Generalized structure of higher order nonclassicality

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Abstract

A generalized notion of higher order nonclassicality (in terms of higher order moments) is introduced. Under this generalized framework of higher order nonclassicality, conditions of higher order squeezing and higher order subpoissonian photon statistics are derived. A simpler form of the Hong-Mandel higher order squeezing criterion is derived under this framework by using an operator ordering theorem introduced by us in [J. Phys. A. **33** (2000) 5607]. It is also generalized for multi-photon Bose operators of Brandt and Greenberg. Similarly, condition for higher order subpoissonian photon statistics is derived by normal ordering of higher powers of number operator. Further, with the help of simple density matrices, it is shown that the higher order antibunching (HOA) and higher order subpoissonian photon statistics (HOSPS) are not the manifestation of the same phenomenon and consequently it is incorrect to use the condition of HOA as a test of HOSPS. It is also shown that the HOA and HOSPS may exist even in absence of the corresponding lower order phenomenon. Binomial state, nonlinear first order excited squeezed state (NLESS) and nonlinear vacuum squeezed state (NLVSS) are used as examples of quantum state and it is shown that these states may show higher order nonclssical characteristics. It is observed that the Binomial state which is always antibunched, is not always higher order squeezed and NLVSS which shows higher order squeezing does not show HOSPS and HOA. The opposite is observed in NLESS and consequently it is established that the HOSPS and HOS are two independent signatures of higher order nonclassicality.

1   Introduction: The generalized notion of higher order nonclassical states

A state which does not have any classical analogue is known as nonclassical state. To be precise, when the Glauber Sudarshan P function of a radiation field become negative or more singular than a delta function then the radiation field is said to be nonclassical. In these situations quasi probability distribution P is not accepted as classical probability and thus we can not obtain an analogous classical state. For example, squeezed state and antibunched state are well known nonclassical states. These two lowest order nonclassical states have been studied since long but the interest in higher order nonclassical states is relatively new. Possibilities of observing higher order nonclassicalities in different physical systems have been investigated in recent past [1-15]. For example, i) higher order squeezed state of Hong Mandel type [1-3], ii) higher order squeezed state of Hillery type [4,5], iii) higher order subpoissonian photon state [6-8] and iv) higher order antibunched state [9-15] are recently studied in different physical systems. But the general nature of higher nonclassicality and the mutual relation between these higher order nonclassical states have not been studied till now. Present work aims to provide a general and simplified frame work for the study of higher order nonclassical state.

Commonly, second order moment (standard deviation) of an observable is considered to be the most natural measure of quantum fluctuation [16] associated with that observable and the reduction of quantum fluctuation below the coherent state (poissonian state) level corresponds to lowest order nonclassical state. For example, an electromagnetic field is said to be electrically squeezed field if uncertainties in the quadrature phase observable X reduces below the coherent state level (i.e. $(\Delta X)^2 < \frac{1}{2}$) and antibunching is defined as a phenomenon in which the fluctuations in photon number reduces below the Poisson level (i.e. $(\Delta N)^2 < \langle N \rangle$) [17,18]. In Essenes, if we consider an arbitrary quantum mechanical operator A and a quantum mechanical state $|\psi\rangle$ then the state $|\psi\rangle$ is lowest order nonclassical with respect to the operator A if

$$(\Delta A)^2_{|\psi\rangle} < (\Delta A)^2_{|poissonian\rangle}. \tag{1}$$

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If $|\psi\rangle$ corresponds to an electromagnetic field, this condition will mean that the radiation field is nonclassical. This condition can now be further generalized and we can say that a state $|\psi\rangle$ has nth order nonclassicality with respect to the operator A if the nth order moment of A in that state reduces below the value of the nth order moment of A in a poissonian state, i.e. the condition of nth order nonclassicality is

$$(\Delta A)^n_{|\psi\rangle} < (\Delta A)^n_{|poissonian\rangle}, \tag{2}$$

where $(\Delta A)^n$ is the nth order moment defined as

$$\langle (\Delta A)^n \rangle = \sum_{r=0}^{n} {}^nC_r(-1)^r \overline{A^r} \ \bar{A}^{n-r}. \tag{3}$$

