$f_\xi(X_\xi|X) \doteq \int_{\mathfrak{R}_\xi} d\nu(\kappa) c_{\kappa,\xi}(X) f_\kappa(X_\xi). \quad (11)$$

Notice that in terms of the spectral decomposition of X_ξ we can write

$$X = \int_{\mathfrak{X}} d\mu(\xi) \int_{\mathcal{X}_\xi} dE_\xi(x) f_\xi(x|X), \quad (12)$$

and since this expansion is linear in the spectral measure $dE_\xi(x)$, the latter can be regarded itself as an observable spanning-set. Indeed, upon introducing the spectral density $dE_\xi(x) \doteq Z_{\xi,x} dx$, and the density $d\mu(\xi) \doteq m(\xi) d\xi$, and renaming $\zeta = (\xi, x)$ and $\mathfrak{Z} \doteq \{(\xi, x), x \in \mathcal{X}_\xi, \xi \in \mathfrak{X}\}$ Eq. (12) can be rewritten in the same form of Eq. (9), namely

$$X = \int_{\mathfrak{Z}} d\zeta c'_\zeta(X) Z_\zeta, \quad (13)$$

where the new expansion coefficients are now given by

$$c'_{\xi,x}(X) = m(\xi) f_\xi(x|X) = m(\xi) \int_{\mathfrak{R}_\xi} d\nu(\kappa) c_{\kappa,\xi}(X) f_\kappa(x). \quad (14)$$

For homodyne tomography the above quantities are explicitly given in Table 1.

For F_ω an operator frame, the coefficients of the expansion (9) can be rewritten in form of a pairing $(\cdot|\cdot)$ with a dual frame G_ω as follows

$$c_\omega(X) = (G_\omega|X), \quad (15)$$

in terms of which Eq. (14) rewrites in the pairing form

$$c'_{\xi,x}(X) = m(\xi) (W_{\xi,x}|X), \quad (16)$$

with dual frame

$$W_{\xi,x} = \int_{\mathfrak{R}_\xi} d\nu(\kappa) G_{\xi,x} f_\kappa^*(x). \quad (17)$$

From the last equation it follows that the estimator itself can be written using the pairing

$$f_\xi(x|X) = (W_{\xi,x}|X), \quad (18)$$

or, in terms of the original observable frame

$$f_\xi(x|X) = \int_{\mathfrak{R}_\xi} d\nu(\kappa) (G_{\xi,x}|X) f_\kappa(x). \quad (19)$$

general	homodyne	general	homodyne	general	homodyne
X_ξ	X_ϕ	$c_\omega(X)$	$\text{Tr}[D^\dagger(\alpha)X]$	$d\mu(\xi)$	$\frac{d\phi}{\pi}$
\mathfrak{X}	$[0, \pi)$	ω	α	$Z_{\xi,x}$	$ x\rangle_{\phi\phi}\langle x $
ξ	ϕ	\mathfrak{R}_ξ	\mathbb{R}	\mathcal{X}_ξ	\mathbb{R}
$d\mu(\xi)$	$\frac{d\phi}{\pi}$	$d\nu(k)$	$\frac{1}{4} dk k $	\mathfrak{Z}	$[0, \pi) \times \mathbb{R}$
ω	α	$c_{\xi,\kappa}(X)$	$\text{Tr}[e^{-i\kappa X_\phi} X]$	$m(\xi)$	$\frac{1}{\pi}$
F_ω	$D(\alpha)$	$f_\kappa(X_\xi X)$	$e^{i\kappa X_\phi}$	$W_{\xi,x}$	$-\frac{1}{4\pi} \frac{P}{(x-X_\phi)^2}$

TABLE 1. Table of correspondence for homodyne tomography.

2.1. Unbiasing noise. It is possible to estimate the *ideal* ensemble average $\langle X \rangle$ by measuring the quorum in the presence of instrumental noise, when the noise map \mathcal{N} is invertible, or, more generally, if there exists the right inverse of \mathcal{N} . In terms of observable frames this just corresponds to using a different dual frame. More precisely, one has:

$$\begin{aligned} X &= \mathcal{N} \mathcal{N}^{-1}(X) = \int_{\mathfrak{X}} d\mu(\xi) \mathcal{N}[f_{\xi}(X_{\xi} | \mathcal{N}^{-1}(X))] \\ &= \int_{\mathfrak{X}} d\mu(\xi) \int_{\mathcal{X}_{\xi}} \mathcal{N}(dE_{\xi}(x)) f_{\xi}(x | \mathcal{N}^{-1}(X)). \end{aligned} \quad (20)$$

Notice that Eq. (20) means that

$$\langle X \rangle = \int_{\mathfrak{X}} d\mu(\xi) \int_{\mathcal{X}_{\xi}} \langle dE_{\xi}(x) \rangle_{\mathcal{N}} f_{\xi}(x | \mathcal{N}^{-1}(X)), \quad (21)$$

where $\langle \cdot \rangle \doteq \text{Tr}[\rho \cdot]$ denotes the ideal ensemble average, whereas $\langle \cdot \rangle_{\mathcal{N}} \doteq \text{Tr}[\mathcal{N}^{\tau}(\rho) \cdot]$ denotes the experimental ensemble average, \mathcal{N}_{*} being the predual map of \mathcal{N} (Schroedinger versus Heisenberg picture). This also means that for left invertible map \mathcal{N} the noisy spectral measures $\mathcal{N}(dE_{\xi}(x))$ are still a quorum. In terms of the pairing in Eq. (18) unbiasing the noise is equivalent to use the new dual frame $\mathcal{N}^{-1\dagger}(W_{\xi,x})$. When \mathcal{N} is not right-invertible one can still estimate the ensemble average of operators in the range of the map. Moreover, in infinite dimension, when the noise CP map \mathcal{N} is compact its inverse map is unbounded, and one generally cannot unbiased the noise without restricting the space of reconstructed operators. Otherwise, one has a Hadamard ill-posed problem, for which there are biased compromises, such as putting a cutoff on the vanishing singular values of \mathcal{N} .

Example 2 (Pauli tomography in a Pauli channel). [7] *The operators $[\sigma_{\alpha}/\sqrt{2}]$ make an observable orthonormal basis for $\mathbb{C}^{\otimes 2}$. We consider now the noise described by the depolarizing Pauli channel*

$$\mathcal{N} = (1-p)\mathcal{I} + \frac{p}{2}\mathcal{T}, \quad (22)$$

where \mathcal{I} denotes the identity map and \mathcal{T} the trace map $\mathcal{T}(X) \doteq I \text{Tr}(X)$. This noise can be simply unbiased via noise-map inversion:

$$\mathcal{N}^{-1\dagger} = \frac{1}{1-p}\mathcal{I} - \frac{p}{2(1-p)}\mathcal{T}. \quad (23)$$

Example 3 (Homodyne tomography with $\eta < 1$). [7] *The set of displacements operators $D(\alpha) := e^{\alpha a^{\dagger} - \alpha^{*} a}$, $\alpha \in \mathbb{C}$ make an observable Dirac-orthonormal frame for $\mathbb{T}_{1/2}$, where*

$$\mathbb{T}_s = \{X \in \mathbb{T}, X = :f(a, a^{\dagger}):, \text{ s.t. } \lim_{\alpha \rightarrow \infty} f(\alpha, \bar{\alpha}) e^{s|\alpha|^2} = 0\} \quad (24)$$

where $: \cdot :$ denotes normal ordering. In the presence of noise from nonunit quantum efficiency η , the unbiased reconstruction is possible for operators in \mathbb{T}_s if $\eta \geq (2s)^{-1}$. In fact, one uses the new dual:

$$D(\alpha) \rightarrow \mathcal{N}^{-1\dagger}(D(\alpha)) = \eta D(\eta^{1/2}\alpha) e^{\frac{1-\eta}{2}|\alpha|^2}. \quad (25)$$

Example 4 (Homodyne tomography in Gaussian noise). [7] *As for quantum efficiency, Gaussian noise can be unbiased for mean thermal photon number $\bar{n} \leq s - \frac{1}{2}$. One has the new dual:*

$$D(\alpha) \rightarrow \mathcal{N}^{-1\dagger}(D(\alpha)) = D(\alpha) e^{\bar{n}|\alpha|^2}. \quad (26)$$

3. THE CASE OF HOMODYNE TOMOGRAPHY

Before addressing the general problem of deriving a general tomographic rule for unbounded operators, in this section we will re-derive the known pattern function of homodyne tomography in order to illustrate the general concepts introduced in the previous section.

