

Nonlinear Two-Mode Squeezed Vacuum States as Realization of $SU(1,1)$ Lie Algebra

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Abstract

Nonlinear extensions of the two-mode squeezed vacuum states (NTMSVS's) are constructed and special cases of these states are discussed. We have constructed the NTMSVS's realization of $SU(1,1)$ Lie algebra. Two cases of the definition are considered for unitary and non-unitary deformation operator functions. Some nonclassical properties of these states are discussed.

Key Words: Non-classical field states, two-mode squeezed vacuum states, nonlinear two-mode coherent states, nonlinear pair coherent states, squeezing, two-mode $SU(1,1)$ lie algebra coherent states.

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1. Introduction

The theoretical analysis [1–7] and experimental [8–12] realization of squeezed states of radiation continue to receive a great deal of attention, due to their potential applications in quantum optical communication, gravitational wave detection, and mesoscopic electric circuit quantization. While much of the work so far has concerned itself with single-mode situations [5, 10, 13] some analysis of two-mode states has also been presented [14–16]. A general invariant squeezing criterion for n -mode systems has been developed, and squeezing transformations based on symplectic group $Sp(4, \mathbb{R})$ have been investigated [17]. The two-mode squeezed state (TMSS) generated from a parametric down-conversion amplifier [18], composing of an idler mode and a signal mode, is an entangled state in the frequency domain.

Multi-mode squeezed states [19] have gained much attention due to the fact that they contain quantum correlations between the different modes that make up the field [20]. A specific case of such states is the TMSS, which play a central role because they can be readily produced from reliable sources and controlled experimentally using accessible sets of operations such as beam splitters, phase shifters and squeezers, and efficient detection systems [21]. The practical implementation of the quantum teleportation for continuous variable state has been realized experimentally using the TMSS [22]. The TMSS, which was used in experiments on quantum teleportation [23], is a fundamental entanglement resource in continuous variable system [24, 25]. A TMSS is an entangled Gaussian continuous variable state that becomes maximally entangled when it is infinitely squeezed [26]. The transmission through the ideal channel with assistance of TMSS, employing encoding based on phase space displacement operators and separable (unentangled) measurement with homodyne detection has been considered [23]. Deterministic signal security communications (cryptography) schemes using TMSS have been proposed [27]. The security evaluation of the quantum key distribution

system with TMSS has been demonstrated [25] and investigated based on the increase in error probability [27]. A test of nonlocality for continuous variable using a TMSS as the source of nonlocal correlations and a measurement scheme based on conditional homodyne detection has been discussed [24].

Squeezed vacuum state (SVS) is important because it can be readily generated using a parametric down-converter. It has less noise in one quadrature than a coherent state. The nonlinear squeezed vacuum state (NSVS) has been introduced and investigated [28–31]. The f-analogies of the squeezed vacuum state, using properties of the inverses of annihilation and creation operators of the f-oscillator, have been constructed [28] and the Pegg-Barnett phase formalism discussed. The NTMSVS has been studied [29]. The nonlinear extension of the single-mode SVS and some of their statistical properties have been considered [30]. Nonlinear squeezing of the vacuum and one-photon states as realizations of the SU(1,1) group have been constructed and some of their properties investigated [31]. The nonlinear single-mode squeezed coherent states (NLSCS's) have been studied [32, 33].

The SU(1,1) coherent states (CS's) as defined by Perelomov [34] have previously been discussed in connection with squeezed states of a single-mode field [35] and are special case of the two-photon CS's of Yuen [2], namely, the SVS [9–11]. The particular realization of the SU(1,1) Lie algebra required for these states involves bilinear and quadratic products of field mode. Such states may be produced out of vacuum by a degenerate parametric amplifier whose Hamiltonian is linear in the SU(1,1) generators. However, it has long been known that the Lie algebra of SU(1,1) can be realized in terms of bilinear products of the annihilation and creation operators of two boson modes [36–43]. The relevant unitary irreducible representations, however, are different from those related to the case of the single-mode field. In fact a special case of these states, known as the two-mode SVS's have been much studied and may be produced by a nondegenerate parametric amplifier acting on a two-mode vacuum state [41, 42].