If A is a field operator then it can be expressed as a function of creation and annihilation operators a and $a^\dagger$ and consequently further simplification of (2) is possible by using the identity

$$\langle :\left(A(a,a^\dagger)\right)^k :\rangle_{|poisonian\rangle} = \langle \left(A(a,a^\dagger)\right)\rangle^k_{|poisonian\rangle} \tag{4}$$

where, the notation $:\left(A(a,a^\dagger)\right)^k:$ is simply a binomial expansion in which powers of the $a^\dagger$ are always kept to the left of the powers of the a. Here it would be interesting to note that (4) helps us to show that that the Glauber Sudarshan P function is negative for condition (2). It is clear from (3) that the problem of finding out the nth order moment of the operator A essentially reduces to a problem of operator ordering (normal ordering) of A^r. Here, we would like to note that we observe the lowest order nonclassicality for $n = 2$. And in this particular case ($n = 2$) we obtain the condition of squeezing of electric field, if $A = X = \frac{1}{\sqrt{2}}(a + a^\dagger)$ and obtain the condition of antibunching if $A = N = a^\dagger a$. Now if we need to generalize the idea of these well known lower order nonclassical effects we have to find out normal ordered form of X^r and N^r. In section 2 we start with an operator ordering theorem which provides a normal ordered form of X^r and obtain a simplified expression of higher order squeezing[3]. We have also generalized that expression for multi-photon Brandt-Greenberg Bose operators. In the section 3 we have provided an operator ordered form of N^r and consequently obtained a condition for higher order subpoissonian photon statistics. In same section we have also discussed the relation between different criteria of higher order nonclassicalities. In section 4, Binomial state, nonlinear first order excited squeezed state (NLESS) and nonlinear vacuum squeezed state (NLVSS) are used as examples of quantum state and it is shown that these states may show higher order nonclssical characteristics. Finally section 5 is dedicated to conclusions.

2 Simplified condition for higher order squeezing:

To obtain the condition for higher order squeezing (HOS) we need to use the following operator order ordering theorem introduced by us in [19]:
Theorem 1: If any two bosonic annihilation and creation operators a and $a^\dagger$ satisfy the commutation relation

$$[a, a^\dagger] = 1. \tag{5}$$

Then for any integral values of m

$$(a^\dagger + a)^m_N = \sum_{r=0}^{\frac{m}{2}} t_{2r} \ {}^mC_{2r} : (a^\dagger + a)^{m-2r} : \tag{6}$$

with

$$t_{2r} = \frac{2r!}{2^r(r)!} = (2r-1)!! = 2^r \left(\frac{1}{2}\right)_r \tag{7}$$

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[3] According to this notion of higher order squeezing Hillery type amplitude powered squeezing is lower order squeezing of nonlinear bosonic operators (Y_1 and Y_2). This is so because the amplitude powered squeezing is described by the reduction (with respect to the poissonian state) of second order moment of the corresponding quadrature variable. One can easily extend the existing notion of Hillery type squeezing and obtain a new kind of higher order nonclassicality, namely, Hong Madel type squeezing of Hillery type operator.

where, the subscript N stands for the normal ordering [4], $(x)_r$ is conventional Pochhammer symbol and the double factorial is defined as

$$n!! = \begin{cases} n(n-2)..5.3.1 \; for \; n > 0 \; odd \\ n(n-2)..6.4.2 \; for \; n > 0 \; even \\ 1 \; for \; n = -1, 0 \end{cases} . \tag{8}$$

This theorem of normal ordering is not restricted to the ordering of annihilation and creation operator rather it is valid for any arbitrary operator E^+ and its conjugate E^- which satisfy,

$$[E^+, E^-] = C. \tag{9}$$

This is so because (9) can easily be reduced to the form of (5) as $[\frac{E^+}{\sqrt{C}}, \frac{E^-}{\sqrt{C}}] = 1$. Apart from theorem 1 we need the following identity to proceed further:

$$^nC_r \, ^rC_j = \frac{n!}{(n-r)!r!}\frac{r!}{(r-j)!j!} = \frac{n!}{(n-j)!j!}\frac{(n-j)!}{[(n-j)-(r-j)]!(r-j)!} = {}^nC_j \, {}^{n-j}C_{r-j}. \tag{10}$$