The starting point is the observable frame $\{D(\alpha)\}_{\alpha \in \mathbb{C}}$, in terms of which the decomposition (9) for Hilbert-Schmidt operators rewrites as follows

$$X = \int_{\mathbb{C}} \frac{d^2\alpha}{\pi} \text{Tr}[D^\dagger(\alpha)X]D(\alpha). \quad (27)$$

By changing to polar variables $\alpha = (-i/2)ke^{i\phi}$, Eq. (27) becomes

$$X = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} \frac{dk|k|}{4} \text{Tr}[X e^{ikX_\phi}] e^{-ikX_\phi}. \quad (28)$$

Eq. (28) can be used only for Hilbert-Schmidt operators, for which the trace under the integral in Eq. (28) exists. In terms of the quadrature spectral measure, one has

$$X = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} \frac{dk|k|}{4} \text{Tr}[X e^{ikX_\phi}] e^{-ikX_\phi} = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dE_\phi(x) \text{Tr}[XW_{\phi,x}], \quad (29)$$

where

$$W_{\phi,x} = e^{-i\phi a^\dagger} D(x) W_{0,0} D^\dagger(x) e^{i\phi a^\dagger}, \quad W_{0,0} = -\frac{1}{2} \text{P} \frac{1}{X_0^2}, \quad (30)$$

P denoting the Cauchy principal value. On the other hand, in the next section we will show that for unbounded operators we also have the expansion

$$\begin{aligned} X &= \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dt \text{Tr}[G^\dagger(t, \phi)X]F(t, \phi), \\ F(t, \phi) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-i2X_\phi)^2}, \quad G(t, \phi) = \frac{d}{dt} t e^{\frac{t^2}{2}} \int_0^1 d\theta |i(1-\theta)te^{i\phi}\rangle \langle -i\theta te^{i\phi}|. \end{aligned} \quad (31)$$

which in terms of the the quadrature spectral measure reads

$$X = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dE_\phi(x) \text{Tr}[XW'_{\phi,x}], \quad (32)$$

where now

$$W'_{\phi,x} = \int_{-\infty}^{\infty} dt \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-i2x)^2} \frac{d}{dt} t e^{\frac{t^2}{2}} \int_0^1 d\theta |i(1-\theta)te^{i\phi}\rangle \langle -i\theta te^{i\phi}|. \quad (33)$$

Alternatively, as shown in Sec. (9) by means of the frame of normal-ordered moments, one has the expansion

$$\begin{aligned} X &= \sum_{n,m=0}^{\infty} a^{\dagger n} a^m \text{Tr}[g_{n,m}^\dagger X] \\ &= \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dE_\phi(x) \text{Tr} \left[X \sum_{n,m=0}^{\infty} g_{n,m}^\dagger \binom{n+m}{n}^{-1} \frac{H_{n+m}(\sqrt{2}x)}{\sqrt{2^{n+m}}} \right] \end{aligned} \quad (34)$$

where

$$g_{n,m} = \sum_{j=0}^{\min(n,m)} \frac{(-1)^j}{j! \sqrt{(n-j)!(m-j)!}} |n-j\rangle \langle m-j|. \quad (35)$$

The above expansions in Eqs. (32) and (34) are just examples of alternate expansions which are equivalent for the estimation of the expectation values of (even unbounded) observables, but can be very different as regards the statistical noise affecting such estimation. As a matter of fact, the problem of classifying all possible expansions has been never solved, and, hopefully, the results of the present paper may suggest a unifying approach to the solution of such a difficult problem. As we will see in the next section, the existence of many alternate expansions is due to the symmetry of the quorum of quadrature operators, and the resulting properties of null estimator functions.

4. CALCULUS WITH NULL FUNCTIONS

We first notice that to a null estimator function $n_\xi \simeq 0$ it corresponds a null expansion over the quorum, namely

$$\int_{\mathfrak{X}} d\mu(\xi) \int_{\mathcal{X}} dE_\xi(x) n_\xi(x) = 0 \iff \int_{\mathfrak{X}} d\mu(\xi) n_\xi(X_\xi) = 0. \quad (36)$$

Let us recall the ordering relation [14]

$$:a^{\dagger k} a^l:_s = \sum_{j=0}^{\{k,l\}} \frac{k!l!}{j!(k-j)!(l-j)!} \left(\frac{s-r}{2}\right)^j :a^{\dagger k-j} a^{l-j}:_r, \quad (37)$$

where $\{k, l\} := \min(k, l)$, and $s = -1, 0$, and 1 correspond to normal, symmetrical, and anti-normal ordering, respectively. We will also write the symmetrical ordering as $S\{a^{\dagger k} a^l\} := :a^{\dagger k} a^l:_0$, and the normal ordering as $:a^{\dagger k} a^l := :a^{\dagger k} a^l:_{-1}$.

Then we have:

Lemma 1 (Main equivalence relation). [13] *The following equivalence relation holds*

$$x^k e^{\pm i(k+2n+2)\phi} \simeq 0, \quad \forall k, n \geq 0. \quad (38)$$

Proof. Since $X_\phi^k = \frac{1}{2^k} \sum_{l=0}^k \binom{k}{l} S\{a^{\dagger l} a^{k-l}\} e^{i\phi(2l-k)}$, one has

$$\int_0^\pi \frac{d\phi}{\pi} e^{\pm i(k+2+2n)\phi} X_\phi^k = 0, \quad \forall k, n \geq 0, \quad (39)$$

which is equivalent to (38).

Stated differently:

Lemma 2. *The following equivalence relation holds*

$$H_k(\sqrt{2}x) e^{\pm i(k+2n+2)\phi} \simeq 0, \quad \forall k, n \geq 0, \quad (40)$$

where $H_k(x)$ denote the k -th Hermite polynomial.

Proof. From the definition of Hermite polynomials one has

$$\begin{aligned} \frac{1}{\sqrt{2^n}} H_n(\sqrt{2}X_\phi) &= \frac{1}{\sqrt{2^n}} \left. \frac{\partial^n}{\partial t^n} \right|_{t=0} e^{-t^2 + \sqrt{2}t(a^\dagger e^{i\phi} + a e^{-i\phi})} \\ &= \frac{1}{\sqrt{2^n}} \left. \frac{\partial^n}{\partial t^n} \right|_{t=0} e^{\sqrt{2}ta^\dagger e^{i\phi}} e^{\sqrt{2}ta e^{-i\phi}} = \sum_{k=0}^n \binom{n}{k} a^{\dagger k} a^{n-k} e^{i\phi(2k-n)} = 2^n :X_\phi^n: . \end{aligned} \quad (41)$$

Then, it follows that

$$\int_0^\pi \frac{d\phi}{\pi} e^{\pm i(k+2+2n)\phi} H_k(\sqrt{2}X_\phi) = 0, \quad \forall k, n \geq 0, \quad (42)$$

which is equivalent to (40).

Moreover, we also have

Lemma 3 (Equivalence of truncated Hermite polynomials). *The following equivalence relations hold:*

$$e^{\pm in\phi} H_{2l+n}^{(l)}(\kappa x) \simeq e^{\pm in\phi} H_{2l+n}(\kappa x), \quad (43)$$

where we introduced the truncated Hermite polynomial

$$H_n^{(l)}(z) = \sum_{m=0}^l \frac{(-)^m n!(2z)^{n-2m}}{m!(n-2m)!}. \quad n \geq 2l. \quad (44)$$

Proof. The two identities are just the complex conjugated of each other. Therefore, it is sufficient to prove the identity with the plus sign. The latter is a simple consequence of identity (38), namely $x^k e^{\pm i(k+2n+2)\phi} \simeq 0$. By using the un-truncated Hermite polynomial, we have

$$e^{in\phi} H_{2l+n}(\kappa x) = e^{in\phi} \sum_{m=0}^{l+[n/2]} \frac{(-)^m (2l+n)! (2\kappa)^{2l+n-2m}}{m! (2l+n-2m)!} \times x^{2l+n-2m} e^{i[n+2(l-m)+2(m-l)]\phi}, \quad (45)$$

from which it follows that all terms with $m > l$ are equivalent to zero.

Finally, one can show the Poisson identities (whose proof can be found in the Appendix)

Lemma 4 (Poisson identities). *The following identities hold*

$$\begin{aligned} f(x^2) \delta_\pi(\phi) &\simeq \frac{1}{\pi} \left[\frac{f(x^2 e^{2i\phi})}{1 - e^{-2i\phi}} + \frac{f(x^2 e^{-2i\phi})}{1 - e^{2i\phi}} \right], \\ x f(x^2) \delta_\pi(\phi) &\simeq \frac{1}{\pi} \left[\frac{x e^{i\phi} f(x^2 e^{2i\phi})}{1 - e^{-2i\phi}} + \frac{x e^{-i\phi} f(x^2 e^{-2i\phi})}{1 - e^{2i\phi}} \right], \end{aligned} \quad (46)$$

In particular, we have the identity

$$\delta_\pi(\phi) \simeq \frac{1}{\pi}. \quad (47)$$

5. THE KOLMOGOROV CONSTRUCTION

In this Section we present the Kolmogorov construction, and its relation with the fundamental identity of quantum tomography.

In the following, by \mathbf{l}_2 we denote the Hilbert space of square summable sequences of complex numbers, and by $\mathbf{L}_2(\mathfrak{X})$ we denote the Hilbert space of square summable functions over the space \mathfrak{X} . For example, $\mathfrak{X} = \mathbb{R}$, and $\mathbf{L}_2(\mathfrak{X})$ the Hilbert space of square summable functions on the real axis, or $\mathfrak{X} = S^1$, and $\mathbf{L}_2(\mathfrak{X})$ is the Hardy space of square-summable complex functions on the circle. In the following we will focus attention only to the case $\mathfrak{X} \equiv \mathbb{R}$. Consider now a complete orthonormal set of functions $[v_n(x)]$ for $\mathbf{L}_2(\mathfrak{X})$. The completeness of the set corresponds to the following distribution identity

$$\sum_n v_n(x)^* v_n(y) = \delta(x - y), \quad (48)$$

where δ denotes the usual Dirac-delta. Consider now a (infinite-dimensional) Hilbert space \mathbf{H} and denote by $[w_n]$ an orthonormal basis for it. The following vector

$$|v(x)\rangle = \sum_n v_n(x) |w_n\rangle, \quad (49)$$

is Dirac-normalizable, in the sense that

$$\langle v(y) | v(x) \rangle = \delta(x - y). \quad (50)$$

Consider now another (infinite-dimensional) Hilbert space $\mathbf{K} \simeq \mathbf{H}$. To the orthonormal basis $[v_n(x)]$ for $\mathbf{L}_2(\mathfrak{X})$ and $[z_n]$ for \mathbf{K} we associate a map from the observables $\mathbf{O}_\mathfrak{X}$ with spectrum \mathfrak{X} on \mathbf{H} to operators in $\mathbf{B}(\mathbf{H}, \mathbf{H} \otimes \mathbf{K})$ given by

$$v(X) = \sum_n v_n(X) \otimes |z_n\rangle, \quad (51)$$

where, as usual, we define the operators $v_n(X)$ in terms of the spectral resolution of X as follows

$$v_n(X) = \int_{\mathfrak{X}} dE_X(x) v_n(x), \quad (52)$$

where $dE_X(x)$ denotes the spectral measure of X .