The SU(1,1) Lie algebra is spanned by the three generators K_1, K_2, K_3 :

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2. \quad (1.1)$$

It is convenient to use the raising and lowering generators $K_{\pm} = K_1 \pm iK_2$, which satisfy

$$[K_3, K_{\pm}] = \pm K_{\pm}, \quad [K_-, K_+] = 2K_3. \quad (1.2)$$

The Casimir operator $K^2 = K_3^2 - K_1^2 - K_2^2$ for any irreducible representation is given by $K^2 = k(k-1)I$ where k is constant. Thus a representation of SU(1,1) is determined by the parameter k which is the so-called Bargmann index. For SU(1,1) there are many unitary irreducible representations, and because SU(1,1) is a noncompact group, they are all of infinite dimensions. Some of the representations are, in fact, continuous but here we shall only deal with the representations known as the positive discrete series for which the operator K_3 is diagonal and has a discrete spectrum. Its discrete representation satisfies the following relation:

$$\begin{aligned} K_+|n, k\rangle &= \sqrt{(n+1)(2k+n)}|n+1, k\rangle, \\ K_-|n, k\rangle &= \sqrt{n(2k+n-1)}|n-1, k\rangle, \\ K_3|n, k\rangle &= (n+k)|n, k\rangle, \end{aligned} \quad (1.3)$$

where $(n = 0, 1, 2, \dots)$. The ground state (the cyclic vector) of the representation is given by the condition $K_-|0, k\rangle = 0$.

The Perelomov coherent state (PCS) $|\alpha, k\rangle_{\text{Per}}$ for SU(1,1) Lie algebra may be obtained by applying the unitary operator $D_{\text{Per}}(\xi)$ to the ground state $|n = 0, k\rangle$ [34], that is

$$\begin{aligned} |\alpha, k\rangle_{\text{Per}} &= D_{\text{Per}}(\xi)|0, k\rangle \\ &= (1 - |\alpha|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{n!\Gamma(2k)}} \alpha^n |n, k\rangle, \end{aligned} \quad (1.4)$$

where $\xi = |\xi|e^{i\theta_0}$ is a complex number, with $\alpha = e^{i\theta_0} \tanh |\xi|$, and

$$\begin{aligned} D_{\text{Per}}(\xi) &= \exp(\xi K_+ - \xi^* K_-) \\ &= \exp(\alpha K_+)(1 - |\alpha|^2)^{K_3} \exp(-\alpha^* K_-) \end{aligned} \quad (1.5)$$

is the SU(1,1) displacement operator. The PCS's form an overcomplete set of states.

The aim of this article is to define the nonlinear two-mode squeezed vacuum states (NTMSVS's) and construct the realizations of SU(1,1) Lie algebra. The remainder of this paper is organized as follows. In section 2 we introduce the definition of NTMSVS's and some special cases are fully expressed. In section 3 we discuss realizations of SU(1,1) Lie algebra for these states.

2. Definition of nonlinear two-mode squeezed vacuum states

The coherent state satisfies the eigenvalue equation $a|\alpha\rangle = \alpha|\alpha\rangle$, with $\alpha = |\alpha| \exp(i\theta)$ and a is the annihilation operator for bosons. However, the coherent state $|\alpha\rangle$ parameterized by α can be cast as the result of the action of the displacement operator $D(\alpha)$ on the ground state $|0\rangle$, with $D(\alpha) = \exp(\alpha a^+ - \alpha^* a)$ [44], where a^+ is the creation operator for bosons being the hermitian conjugate of a , and α^* the complex conjugate of α . The NLCS's $|\alpha\rangle_f$, are right-hand eigenstates of the product of the boson annihilation operator a and the operator valued function $f(a^+ a)$ of the number operator $N = a^+ a$. They satisfy $a f(N)|\alpha\rangle_f = \alpha|\alpha\rangle_f$, the nature of the nonlinearity depends on the choice of the function $f(N)$ [45].

The TMSVS is obtained by acting by the two-mode squeezing operator on a two-mode vacuum. In its number state representation, it takes the form

$$|z\rangle = S_{12}(z)|0_a, 0_b\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (e^{i\phi} \tanh r)^n |n, n\rangle \quad (2.1)$$

where

$$S_{12}(z) = \exp(z a^+ b^+ - z^* a b). \quad (2.2)$$

Here, a and b are standard annihilation operators, i.e., $[a, a^+] = 1$, $[b, b^+] = 1$, with $z = r \exp(i\phi)$.