For $X, Y \in \mathbf{O}_\mathfrak{X}$ formally we write

$$v(X)^\dagger v(Y) = \sum_n v_n(X)^\dagger v_n(Y), \quad (53)$$

Consider the integral kernel $K(x, y)$, $x, y \in \mathfrak{X}$ corresponding to the positive operator $K \in \mathbf{B}(\mathbf{H})$

$$K(x, y) = \langle v(x) | K | v(y) \rangle. \quad (54)$$

For any two self-adjoint operators X, Y on $\mathbf{L}_2(\mathbb{R})$, the expression $K(X, Y)$ is well defined in the following sense

$$K(X, Y) \doteq v(X)^\dagger (I \otimes \tilde{K}) v(Y), \quad (55)$$

where $\tilde{K} \in \mathbf{B}(\mathbf{K})$ is given by

$$\tilde{K} = \sum_{n,m} |z_n\rangle \langle w_n | K | w_m \rangle \langle z_m|, \quad (56)$$

namely we can also write

$$K(X, Y) = \sum_{n,m} v_n(X)^\dagger \langle w_n | K | w_m \rangle v_m(Y) = \int_{\mathfrak{X}} dE_X(x) \int_{\mathfrak{X}} dE_X(y) K(x, y). \quad (57)$$

This is also equivalent to say that for any expansion of $K(x, y)$ in series of products of functions of single variable, $K(X, Y)$ is defined as the same expansion, ordered with the functions of X on the left and the functions of Y on the right. For commuting X, Y , then $K(X, Y)$ simply represents the same analytic expression of $K(x, y)$, now substituting the operators in place of the variables. As an example, the identity operator $K = I$ corresponds to the Dirac-delta kernel, and for commuting $X, Y \in \mathbf{O}_{\mathfrak{X}}$ we have

$$v(X)^\dagger v(Y) = \delta(X - Y). \quad (58)$$

By replacing now $\mathbf{H} \rightarrow \mathbf{H}^{\otimes 2}$, even for non commuting X and Y , one has

$$(v(X)^\dagger \otimes I)(I \otimes v(Y)) = \delta(X \otimes I - I \otimes Y). \quad (59)$$

Moreover, similarly to Eq. (57), one has

$$\begin{aligned} K(X \otimes I, I \otimes Y) &\doteq (v(X)^\dagger \otimes I)(I_{\mathbf{H}^{\otimes 2}} \otimes \tilde{K})(I \otimes v(Y)) \\ &= \sum_{n,m} v_n(X)^\dagger \otimes v_m(Y) \langle w_n | K | w_m \rangle \\ &= \int_{\mathfrak{X}} dE_X(x) \otimes \int_{\mathfrak{X}} dE_X(y) K(x, y). \end{aligned} \quad (60)$$

The fundamental identities of quantum tomography are obtained as an expansion of the swap operator E over the *quorum*, since for any state ρ and observable A one has $\text{Tr}[\rho A] = \text{Tr}[(\rho \otimes A)E]$, where $E|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$.

5.1. Homodyne tomography. From Eq. (28), it is clear the swap operator can be written as

$$E = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} \frac{dk |k|}{4} e^{-ikX_\phi} \otimes e^{ikX_\phi}. \quad (61)$$

Then, the usual homodyne tomographic formula can be obtained by the Kolmogorov construction in writing the swap operator as follows

$$E = \int_0^\pi \frac{d\phi}{\pi} (v(X_\phi)^\dagger \otimes I)(I_{\mathbf{H}^{\otimes 2}} \otimes \tilde{K}) I \otimes v(X_\phi), \quad (62)$$

corresponding to the positive kernel

$$K(x, x') = \frac{\pi}{2} \langle x || Y || x' \rangle = \int_{\mathbb{R}} \frac{dk}{4} |k| e^{ik(x'-x)} = -\frac{\mathbf{P}}{2} \frac{1}{(x-x')^2}, \quad (63)$$

where Y is the quadrature conjugated to X , with $[X, Y] = \frac{i}{2}$. The kernel is clearly positive, since one has

$$\sum_{i,j} K(x_i, x_j) \xi_i^* \xi_j = \int_{\mathbb{R}} \frac{dk}{4} |k| \left| \sum_j e^{ix_j} \xi_j \right|^2 \quad (64)$$

The tomographic formula consists in the following identity

$$E = \int_0^\pi \frac{d\phi}{\pi} K(X_\phi \otimes I, I \otimes X_\phi) \quad (65)$$

Using Eq. (60), one can also write

$$E = \int_0^\pi \frac{d\phi}{\pi} \sum_n v_n(X_\phi)^\dagger \otimes u_n(X_\phi), \quad (66)$$

with

$$u_n(X_\phi) = \sum_m v_m(X_\phi) \langle n || Y || m \rangle. \quad (67)$$

The existence of null estimator functions can be taken into account by considering any operator $N_{n,\phi}$ such that

$$\int_0^\pi \frac{d\phi}{\pi} v_n(X_\phi)^\dagger \otimes N_{n,\phi} = 0, \quad (68)$$

and any estimation rule can be obtained by the swap operator

$$E = \int_0^\pi \frac{d\phi}{\pi} \sum_n v_n(X_\phi)^\dagger \otimes D_{n,\phi}, \quad (69)$$

with $D_{n,\phi} = u_n(X_\phi) + N_{n,\phi}$, as follows

$$\text{Tr}[\rho X] = \int_0^\pi \frac{d\phi}{\pi} \sum_n \text{Tr}[\rho v_n(X_\phi)^\dagger] \text{Tr}[D_{n,\phi} X]. \quad (70)$$

5.2. Spin tomography. For spin tomography the swap operator writes as follows

$$E = \frac{2J+1}{2\pi} \int \frac{d\vec{n}}{4\pi} \int_0^{2\pi} d\psi \sin^2 \frac{\psi}{2} e^{i(\vec{J}_1 - \vec{J}_2) \cdot \vec{n}\psi} = \int \frac{d\vec{n}}{4\pi} K(\vec{J}_1 \cdot \vec{n} \otimes I, I \otimes \vec{J}_2 \cdot \vec{n}), \quad (71)$$

and we immediately see that the kernel can be written as follows

$$K(r, s) = \frac{1}{2} \left(J + \frac{1}{2} \right) \langle r + J | 2 - e_+ - e_- | s + J \rangle, \quad (72)$$

where $r, s = -J, -J+1 \dots J$, and $\{|n\rangle\}$ denote any orthonormal basis of the infinite dimensional Hilbert space \mathbb{H} . Such a basis can be conveniently regarded as the Hardy space of functions on the unit circle, with $\langle n | z \rangle = z^n$, $|z| = 1$, and

$$\oint \frac{dz}{2\pi iz} |z\rangle \langle z| \equiv \int_0^{2\pi} \frac{d\psi}{2\pi} |e^{i\psi}\rangle \langle e^{i\psi}|, \quad (73)$$

e_- denoting the shift operator $e_- |n\rangle = |n-1\rangle$, and $e_+ = e_-^\dagger$. By introducing the vectors $|v(m)\rangle \doteq |m+J\rangle$, we can write

$$E = \int \frac{d\vec{n}}{4\pi} (v(\vec{J}_1 \cdot \vec{n})^\dagger \otimes I) (I_{\mathbb{H} \otimes 2} \otimes K) (I \otimes v(\vec{J}_2 \cdot \vec{n})), \quad (74)$$

where the operator $K \in \mathbb{B}(\mathbb{K})$ is given by

$$K = \left(J + \frac{1}{2} \right) (1 - C), \quad (75)$$

where $C = \frac{1}{2}(e_+ + e_-)$ is the *cosine operator*.

5.3. **Alternate expansions.** The general form of the swap operator is

$$E = \sum_{\nu} (v(X_{\nu})^{\dagger} \otimes I)(I_{\mathbb{H} \otimes 2} \otimes K)(I \otimes v(X_{\nu})). \quad (76)$$

Introducing an invertible operator $L \in \mathbb{B}(\mathbb{K})$, we can write

$$\begin{aligned} E &= \sum_{\nu} (v(X_{\nu})^{\dagger} \otimes I)(I_{\mathbb{H} \otimes 2} \otimes K^{\frac{1}{2}} L^{-1} L K^{\frac{1}{2}})(I \otimes v(X_{\nu})), \\ &= \sum_{\nu} \sum_{n, m, l} v_n(X_{\nu})^{\dagger} \otimes v_m(X_{\nu}) \langle n | K^{\frac{1}{2}} L^{-1} | z(l) \rangle \langle z(l) | L K^{\frac{1}{2}} | m \rangle, \end{aligned} \quad (77)$$

where $\{|z(l)\rangle\}$ is any orthonormal basis for \mathbb{K} . Therefore, we have all the alternate expansions on the quorum

$$Z = \sum_{\nu} \sum_l \text{Tr}[L_l(X_{\nu})^{\dagger} Z] M_l(X_{\nu}), \quad (78)$$

where

$$\begin{aligned} M_l(x) &= \sum_m v_m(x) \langle z(l) | L K^{\frac{1}{2}} | m \rangle = \langle z(l) | L K^{\frac{1}{2}} | v(x) \rangle, \\ L_l(x)^* &= \sum_m v_m^*(x) \langle m | K^{\frac{1}{2}} L^{-1} | z(l) \rangle = \langle v(x) | K^{\frac{1}{2}} L^{-1} | z(l) \rangle. \end{aligned} \quad (79)$$

6. EXPANSION OF UNBOUNDED OPERATORS OVER THE QUADRATURES

As already noticed, the swap operator in Eq. (61) provides the estimation rule just for trace-class operators. However, it is known since Richter [12] the following formula

$$a^{\dagger n} a^m = \binom{n+m}{n}^{-1} \int_0^{\pi} \frac{d\phi}{\pi} \frac{1}{\sqrt{2^{n+m}}} H_{n+m}(\sqrt{2} X_{\phi}) e^{i\phi(m-n)}. \quad (80)$$

Eq. (80) was originally derived by using nontrivial identities involving trilinear products of Hermite polynomials.

Here, we provide a much simpler derivation as follows. Using the definition of Hermite polynomials in Eq. (41), one has

$$\begin{aligned} a^{\dagger n} a^m &= \binom{n+m}{n}^{-1} \sum_{k=0}^{n+m} \binom{n+m}{k} a^{\dagger k} a^{n+m-k} \delta_{k,n} \\ &= \binom{n+m}{n}^{-1} \int_0^{\pi} \frac{d\phi}{\pi} \sum_{k=0}^{n+m} \binom{n+m}{k} a^{\dagger k} a^{n+m-k} e^{i\phi(2k-n-m)} e^{i\phi(m-n)} \\ &= \binom{n+m}{n}^{-1} \int_0^{\pi} \frac{d\phi}{\pi} :X_{\phi}^{n+m}: e^{i\phi(m-n)} \\ &= \binom{n+m}{n}^{-1} \int_0^{\pi} \frac{d\phi}{\pi} \frac{1}{\sqrt{2^{n+m}}} H_{n+m}(\sqrt{2} X_{\phi}) e^{i\phi(m-n)} \end{aligned} \quad (81)$$

Similarly, for the symmetrical ordering, one derives the identity

$$\begin{aligned} S\{a^{\dagger n} a^m\} &= \binom{n+m}{m}^{-1} \sum_{l=0}^{n+m} \binom{n+m}{l} S\{a^{\dagger n+m-l} a^l\} \delta_{lm} \\ &= \binom{n+m}{m}^{-1} \int_0^{\pi} \frac{d\phi}{\pi} e^{i\phi(m-n)} \sum_{l=0}^{n+m} \binom{n+m}{l} S\{(a^{\dagger} e^{i\phi})^{n+m-l} (a e^{-i\phi})^l\} \\ &= \binom{n+m}{m}^{-1} \int_0^{\pi} \frac{d\phi}{\pi} e^{i\phi(m-n)} 2^{n+m} X_{\phi}^{n+k}. \end{aligned} \quad (82)$$

For arbitrary ordering, using Eq. (37), one obtains

$$\begin{aligned}
: a^\dagger{}^k a^l :_s &= \sum_{j=0}^{\{k,l\}} \frac{k!l!}{j!(k-j)!(l-j)!} \left(\frac{s}{2}\right)^j S\{a^\dagger{}^{k-j} a^{l-j}\} \\
&= \int_0^\pi \frac{d\phi}{\pi} e^{i\phi(l-k)} \sum_{j=0}^{\{k,l\}} \frac{k!l!}{j!(k+l-2j)!} \left(\sqrt{\frac{s}{2}}\right)^{k+l} \left(2\sqrt{\frac{2}{s}} X_\phi\right)^{k+l-2j} \\
&= \binom{k+l}{k}^{-1} \int_0^\pi \frac{d\phi}{\pi} e^{i\phi(l-k)} \left(\sqrt{\frac{s}{2}}\right)^{k+l} H_{k+l}^{(k,l)} \left(\sqrt{\frac{2}{s}} X_\phi\right). \tag{83}
\end{aligned}$$

Using Lemma 3, one has the equivalent identity

$$: a^\dagger{}^k a^l :_s = \binom{k+l}{k}^{-1} \int_0^\pi \frac{d\phi}{\pi} e^{i\phi(l-k)} \left(\sqrt{\frac{s}{2}}\right)^{k+l} H_{k+l} \left(\sqrt{\frac{2}{s}} X_\phi\right). \tag{84}$$

In a similar way, one can derive the useful relation

$$(\mu a + \nu a^\dagger)^n = \int_0^\pi \frac{d\phi}{\pi} (2X_\phi)^n \frac{(\nu e^{-i\phi})^{n+1} - (\mu e^{i\phi})^{n+1}}{\nu e^{-i\phi} - \mu e^{i\phi}}, \tag{85}$$

whence

$$X_\varphi^n = \int_0^\pi \frac{d\phi}{\pi} X_\phi^n \frac{\sin[(\phi - \varphi)(n+1)]}{\sin(\phi - \varphi)}. \tag{86}$$

Using Eq. (82), for the displacement operator one obtains

$$\begin{aligned}
D(\alpha) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha a^\dagger - \alpha^* a)^n = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} (-\alpha^*)^k S\{a^\dagger{}^{n-k} a^k\} \\
&= \int_0^\pi \frac{d\phi}{\pi} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2\alpha e^{-i\phi} X_\phi)^n (-2\alpha^* e^{i\phi} X_\phi)^k}{(n+k)!}. \tag{87}
\end{aligned}$$

The last equation can be summed using the identity

$$\begin{aligned}
\sum_{n,m=0}^{\infty} \frac{z^n (-z^*)^m}{(n+m)!} &= \sum_{s=0}^{\infty} \frac{z^s}{s!} \sum_{d=0}^s \binom{s}{d} \left(-\frac{z^*}{z}\right)^d = \sum_{s=0}^{\infty} \frac{z^s}{s!} \frac{1 - \left(-\frac{z^*}{z}\right)^{s+1}}{1 + \frac{z^*}{z}} \\
&= \sum_{s=0}^{\infty} \frac{1}{s!} \frac{z^{s+1} - (-z^*)^{s+1}}{z + z^*} = \frac{ze^z + z^* e^{-z^*}}{z + z^*}
\end{aligned} \tag{88}$$

which gives the estimation rule

$$f_\phi(X_\phi|D(\alpha)) = \frac{e^{-i\phi} \alpha e^{2X_\phi} e^{-i\phi} \alpha + e^{i\phi} \alpha^* e^{-2X_\phi} e^{i\phi} \alpha^*}{e^{-i\phi} \alpha + e^{i\phi} \alpha^*}. \tag{89}$$

Identity (89) should be compared with the equivalent estimator given in Ref. [6]. The identity in Eq. (87) can be also derived by explicitly using the properties of null estimator functions, as shown in the Appendix.

All estimation rules $f_\phi(X_\phi|X)$ given in the present section do not correspond to an expectation of X as in Eq. (70). However, we can suitably recover an expectation rule—which is generally not unique—for any observable. Consider, for example, Eq. (87). Using the following identity [15]

$$\int_0^1 d\theta \theta^n (1-\theta)^m = B(n+1, m+1) = \frac{n!m!}{(n+m+1)!}, \tag{90}$$

along with

$$(m+n+1) = \left. \frac{d}{dt} t^{m+n+1} \right|_{t=1}, \tag{91}$$

one obtains the integral form for the inverse binomial coefficient

$$\binom{m+n}{m}^{-1} = \frac{d}{dt} \Big|_{t=1} t \int_0^1 d\theta (t\theta)^n (t(1-\theta))^m. \quad (92)$$

Then, the estimator (87) becomes

$$f_\phi(X_\phi|D(\alpha)) = \frac{d}{dt} \Big|_{t=1} t \int_0^1 d\theta \exp(-2X_\phi e^{i\phi} \alpha^* t\theta) \exp(2X_\phi e^{-i\phi} \alpha t(1-\theta)), \quad (93)$$

and for its spectral kernel one has

$$f_\phi(x|D(\alpha)) = \frac{d}{dx} x \int_0^1 d\theta \exp(-2xe^{i\phi} \alpha^* \theta) \exp(2xe^{-i\phi} \alpha(1-\theta)). \quad (94)$$

We can rewrite the estimator in form of expectation

$$f_\phi(x|D(\alpha)) = \frac{d}{dx} x \int_0^1 d\theta \langle 2xe^{i\phi}(1-\theta) | e^{\alpha a^\dagger} e^{-\alpha^* a} | 2xe^{i\phi}\theta \rangle, \quad (95)$$

where the vectors are coherent states. Eq. (95) corresponds to the functional form

$$f_\phi(x|D(\alpha)) = \text{Tr}[W_{\phi,x}:D(\alpha):], \quad (96)$$

where

$$W_{\phi,x} = \frac{d}{dx} x \int_0^1 d\theta |2xe^{i\phi}\theta\rangle \langle 2xe^{i\phi}(1-\theta)|. \quad (97)$$

In order to give an expectation rule corresponding to Eq. (81), we can proceed as follows. From the integral representation [15]

$$H_n(x) = (-2i)^n \int_{-\infty}^{\infty} \frac{dt}{\sqrt{\pi}} e^{-(t-ix)^2} t^n, \quad (98)$$

using identity (92), Eq. (81) for normal ordering rewrites as follows

$$\begin{aligned} a^{\dagger m} a^n &= \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-i2X_\phi)^2} \frac{d}{dt} t \int_0^1 d\theta (-i\theta t e^{i\phi})^n [-i(1-\theta)t e^{-i\phi}]^m \\ &= \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-i2X_\phi)^2} \\ &\quad \times \frac{d}{dt} t \int_0^1 d\theta \text{Tr}[a^{\dagger m} a^n | -i\theta t e^{i\phi} \rangle \langle i(1-\theta)t e^{i\phi} |] e^{\frac{t^2}{2}}, \end{aligned} \quad (99)$$

where we have used matrix elements on coherent states

$$\langle \beta | a^{\dagger m} a^n | \alpha \rangle = \alpha^n \beta^* m e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha\beta^*)}. \quad (100)$$