In this section we extend the investigation to the NTMSVS's. We first start with the case when the operators valued functions are unitary.

2.1. The definition for unitary nonlinear functions

The annihilation and creation for two-mode f-oscillators are defined [29] as

$$a_f = a f_1(N_a) \quad \text{and} \quad b_f = b f_2(N_b), \quad (2.3)$$

where the following commutation relations hold

$$[a_f, N_a] = a_f, \quad [b_f, N_b] = b_f, \quad [a_f^+, N_a] = -a_f^+, \quad [b_f^+, N_b] = -b_f^+ \quad (2.4)$$

and

$$[a_f, a_f^+] = (N_a + 1) f(N_a + 1) f^+(N_a + 1) - N_a f^+(N_a) f(N_a) \quad (2.5)$$

where $N_a = a^+ a$ and $N_b = b^+ b$ are number operators in the mode a and mode b , respectively. Here, the nonlinear functions $f_1(N_a)$ and $f_2(N_b)$ are assumed unitary, i.e., $f_i^+ = f_i^{-1}$, $i = 1, 2$. Therefore, the corresponding squeezing operator is written as

$$S_{12}(z, f_1, f_2) = \exp(z a_f^+ b_f^+ - z^* a_f b_f) \quad (2.6)$$

and satisfies the identities

$$S_{12}^{-1}(z, f_1, f_2) = S_{12}^+(z, f_1, f_2) = S_{12}(-z, f_1, f_2). \quad (2.7)$$

The NTMSVS's are defined by acting by the nonlinear two-mode squeezing operator on a two-mode vacuum:

$$|z\rangle_f = S_{12}(z, f_1, f_2)|0_a, 0_b\rangle. \quad (2.8)$$

Consequently the NTMSVS's are given by

$$|z\rangle_f = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (e^{i\phi} \tanh r)^n (f_1(n)!)(f_2(n)!)|n, n\rangle, \quad (2.9)$$

where $f(0) = 1$, and $f(n)! = \prod_{i=0}^n f(i)$.

2.2. The definition for non-unitary nonlinear functions

Even if the operator functions $f_1(N_a)$ and $f_2(N_b)$ are not unitary operator function one can still define a NTMSVS's in a similar way as that outlined in [28–33]. The steps towards this depends on using a canonical conjugate operators. If we have

$$a_f = (af(N_a)), \quad a_f^+ = ([f(N_a)]^+ a^+), \quad \text{and} \quad b_f = (bf(N_b)), \quad b_f^+ = ([f(N_b)]^+ b^+), \quad (2.10)$$

then the canonical conjugate operators are

$$A^+ = \frac{1}{f(N_a)} a^+, \quad A = a \frac{1}{[f(N_a)]^+}, \quad B^+ = \frac{1}{f(N_b)} b^+, \quad \text{and} \quad B = b \frac{1}{[f(N_b)]^+}. \quad (2.11)$$

The operators a_f, b_f, A and B satisfy the commutation relations

$$[a_f, A^+] = 1, \quad [A, a_f^+] = 1, \quad [b_f, B^+] = 1, \quad \text{and} \quad [B, b_f^+] = 1. \quad (2.12)$$

In what follows the operator valued function f_i is assumed to be a well-behaved real function. The use of the operators $a_f b_f$ and $A^+ B^+$ (instead of $a_f^+ b_f^+$) does not insure the operator $S_{12}(z)$ being unitary, thus one looks for the eigenfunctions of the operators

$$C_1 = \frac{1}{\sqrt{1 - |\xi_1|^2}} (a_f b_f - \xi_1 A^+ B^+) \quad \text{or} \quad C_2 = \frac{1}{\sqrt{1 - |\xi_2|^2}} (AB - \xi_2 a_f^+ b_f^+) \quad (2.13)$$

with the eigenvalue zero, i.e., the nonlinear squeezed vacuum states are the solutions of the equations

$$C_i |\Psi_i\rangle_f = 0, \quad i = 1, 2. \quad (2.14)$$

By assuming $|\Psi_i\rangle_f$ is the eigenfunction for the operator C_1 , in the form