Eq. (99) is equivalent to the following expansion for operators X admitting normal ordered form

$$\begin{aligned} X &= \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dt \text{Tr}[G^\dagger(t, \phi) X] F(t, \phi), \\ F(t, \phi) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-i2X_\phi)^2}, \quad G(t, \phi) = \frac{d}{dt} t e^{\frac{t^2}{2}} \int_0^1 d\theta |i(1-\theta)t e^{i\phi} \rangle \langle -i\theta t e^{i\phi} |. \end{aligned} \quad (101)$$

7. CANONICAL DUAL FOR HOMODYNE TOMOGRAPHY

The frame theory approach to quantum homodyne tomography gives further insight to the structure of the *quorum* of quadrature observables. Given a set of vectors $|v_n\rangle$ in a Hilbert space, if the positive operator $F = \sum_n |v_n\rangle \langle v_n|$ is invertible, then the scalar product between two arbitrary vectors can be written as

$$\langle \psi | \eta \rangle = \sum_n \langle \psi | u_n \rangle \langle v_n | \eta \rangle, \quad (102)$$

where the set of vectors $\{u_n \equiv F^{-1}|v_n\rangle\}$ is called ‘‘canonical dual’’ of the set $\{|v_n\rangle\}$, and F is denoted as ‘‘frame operator’’. In other words, the set $\{|v_n\rangle\}$, along with its dual $\{|u_n\rangle\}$, is a spanning

set for the Hilbert space, and provide a generalized resolution of the identity. In this section we show that the set of (generalized) projectors $|x\rangle_{\phi\phi}\langle x|$ over the quadratures X_{ϕ} give a frame when varying ϕ , and the expansion for trace-class operators in Eq. (28) corresponds to using the canonical dual for the estimation rule.

In the following, we will make extensive use of the isomorphism between the Hilbert space of the Hilbert-Schmidt operators A, B on \mathbb{H} , with scalar product $\langle A, B \rangle = \text{Tr}[A^{\dagger}B]$, and the Hilbert space of bipartite vectors $|A\rangle, |B\rangle \in \mathbb{H} \otimes \mathbb{H}$, with $\langle\langle A|B\rangle\rangle \equiv \langle A, B \rangle$, and

$$|A\rangle\rangle = \sum_n \sum_m A_{nm} |n\rangle \otimes |m\rangle, \quad (103)$$

where $|n\rangle$ and $|m\rangle$ are fixed orthonormal bases for \mathbb{H} , and $A_{nm} = \langle n|A|m\rangle$. Notice the identities

$$A \otimes B |C\rangle\rangle = |ACB^{\tau}\rangle\rangle, \quad A \otimes B^{\dagger} |C\rangle\rangle = |ACB^*\rangle\rangle, \quad (104)$$

where τ and $*$ denote transposition and complex conjugation with respect to the fixed bases in Eqs. (103).

By taking $\langle n|x\rangle_0$ as real, in the $|\rangle\rangle$ notation the spanning set $|x\rangle_{\phi\phi}\langle x|$ corresponds to the vectors on $\mathbb{H}_a \otimes \mathbb{H}_b$ of modes a and b

$$|(|x\rangle_{\phi\phi}\langle x|)\rangle\rangle = e^{i\phi(a^{\dagger}a - b^{\dagger}b)} |x\rangle_0 |x\rangle_0. \quad (105)$$

Notice the identities

$$\langle\langle D(z)|x\rangle_0 |x\rangle_0 = \exp(2ix\text{Im}z) \delta(\text{Re}z), \quad (106)$$

and

$$e^{i\phi(a^{\dagger}a - b^{\dagger}b)} |D(z)\rangle\rangle = |D(ze^{i\phi})\rangle\rangle. \quad (107)$$

The frame operator can be evaluated as follows

$$\begin{aligned} F &= \int_0^{\pi} \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dx e^{i\phi(a^{\dagger}a - b^{\dagger}b)} |x\rangle_0 |x\rangle_0 \langle x|_0 \langle x|_0 e^{-i\phi(a^{\dagger}a - b^{\dagger}b)} \\ &= \int_0^{\pi} \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dx \int \frac{d^2 z}{\pi} \int \frac{d^2 w}{\pi} |D(z)\rangle\rangle \langle\langle D(ze^{-i\phi})|x\rangle_0 |x\rangle_0 \langle x|_0 \langle x|_0 D(we^{-i\phi})\rangle\rangle \langle\langle D(w)| \\ &= \int_0^{\pi} \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dx \int \frac{d^2 z}{\pi} \int \frac{d^2 w}{\pi} |D(ze^{i\phi})\rangle\rangle \langle\langle D(we^{i\phi})|e^{2ix\text{Im}z - 2ix\text{Im}w} \delta(\text{Re}z) \delta(\text{Re}w) \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dt |D(ite^{i\phi})\rangle\rangle \langle\langle D(ite^{i\phi})| = \frac{1}{\pi} \int \frac{d^2 z}{\pi} \frac{1}{|z|} |D(z)\rangle\rangle \langle\langle D(z)| \\ &= \frac{1}{\pi|a - b^{\dagger}|}, \end{aligned} \quad (108)$$

where we used the eigenvalue equation $(a - b^{\dagger})|D(z)\rangle\rangle = z|D(z)\rangle\rangle$. The inverse of F is simply given by

$$F^{-1} = \pi|a - b^{\dagger}| = \int d^2 z |z| |D(z)\rangle\rangle \langle\langle D(z)|. \quad (109)$$

The canonical dual is obtained as follows

$$\begin{aligned} F^{-1}|(|x\rangle_{\phi\phi}\langle x|)\rangle\rangle &= \int d^2 z |z| |D(z)\rangle\rangle \langle\langle D(z)|x\rangle_{\phi}\langle x|_{\phi} \\ &= \int d^2 z |z| |D(ze^{i\phi})\rangle\rangle e^{2ix\text{Im}z} \delta(\text{Re}z) = \int_{-\infty}^{\infty} dk |k| |D(ike^{i\phi})\rangle\rangle e^{2ikx} \\ &= \frac{1}{4} \int_{-\infty}^{\infty} dk |k| |e^{ik(X_{\phi} - x)}\rangle\rangle. \end{aligned} \quad (110)$$

Hence, it follows that the usual Kernel operator corresponds to the canonical dual.

7.1. Alternate dual frames. The dual of the quadrature projectors is not unique. However, the formula of Li [11] for characterize all possible alternate duals for bounded frames and discrete indexes cannot provide any new dual set. By denoting the frame as $\{|\Xi(x, \phi)\rangle\rangle\}$ with $\Xi(x, \phi) = \delta(X_\phi - x)$, such a formula can be formally written in the form

$$|\Theta(x, \phi)\rangle\rangle = F^{-1}|\Xi(x, \phi)\rangle\rangle + |f(x, \phi)\rangle\rangle - \int_0^\pi \frac{d\phi'}{\pi} \int_{-\infty}^\infty dx' \langle\langle \Xi(x', \phi') | F^{-1} |\Xi(x, \phi)\rangle\rangle |f(x', \phi')\rangle\rangle, \quad (111)$$

where $\{F^{-1}|\Xi(x, \phi)\rangle\rangle\}$ is the canonical dual, and $\{f(x, \phi)\}$ is an arbitrary Bessel set, namely

$$\int_{-\infty}^\infty dx \int_0^\pi \frac{d\phi}{\pi} |f(x, \phi)|^2 \leq \infty. \quad (112)$$

The scalar product that appears in the integral of Eq. (111) writes

$$\begin{aligned} & \langle\langle \Xi(x', \phi') | F^{-1} |\Xi(x, \phi)\rangle\rangle \\ &= \int d^2z |z\rangle_0 \langle x' | {}_0 \langle x' | |D(z)\rangle\rangle \langle\langle D(ze^{i(\psi-\phi)}) | x\rangle_0 | x\rangle_0 \\ &= \int d^2z |z| e^{-i\text{Im}z(\text{Re}z+2x')} \delta(\text{Re}z) e^{i\text{Im}(ze^{i(\psi-\phi)})(\text{Re}(ze^{i(\psi-\phi)}+2x))} \delta(\text{Re}(ze^{i(\psi-\phi)})) \\ &= \int_{-\infty}^\infty dk |k| e^{2ik[x \cos(\psi-\phi) - x']} \delta(k \sin(\psi - \phi)) \\ &= \int_{-\infty}^\infty dk e^{2ik[x \cos(\psi-\phi) - x']} \delta_\pi(\psi - \phi) \\ &= \pi \delta(x - x') \delta_\pi(\psi - \phi). \end{aligned} \quad (114)$$

This bi-orthogonality relation implies that the formula (111) cannot reveal any new dual set.

8. FRAMES OF NORMAL-ORDERED MOMENTS

In this Section, by simply applying the frame theory, we recover some results of Refs. [16, 17], where the set of normally ordered moments $\{a^{\dagger k} a^l\}$ is shown to be complete, and related to a biorthogonal set given on the basis of Fock states.

From the set $\{a^{\dagger k} a^l\}$ we immediately write the frame operator

$$\tilde{F} = \sum_{k,l=0}^\infty |a^{\dagger k} a^l\rangle\rangle \langle\langle a^{\dagger k} a^l|. \quad (115)$$

On the Fock basis one has

$$\begin{aligned} \tilde{F} &= \sum_{k,l,n,j=0}^\infty \frac{\sqrt{(k+n)!(l+n)!(k+j)!(l+j)!}}{n!j!} |k+n\rangle |l+n\rangle \langle k+j| \langle l+j| \\ &= \sum_{n=0}^\infty \frac{a^n b^n}{n!} \left(\sum_{k,l=0}^\infty k!l! |k\rangle \langle k| \otimes |l\rangle \langle l| \right) \sum_{j=0}^\infty \frac{a^{\dagger j} b^{\dagger j}}{j!} \\ &= e^{ab} (a^\dagger a! \otimes b^\dagger b!) e^{a^\dagger b^\dagger}. \end{aligned} \quad (116)$$

The inverse of \tilde{F} simply writes

$$\tilde{F}^{-1} = e^{-a^\dagger b^\dagger} \left(\frac{1}{a^\dagger a!} \otimes \frac{1}{b^\dagger b!} \right) e^{-ab}. \quad (117)$$

The dual set $g_{k,l}$ can be obtained as follows

$$\begin{aligned}
|g_{k,l}\rangle &\equiv \tilde{F}^{-1}|a^{\dagger k}a^l\rangle = \sum_{n,m,t,j=0}^{\infty} |n\rangle|m\rangle \frac{(-1)^{t+j}}{t!j!} \langle n|\langle m|a^t b^t \frac{1}{a^\dagger a!} \otimes \frac{1}{b^\dagger b!} a^{\dagger j} b^{\dagger j} |a^{\dagger k}a^l\rangle \\
&= \sum_{n,m,t,j,s=0}^{\infty} |n\rangle|m\rangle \frac{(-1)^{t+j}}{t!j!} \langle n+t|k+j+s\rangle \langle m+t|l+j+s\rangle \frac{\sqrt{(k+j+s)!(l+j+s)!}}{s!\sqrt{n!m!(n+t)!(m+t)!}} \\
&= \sum_{n,m,t,j,s=0}^{\infty} |n\rangle|m\rangle \frac{(-1)^{t+j}}{t!j!} \delta_{n+t,k+j+s} \delta_{m+t,l+j+s} \frac{\sqrt{(k+j+s)!(l+j+s)!}}{s!\sqrt{n!m!(n+t)!(m+t)!}} \\
&= \sum_{n,m,t=0}^{\infty} \frac{1}{\sqrt{n!m!}} |n\rangle|m\rangle \frac{(-1)^t}{t!} \delta_{m,l+n-k} \frac{1}{(n+t-k)!} \sum_{j=0}^{n+t-k} \frac{(-1)^j}{j!} \frac{(n+t-k)!}{(n+t-k-j)!} \\
&= \sum_{n,m,t=0}^{\infty} \frac{1}{\sqrt{n!m!}} |n\rangle|m\rangle \frac{(-1)^t}{t!} \delta_{n,k-t} \delta_{m,l-t} \frac{1}{(n+t-k)!} \\
&= \sum_{t=0}^{\min(k,l)} \frac{(-1)^t}{t!\sqrt{(k-t)!(l-t)!}} |k-t\rangle|l-t\rangle.
\end{aligned} \tag{118}$$

The dual set is then given by

$$g_{k,l} = \sum_{t=0}^{\min(k,l)} \frac{(-1)^t}{t!\sqrt{(k-t)!(l-t)!}} |k-t\rangle\langle l-t| = \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} a^{\dagger k-t}|0\rangle\langle 0|a^{l-t}. \tag{119}$$

The dual set is unique, and in fact one has the biorthogonal relation

$$\text{Tr}[g_{k',l'}^\dagger a^{\dagger k}a^l] = \delta_{k,k'} \delta_{l,l'}. \tag{120}$$

9. FRAME OF MOMENTS VERSUS QUADRATURE DISTRIBUTION

The frame of moments allows to recover the estimation rule for unbounded operators as an expectation rule with a dual operator of the quadrature projectors $|x\rangle_{\phi\phi}\langle x|$. In other words, by inspecting Eq. (81), we would like to write an operator $G(x, \phi)$ such that

$$\text{Tr}[G^\dagger(x, \phi)a^{\dagger n}a^m] = \binom{n+m}{n}^{-1} \frac{1}{\sqrt{2^{n+m}}} H_{n+m}(\sqrt{2}x) e^{i\phi(m-n)}. \tag{121}$$

In this case, the operator $G(x, \phi)$ is a dual of the quadrature projectors $|x\rangle_{\phi\phi}\langle x|$, but is different from the canonical dual (110), which is divergent for unbounded operators. One has

$$\begin{aligned}
|G^\dagger(x, \phi)\rangle &= \tilde{F}^{-1}\tilde{F}|G^\dagger(x, \phi)\rangle = \tilde{F}^{-1} \sum_{k,l=0}^{\infty} |a^{\dagger k}a^l\rangle \langle\langle a^{\dagger k}a^l || G^\dagger(x, \phi)\rangle\rangle \\
&= \sum_{k,l=0}^{\infty} |g_{k,l}\rangle \text{Tr}[a^{\dagger l}a^k G^\dagger(x, \phi)].
\end{aligned} \tag{122}$$

Then we obtain

$$G^\dagger(x, \phi) = e^{i\phi a^\dagger a} G^\dagger(x, 0) e^{-i\phi a^\dagger a}, \tag{123}$$

with

$$\begin{aligned}
G^\dagger(x, 0) &= \sum_{k,l=0}^{\infty} g_{k,l} \binom{k+l}{k}^{-1} \frac{1}{\sqrt{2^{k+l}}} H_{k+l}(\sqrt{2}x) \\
&= \sum_{k,l=0}^{\infty} \sum_{t=0}^{\min(k,l)} \frac{(-1)^t}{t! \sqrt{k!l!}} a^t |k\rangle \langle l| a^{\dagger t} \binom{k+l}{k}^{-1} \frac{1}{\sqrt{2^{k+l}}} H_{k+l}(\sqrt{2}x) \\
&= \sum_{k,l=0}^{\infty} \sum_{t=0}^{\min(k,l)} \frac{(-1)^t}{t! \sqrt{(k-t)!(l-t)!}} |k-t\rangle \langle l-t| \binom{k+l}{k}^{-1} \frac{1}{\sqrt{2^{k+l}}} H_{k+l}(\sqrt{2}x).
\end{aligned} \tag{124}$$

The dual $G(x, \phi)$ provides also new pattern functions for the matrix elements, as previously noticed in Refs. [12, 18]. For example, for the vacuum component one has

$$\text{Tr}[G^\dagger(x, \phi)|0\rangle\langle 0|] = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \frac{k!}{(2k)!} H_{2k}(\sqrt{2}x). \tag{125}$$

Notice that such pattern functions are generally no longer bounded for $x \rightarrow \pm\infty$, even for $\eta > 0.5$.

One can check that the set $G(x, \phi)$ is dual to the set of quadrature projectors $|x\rangle_{\phi\phi}\langle x|$ as follows

$$\begin{aligned}
&\int_{-\infty}^{+\infty} dx \int_0^\pi \frac{d\phi}{\pi} |\delta(X_\phi - x)\rangle \langle\langle G(x, \phi) | \\
&= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' \int_0^\pi \frac{d\phi}{\pi} ||x'\rangle_{\phi\phi}\langle x'| \rangle \langle\langle G(x, \phi) | \delta(x - x') \\
&= \int_0^\pi \frac{d\phi}{\pi} \sum_{k,l=0}^{\infty} \binom{k+l}{k}^{-1} \frac{1}{\sqrt{2^{k+l}}} e^{i\phi(k-l)} |H_{k+l}(\sqrt{2}X_\phi)\rangle \langle\langle g_{k,l}^\dagger | \\
&= \sum_{k,l=0}^{\infty} |a^{\dagger l} a^k\rangle \langle\langle g_{k,l}^\dagger | = \sum_{k,l=0}^{\infty} |a^{\dagger l} a^k\rangle \langle\langle g_{l,k} | = \tilde{F} \tilde{F}^{-1} = I \otimes I.
\end{aligned} \tag{126}$$

10. GENERATING NEW FRAMES

We can generate different frames by changing the function of $a - b^\dagger$ which gives the frame operator in Eq. (108). Explicitly, we have

$$\begin{aligned}
f(|a - b^\dagger|) &= \int \frac{d^2 z}{\pi} f(|z|) |D(z)\rangle \langle\langle D(z) | = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dt |t| f(|t|) |D(ite^{i\phi})\rangle \langle\langle D(ite^{i\phi}) | \\
&= \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\pi} \int d^2 z \int d^2 w |D(ze^{i\phi})\rangle \langle\langle D(we^{i\phi}) | \\
&\times g(|z|) g^*(|w|) e^{2ix\text{Im}z - 2ix\text{Im}w} \delta(\text{Re}z) \delta(\text{Re}w) \\
&= \pi \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dx \int \frac{d^2 z}{\pi} \int \frac{d^2 w}{\pi} |D(z)\rangle \langle\langle D(ze^{-i\phi}) | g(|a^\dagger - b|) |x\rangle_0 |x\rangle_0 \\
&\times {}_0\langle x|_0 \langle x| g^*(|a^\dagger - b|) |D(we^{-i\phi})\rangle \langle\langle D(w) | \\
&= \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dx e^{i\phi(a^\dagger a - b^\dagger b)} \sqrt{\pi} g(|a^\dagger - b|) |x\rangle_0 |x\rangle_0 {}_0\langle x|_0 \langle x| \sqrt{\pi} g^*(|a^\dagger - b|) e^{-i\phi(a^\dagger a - b^\dagger b)},
\end{aligned} \tag{127}$$

where the functions g and f are related as follows

$$|t|f(|t|) = |g(t)|^2. \tag{128}$$

Using the identity

$$(a - b^\dagger)|A\rangle = (|aA\rangle - |Aa^*\rangle) = |[a, A]\rangle, \tag{129}$$

for reference basis such that $a^* \equiv a$, e. g. the photon number basis, we have

$$|a - b^\dagger|^2 |A\rangle\rangle = (a^\dagger - b) |a, A\rangle\rangle = |[a^\dagger, [a, A]]\rangle\rangle. \quad (130)$$

But the double commutator can be written in terms of the (dual) Lindblad super-operator

$$[a^\dagger, [a, A]] = -(\mathcal{L}[a] + \mathcal{L}[a^\dagger])A, \quad (131)$$

where $L[W]A \doteq W^\dagger AW - \frac{1}{2}(W^\dagger WA + AW^\dagger W)$. Remarkably, this is exactly the dissipative super-operator of the displacement Gaussian noise, corresponding to a distributed loss compensated by a phase-insensitive amplification.