$$|\Psi_1\rangle_f = \sum_{n=0}^{\infty} C_{n,n}(f_1, f_2) |n, n\rangle, \quad (2.15)$$

it is straightforward to find the expression

$$|\Psi_1\rangle_f = N_1 \sum_{n=0}^{\infty} (e^{i\phi} \tanh r)^n (f_1(n)!)(f_2(n)!)|n, n\rangle, \quad (2.16)$$

where $\xi = e^{i\phi} \tanh r$ and N_1 the normalization constant, and has the form

$$N_1^{-2} = \sum_{n=0}^{\infty} (\tanh r)^{2n} [(f_1(n)!)(f_2(n)!)]^2. \quad (2.17)$$

Generally, there is a number of NTMSVS's corresponding to various choices of the nonlinearity functions. Noting that the normalization constant in this case of the non-unitary nonlinear functions, depends on the values of these functions, that insures its boundedness.

3. NTMSVS as a realization of SU(1,1) Lie algebra

The physical quantities observed experimentally in many optical effects based on emission and absorption photons can be associated with the creation a^+ and annihilation a operators. Optical effects connected with the two-photon physics, are often related to the SU(1,1) Lie group [34–42]. It has been shown that the single- and two-mode bosonic realizations of the SU(1,1) Lie algebra have immediate relevance to the nonclassical squeezing properties of light [34–43].

3.1. The standard states

First we briefly review the realization two-mode standard case. The squeezed vacuum realization of the SU(1,1) Lie group is considered by taking the K -operators in the form

$$K_+ = a^+b^+, \quad K_- = ab, \quad K_3 = \frac{1}{2}(N_a + N_b + I). \quad (3.1)$$

In this case the boson number difference between the two modes is conserved. The dynamics of the nondegenerate parametric amplifier is described by (3.1) [13]. This two-mode representation of the SU(1,1) Lie algebra plays an important role in the description of linear dissipative processes [40].

The Casimir operator in this case becomes $K^2 = K_3^2 - \frac{1}{2}(K_+K_- + K_-K_+)$ equal $\frac{1}{4}[(N_a - N_b)^2 - I]$. Therefore, the irreducible representation with $k = \frac{(1+q)}{2}$, where $q = 0, 1, 2, \dots$, is the eigenvalue of the difference between the number of quanta in modes one and two, i.e., $N_a - N_b$, [42]. Here, $|m, k\rangle = |n+q, n\rangle$, and $n = 0, 1, 2, \dots$ [42]. The corresponding SU(1,1) Lie algebra CS or Perelomov CS for the two-mode field, written in terms of the two mode number states takes the form [42]

$$|\xi, k = \frac{(1+q)}{2}\rangle = \sum_{n=0}^{\infty} C_n^q |n+q, n\rangle, \quad (3.2a)$$

where

$$C_n^q = (1 - |\xi|^2)^{1/2(1+q)} \left[\frac{(n+q)!}{n!q!} \right]^{1/2} \xi^n. \quad (3.2b)$$

For the special case $q = 0$ this is the TMSVS. For q not equal 0 it is the state obtained by the action of the two-mode squeeze operator on the number state $|q, 0\rangle$.

The squeeze operator

$$S(z) = \exp(zK_+ - z^*K_-) = \exp(za^+b^+ - z^*ab) \quad (3.3)$$

is the unitary group operator for the two-photon realization, where K_+, K_-, K_3 are given by (3.1), with $\xi = \frac{z}{|z|} \tanh |z| = e^{i\phi} \tanh r$.

There is another coherent state of SU(1,1) which is known as the Barut-Girardello coherent states (BGCS's) [46]; it is defined as the eigenstate of the lowering operator K_- ,

$$K_- |\alpha, k\rangle_{BG} = \alpha |\alpha, k\rangle_{BG}, \quad (3.4)$$

and it can be expressed as

$$|\alpha, k\rangle_{BG} = \sqrt{\frac{|\alpha|^{2k-1}}{I_{2k-1}(2|\alpha|)}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n! \Gamma(n+2k)}} |n, k\rangle, \quad (3.5)$$

where $I_\nu(x)$ is the modified Bessel function of the first kind. The BGCS's are normalized but they are not orthogonal to each other. Note that the realization of BGCS's is the pair coherent states [47], or correlated SU(1,1) CS's [42]. These states may be produced by the action of a nondegenerate parametric amplifier on a two-mode state [42].