With the aid of the following commutator rule

$$[a^\dagger, [a, e^{i\lambda X}]] = \frac{1}{4}\lambda^2 e^{i\lambda X}, \quad (132)$$

we easily obtain

$$|a^\dagger - b|^2 |x\rangle_0 |x\rangle_0 = |a - b^\dagger|^2 |x\rangle_0 |x\rangle_0 = -\frac{1}{4}\partial_x^2 |\delta(x - X)\rangle\rangle = |\mathcal{F}[\frac{1}{4}\lambda^2](X - x)\rangle\rangle, \quad (133)$$

where \mathcal{F} denotes the Fourier transform

$$\mathcal{F}[f](x) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda x} f(\lambda). \quad (134)$$

Therefore, we have

$$g(|a^\dagger - b|)|x\rangle_0 |x\rangle_0 = |\mathcal{F}[g(\frac{1}{2}\lambda)](X - x)\rangle\rangle, \quad (135)$$

corresponding to the frame

$$\Xi(x, \phi) = \sqrt{\pi}\mathcal{F}[g(\frac{1}{2}\bullet)](X_\phi - x). \quad (136)$$

For example, if we choose $g(x) = \sqrt{\frac{\sigma^2}{2\pi}} \exp\left(-\frac{\sigma^2}{2}x^2\right)$, we have

$$\Xi(x, \phi) = \sqrt{2\pi} \exp\left[-\frac{1}{\sigma^2}(X_\phi - x)^2\right]. \quad (137)$$

Notice that a function

$$\Xi(x, \phi) = h(X_\phi - x) \quad (138)$$

corresponds to a frame if the function h has Fourier transform which is invertible and bounded. Moreover two functions h and h' will correspond to the same frame operator if their Fourier transform have the same module. More precisely, the frame operator will be given by

$$F = f(|a - b^\dagger|), \quad f(t) = \frac{1}{\pi t} |\mathcal{F}^{-1}[h](2t)|^2. \quad (139)$$

The canonical dual frame operator can be obtained by inverting the frame operator

$$\begin{aligned} F^{-1}(|x\rangle_\phi \langle x|) &= \int \frac{d^2 z}{\pi} \frac{1}{f(|z|)} |D(z)\rangle\rangle \langle\langle D(z)|x\rangle_\phi \langle x|_\phi \\ &= \int \frac{d^2 z}{\pi} \frac{1}{f(|z|)} |D(ze^{i\phi})\rangle\rangle e^{2ix\text{Im}z} \delta(\text{Re}z) = \int_{-\infty}^{\infty} dk \frac{1}{f(|k|)} |D(-ike^{i\phi})\rangle\rangle e^{2ikx} \\ &= \pi \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{f(|k|/2)} |e^{-ikX_\phi}\rangle\rangle e^{ikx} = |h^\vee(x - X_\phi)\rangle\rangle, \end{aligned} \quad (140)$$

where

$$h^\vee(x) = \pi \mathcal{F}\left[\frac{1}{f(|k|/2)}\right](x). \quad (141)$$

Hence, it follows that the usual Kernel operator corresponds to the canonical dual.

11. OTHER FRAMES

Using frame calculus, it is easy to show that the following sets of operators are spanning sets

$$\begin{aligned} A_{n,\phi} &= e^{-\frac{1}{2}X_\phi^2} X_\phi^n \\ B_{n,\phi} &= \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} e^{-X_\phi^2} H_n(\sqrt{2}X_\phi), \end{aligned} \quad (142)$$

with corresponding frame operators

$$\begin{aligned} \int_0^\pi \frac{d\phi}{\pi} \sum_{n=0}^\infty |A_{n,\phi}\rangle\langle A_{n,\phi}| &= \frac{e^{-|Z|^2}}{|Z|}, \\ \int_0^\pi \frac{d\phi}{\pi} \sum_{n=0}^\infty |B_{n,\phi}\rangle\langle B_{n,\phi}| &= \frac{1}{\pi|Z|}, \end{aligned} \quad (143)$$

where $Z = a - b^\dagger$.

APPENDIX

11.1. **Proof of Lemma 4.** Consider the Poisson form of the Dirac delta for the 2π -interval

$$\delta_{2\pi}(\phi) = \lim_{\epsilon \rightarrow 1^-} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \epsilon^{|n|} e^{in\phi}. \quad (144)$$

In the following we will use ϵ to mean $\epsilon = 1_-$. Rescaling ϕ by a factor 2 we obtain

$$\delta_\pi(\phi) = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \epsilon^{|n|} e^{i2n\phi} \equiv \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \epsilon^{2|n|} e^{i2n\phi} \doteq \delta_\pi^{(+)}(\phi), \quad (145)$$

which is also equivalent to the *even folding* relation

$$\delta_\pi^{(+)}(\phi) = \delta_{2\pi}(\phi) + \delta_{2\pi}(\phi + \pi), \quad (146)$$

since

$$\begin{aligned} \delta_{2\pi}(\phi) + \delta_{2\pi}(\phi + \pi) &= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \epsilon^{|n|} \left[e^{in\phi} + (-)^n e^{in\phi} \right] \\ &= \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \epsilon^{2|n|} e^{i2n\phi} \end{aligned} \quad (147)$$

On the other hand, we have the *odd folding* relation

$$e^{i\phi} \delta_\pi^{(-)}(\phi) = \delta_{2\pi}(\phi) - \delta_{2\pi}(\phi + \pi) \quad (148)$$

since

$$\begin{aligned} \delta_{2\pi}(\phi) - \delta_{2\pi}(\phi + \pi) &= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \epsilon^{|n|} \left[e^{in\phi} - (-)^n e^{in\phi} \right] \\ &= \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \epsilon^{|2n+1|} e^{i(2n+1)\phi} \doteq e^{i\phi} \delta_\pi^{(-)}(\phi) \end{aligned} \quad (149)$$

From the equivalence relations (38), we immediately derive the equivalence

$$\begin{aligned} x^{2k} \delta_\pi(\phi) &\simeq \frac{1}{\pi} x^{2k} \sum_{n=-k}^k \epsilon^{|n|} e^{2in\phi} = \frac{1}{\pi} x^{2k} \left[\frac{1 - (\epsilon e^{2i\phi})^{k+1}}{1 - \epsilon e^{2i\phi}} + \frac{1 - (\epsilon e^{-2i\phi})^{k+1}}{1 - \epsilon e^{-2i\phi}} - 1 \right] \\ &= \frac{1}{\pi} x^{2k} \left[\frac{(\epsilon e^{2i\phi})^{k+1}}{\epsilon e^{2i\phi} - 1} + \frac{(\epsilon e^{-2i\phi})^{k+1}}{\epsilon e^{-2i\phi} - 1} + \kappa_\epsilon(\phi) \right], \end{aligned} \quad (150)$$

where the distribution

$$\kappa_\epsilon(\phi) \doteq \frac{1 - \epsilon^2}{1 + \epsilon^2 - \epsilon(e^{2i\phi} + e^{-2i\phi})} \quad (151)$$

with support in $\phi = 0$ gives

$$\begin{aligned} \int_0^\pi \frac{d\phi}{\pi} \kappa_\epsilon(\phi) e^{i2n\phi} &= \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{1 - \epsilon^2}{1 + \epsilon^2 - \epsilon(e^{i\phi} + e^{-i\phi})} e^{in\phi} \\ &= \frac{1}{2\pi i} \oint \frac{dz}{z} \frac{1 - \epsilon^2}{(1 - \epsilon z)(1 - \epsilon z^{-1})} z^n = \frac{1}{2\pi i} \oint dz \frac{\epsilon - \epsilon^{-1}}{(z - \epsilon^{-1})(z - \epsilon)} z^n =: \vartheta(n) \end{aligned} \quad (152)$$

where $\vartheta(n) = 1$ for $n \geq 0$ and $\vartheta(n) = 0$ for $n < 0$, and the integral is performed on the unit circle. Eq. (150), which contains identity (47) as a special case, generalizes as follows

$$f(x^2) \delta_\pi(\phi) \simeq \frac{1}{\pi} \left[\frac{f(\epsilon x^2 e^{2i\phi})}{1 - \epsilon^{-1} e^{-2i\phi}} + \frac{f(\epsilon x^2 e^{-2i\phi})}{1 - \epsilon^{-1} e^{2i\phi}} + \kappa_\epsilon(\phi) f(x^2) \right], \quad (153)$$