3.2. The nonlinear realization

In what follows we mention the NTMSVS's realization of the $SU(1,1)$ Lie algebra by constructing the $K-$ operators in the following way [14],

$$K_+ = [f_1(N_a)]^+ a^+ [f_2(N_b)]^+ b^+, \quad K_- = f_1(N_a) a f_2(N_b) b, \quad (3.6)$$

where the operator valued functions $f_1(N_a)$ and $f_2(N_b)$ are well behaved operator functions of the photon number operators N_a and N_b . For the operator K_3 to be in the form of (3.6), f_i must be a unitary operator $f_i^+ = f_i^{-1}$, $i = 1, 2$. Under this condition

$$K_3 = \frac{1}{2}(N_a + N_b + 1). \quad (3.7)$$

In this case the Perelomov CS which the action of the squeezing operator (3.3), with generators defined by (3.6), on the two-mode vacuum state are the NTSVS's. In other words, the Perelomov $SU(1,1)$ CS are itself the NTSVS's. But the BGCS's are the nonlinear pair coherent states studied in [48].

For non-unitary operator functions $f_1(N_a)$ and $f_2(N_b)$, there are two non-Hermitian two-mode realization of $SU(1,1)$ Lie algebra in terms of the conjugate operators as follows:

$$K_+ = A^+ B^+, \quad K_- = a_f b_f, \quad \text{and} \quad K_3 = \frac{1}{2}(N_a + N_b + 1) \quad (3.8)$$

and

$$\hat{K}_+ = a_f^+ b_f^+, \quad \hat{K}_- = AB, \quad \text{and} \quad \hat{K}_3 = \frac{1}{2}(N_a + N_b + 1). \quad (3.9)$$

Consequently, the NTSVS's are given by solutions of the eigenvalue equations

$$C_i |\Phi_i\rangle = 0, \quad i = 1, 2, \quad (3.10)$$

where C_i given is by (2.3). Carrying out the calculations, it is easy to find that these states are of the same form of (2.16).

The NTSVS's $|\Phi_i\rangle$ may be formulated as results of applications of exponential operators on the state $|0\rangle$. In effect it is easy to find that

$$|\Phi_1\rangle = N_1 e^{\frac{1}{2}\xi_1 A^+ B^+} |0\rangle, \quad |\Phi_2\rangle = \hat{N}_1 e^{\frac{1}{2}\xi_2 a_f^+ b_f^+} |0\rangle \quad (3.10)$$

where N_1 and \hat{N}_1 are the normalization constants.

However before we proceed any further it is necessary to specify the nonlinearity functions $f_1(n_1)$ and $f_2(n_2)$. From equation (2.16), it is clear that for every choice of $f_1(n_1)$ and $f_2(n_2)$ we shall get different NTSVS's states. In the present case, we choose the following nonlinearity functions which have been used in the description of the motion of a trapped ion [49], namely

$$f_i(n_i) = \frac{L_{n_i}^1(\eta_i^2)}{(n_i + 1)L_{n_i}^0(\eta_i^2)}, \quad i = 1, 2. \quad (3.11)$$

Here, η is known as the Lamb-Dicke parameter and $L_n^m(x)$ are associated Laguerre polynomials. Clearly $f_i(n) = 1$ when $\eta_i = 0$ in this case the states of (2.9) and (2.16) become the standard two-mode SCS's. However, when $\eta_i \neq 0$ nonlinearity starts developing, with the degree of nonlinearity depending on the magnitude of the parameters η_i .

In conclusion, in this work we have studied the nonlinear extension of two-mode squeezed vacuum states. We have defined a class of nonlinear two-mode squeezed states. Some basic definitions and properties of $SU(1,1)$ Lie algebra have been considered. Various applications of these results in the context of the two-photon realization of $SU(1,1)$ in quantum optics are also considered. The NLSS's realization of $SU(1,1)$ Lie group have been constructed. These states may find applications in the fields of quantum optics and quantum information.

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