where $f(z)$ denotes any analytic function in z . In a similar way we calculate

$$\begin{aligned} x^{2k+1} e^{i\phi} \delta_\pi(\phi) &\simeq \frac{1}{\pi} x^{2k+1} \sum_{n=-k-1}^k \epsilon^{|n|} e^{i(2n+1)\phi} \\ &= \frac{1}{\pi} x^{2k+1} e^{i\phi} \left[\frac{1 - (\epsilon e^{2i\phi})^{k+1}}{1 - \epsilon e^{2i\phi}} + \frac{1 - (\epsilon e^{-2i\phi})^{k+2}}{1 - \epsilon e^{-2i\phi}} - 1 \right] \\ &= \frac{1}{\pi} x^{2k+1} e^{i\phi} \left[\frac{(\epsilon e^{2i\phi})^{k+1}}{\epsilon e^{2i\phi} - 1} + \frac{(\epsilon e^{-2i\phi})^{k+2}}{\epsilon e^{-2i\phi} - 1} + \kappa_\epsilon(\phi) \right], \end{aligned} \quad (154)$$

which generalizes as follows

$$x f(x^2) e^{i\phi} \delta_\pi(\phi) \simeq \frac{1}{\pi} x \left[\frac{e^{i\phi} f(\epsilon x^2 e^{2i\phi})}{1 - \epsilon^{-1} e^{-2i\phi}} + \frac{e^{-i\phi} f(\epsilon x^2 e^{-2i\phi})}{1 - \epsilon^{-1} e^{2i\phi}} + \kappa_\epsilon(\phi) e^{i\phi} f(x^2) \right]. \quad (155)$$

11.2. Alternative derivation of identity (87). By posing $\alpha = \frac{i}{2} r e^{i\phi}$, we have

$$\begin{aligned} D(\alpha) &= \int_{-\pi}^\pi \frac{d\phi}{2\pi} 2\pi \delta_{2\pi}(\phi - \phi) e^{irX_\phi} \\ &= \int_0^\pi \frac{d\phi}{\pi} \pi \left[\delta_{2\pi}(\phi - \phi) e^{irX_\phi} + \delta_{2\pi}(\phi - \phi - \pi) e^{-irX_\phi} \right] \\ &= \int_0^\pi \frac{d\phi}{\pi} \pi \left[\cos rX_\phi (\delta_{2\pi}(\phi - \phi) + \delta_{2\pi}(\phi - \phi - \pi)) \right. \\ &\quad \left. + i \sin rX_\phi (\delta_{2\pi}(\phi - \phi) - \delta_{2\pi}(\phi - \phi - \pi)) \right] \\ &= \int_0^\pi \frac{d\phi}{\pi} \pi \left[\cos rX_\phi \delta_\pi^{(+)}(\phi - \phi) + i \sin rX_\phi \delta_\pi^{(-)}(\phi - \phi) \epsilon e^{i(\phi - \phi)} \right]. \end{aligned} \quad (156)$$

Now, we evaluate separately the cosine and the sine terms. In the following, we will denote $\psi = \phi - \phi$. The cosine term can be transformed as follows

$$\begin{aligned} \cos rX_\phi \pi \delta_\pi^{(+)}(\psi) &\simeq \sum_{k=0}^\infty \frac{(-)^k}{(2k)!} r^{2k} X_\phi^{2k} \sum_{n=-k}^k \epsilon^{2|n|} e^{2in\psi} \\ &= \sum_{k=0}^\infty \frac{(-)^k}{(2k)!} r^{2k} X_\phi^{2k} \left[\sum_{n=0}^k \epsilon^{2n} (e^{2in\psi} + e^{-2in\psi}) - 1 \right] \\ &= \sum_{n=0}^\infty \epsilon^{2n} (e^{2in\psi} + e^{-2in\psi}) \sum_{k=n}^\infty \frac{(-)^k}{(2k)!} r^{2k} X_\phi^{2k} - \cos(rX_\phi) \\ &= \sum_{n=0}^\infty \epsilon^{2n} (e^{2in\psi} + e^{-2in\psi} - \delta_{n0}) (-)^n r^{2n} X_\phi^{2n} \sum_{k=0}^\infty \frac{(-)^k r^{2k} X_\phi^{2k}}{(2k + 2n)!} \end{aligned} \quad (157)$$

On the other hand, the sine term transforms as follows

$$\begin{aligned}
& \sin r X_\phi \pi \delta_\pi^{(-)}(\psi) \epsilon e^{i\psi} \simeq \sum_{k=0}^{\infty} \frac{(-)^k r^{2k+1} X_\phi^{2k+1}}{(2k+1)!} \sum_{n=-k-1}^k \epsilon^{|2n+1|} e^{i(2n+1)\psi} \\
&= \sum_{k=0}^{\infty} \frac{(-)^k r^{2k+1} X_\phi^{2k+1}}{(2k+1)!} \left(\sum_{n=0}^k \epsilon^{2n+1} e^{i(2n+1)\psi} + \sum_{n=1}^{k+1} \epsilon^{2n-1} e^{-i(2n-1)\psi} \right) \\
&= \sum_{k=0}^{\infty} \frac{(-)^k r^{2k+1} X_\phi^{2k+1}}{(2k+1)!} \left(\sum_{n=0}^k \epsilon^{2n+1} e^{i(2n+1)\psi} + \sum_{n=0}^k \epsilon^{2n+1} e^{-i(2n+1)\psi} \right) \quad (158) \\
&= \sum_{n=0}^{\infty} \epsilon^{2n+1} \left(e^{i(2n+1)\psi} + e^{-i(2n+1)\psi} \right) \sum_{k=n}^{\infty} \frac{(-)^k r^{2k+1} X_\phi^{2k+1}}{(2k+1)!} \\
&= \sum_{n=0}^{\infty} \epsilon^{2n+1} \left(e^{i(2n+1)\psi} + e^{-i(2n+1)\psi} \right) r^{2n} X_\phi^{2n} \sum_{k=0}^{\infty} \frac{(-)^k r^{2k+1} X_\phi^{2k+1}}{(2k+2n+1)!}
\end{aligned}$$

It is convenient now to make the following substitutions

$$r = 2|\alpha|, \quad e^{-i\phi} = i \frac{\alpha^*}{|\alpha|}, \quad e^{i\psi} = i \frac{\alpha^*}{|\alpha|} e^{i\phi}, \quad e^{i\psi} r = 2i\alpha e^{i\phi}. \quad (159)$$

In this way, the cosine term becomes

$$\begin{aligned}
& \cos r X_\phi \pi \delta_\pi^{(+)}(\psi) \\
& \simeq \sum_{n=0}^{\infty} \left((\epsilon \alpha^*)^{2n} e^{i2n\phi} + (\epsilon \alpha)^{2n} e^{-i2n\phi} - \delta_{n0} \right) (2X_\phi)^{2n} \sum_{k=0}^{\infty} \frac{(-)^k (2|\alpha| X_\phi)^{2k}}{(2k+2n)!}, \quad (160)
\end{aligned}$$

and the sine term

$$\begin{aligned}
& i \sin r X_\phi \pi \delta_\pi^{(-)}(\psi) \epsilon e^{i\psi} \\
& \simeq \sum_{n=0}^{\infty} \left(-(\epsilon \alpha^*)^{2n+1} e^{i(2n+1)\phi} + (\epsilon \alpha)^{2n+1} e^{-i(2n+1)\phi} \right) (2X_\phi)^{2n+1} \sum_{k=0}^{\infty} \frac{(-)^k (2|\alpha| X_\phi)^{2k}}{(2k+2n+1)!}, \quad (161)
\end{aligned}$$

and putting the two terms together we have

$$D(\alpha) = \int_0^\pi \frac{d\phi}{\pi} f_\phi^\epsilon(X_\phi | D(\alpha)), \quad (162)$$

with

$$\begin{aligned}
& f_\phi^\epsilon(X_\phi | D(\alpha)) = \sum_{n=0}^{\infty} \left((-\epsilon \alpha^*)^n e^{in\phi} + (\epsilon \alpha)^n e^{-in\phi} - \delta_{n0} \right) (2X_\phi)^n \sum_{k=0}^{\infty} \frac{(-)^k (2|\alpha| X_\phi)^{2k}}{(2k+n)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(\epsilon^n e^{in\phi} (-\alpha^*)^{n+k} \alpha^k + \epsilon^n e^{-in\phi} (-\alpha^*)^k \alpha^{n+k} - \delta_{n0} \right) \frac{(2X_\phi)^{2k+n}}{(2k+n)!} \quad (163)
\end{aligned}$$

In the limit $\epsilon \rightarrow 1^-$ the last expression can be simplified using the reordering rule

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n+2k} (z^{n+k} w^k t^n + z^k w^{n+k} t^{-n} - \delta_{n0} z^k w^k) \\
&= \sum_{k=0}^{\infty} \sum_{h=k}^{\infty} a_{h+k} (z^h w^k t^{h-k} + z^k w^h t^{k-h} - \delta_{hk} z^k w^k) \\
&= \sum_{h=0}^{\infty} \sum_{k=0}^h a_{h+k} (tz)^h (t^{-1}w)^k + \sum_{k=0}^{\infty} \sum_{h=k}^{\infty} a_{h+k} (tz)^k (t^{-1}w)^h - \sum_{h=0}^{\infty} a_{2h} z^h w^h \\
&= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} a_{h+k} (tz)^h (t^{-1}w)^k,
\end{aligned} \tag{164}$$

corresponding to

$$f_{\phi}(X_{\phi}|D(\alpha)) = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-2X_{\phi}e^{i\phi}\alpha^*)^h (2X_{\phi}e^{-i\phi}\alpha)^k}{(h+k)!}. \tag{165}$$

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