Using (3), (6), (7) and (10) the nth order moment of $\Delta E = E - \bar{E}$ (where the quadrature variable $E = (a + a^\dagger)$) can be expressed as

$$\begin{aligned}
\langle (\Delta E)^n \rangle &= \sum_{r=0}^{n} {}^nC_r(-1)^r \overline{E^r} \, \bar{E}^{n-r} \\
&= \sum_{r=0}^{n} {}^nC_r(-1)^r \langle (a^\dagger + a)^r \rangle \langle a^\dagger + a \rangle^{n-r} \\
&= \sum_{r=0}^{n} {}^nC_r(-1)^r \sum_{i=0}^{\frac{r}{2}} t_{2i} \, {}^rC_{2i} \langle : (a^\dagger + a)^{r-2i} : \rangle \langle a^\dagger + a \rangle^{n-r} \\
&= \sum_{r=0}^{n} \sum_{i=0}^{\frac{r}{2}} t_{2i} \, {}^nC_{2i} \, {}^{n-2i}C_{r-2i}(-1)^{r-2i} \langle : (a^\dagger + a)^{r-2i} : \rangle \langle a^\dagger + a \rangle^{\{(n-2i)-(r-2i)\}} \\
&= \sum_{i=0}^{\frac{n}{2}} t_{2i} \, {}^nC_{2i} \langle : (\Delta E)^{n-2i} : \rangle.
\end{aligned} \tag{11}$$

Now if we follow (2), and define nth order squeezed state as a quantum mechanical state in which nth order moment $\langle (\Delta E)^n \rangle$ is shorter than its poissonian sate value then the condition for nth order squeezing reduces to

$$\langle (\Delta E)^n \rangle < t_n = (n-1)!! \tag{12}$$

which can be alternatively written as

$$\sum_{i=0}^{\frac{n}{2}-1} t_{2i} \, {}^nC_{2i} \langle : (\Delta E)^{n-2i} : \rangle < 0. \tag{13}$$

or,

$$\langle (\Delta E)^n \rangle = \sum_{r=0}^{n} \sum_{i=0}^{\frac{r}{2}} \sum_{k=0}^{r-2i} (-1)^r t_{2i} \, {}^{r-2i}C_k \, {}^nC_r \, {}^rC_{2i} \langle a^\dagger + a \rangle^{n-r} \langle a^{\dagger k} a^{r-2i-k} \rangle < (n-1)!!. \tag{14}$$

Conditions (12) and (13) coincide exactly with the definition of Hong Mandel squeezing, reported in earlier works [5] [1, 3] and the equivalent condition (14) considerably simplifies the calculation of HOS. Now instead of E if we calculate the nth order moment for usual quadrature variable X defined as $X = \frac{1}{\sqrt{2}}(a + a^\dagger)$, then we obtain

$$\langle (\Delta X)^n \rangle < \frac{1}{2^{\frac{n}{2}}} t_n = \frac{1}{2^{\frac{n}{2}}}(n-1)!! = \left(\frac{1}{2}\right)_{\frac{n}{2}} \tag{15}$$

or,

$$\sum_{r=0}^{n} \sum_{i=0}^{\frac{r}{2}} \sum_{k=0}^{r-2i} (-1)^r \frac{1}{2^{\frac{n}{2}}} t_{2i} \, {}^{r-2i}C_k \, {}^nC_r \, {}^rC_{2i} \langle a^\dagger + a \rangle^{n-r} \langle a^{\dagger k} a^{r-2i-k} \rangle < \left(\frac{1}{2}\right)_{\frac{n}{2}}. \tag{16}$$

Starting from the generalized notion of higher order nonclassicality (2) we have obtained a closed from expression of Hong-Mandel squeezing with the help of Theorem 1. The use of Theorem 1 not only simplifies

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[4] One can write $f(a, a^\dagger)$ in such a way that all powers of $a^\dagger$ always appear to the left of all powers of a. Then $f(a, a^\dagger)$ is said to be normal ordered.

[5] If we choose $E_1 = E^+ + E^-$ in analogy with Hong and Mandel [1] then (12) reduces to

$$\langle (\Delta E)_1^n \rangle < t_n C^{\frac{n}{2}} = (n-1)!! C^{\frac{n}{2}},$$

where n is even. This is the generalized expression obtained in [1] by using some other trick.

the condition it significantly reduces the calculational difficulties. To be precise, to study the possibility of HOS for an arbitrary quantum state $|\psi\rangle$ we just need to calculate $\langle a^\dagger + a \rangle$ and $\langle a^{\dagger k} a^{r-2i-k} \rangle$. Calculation of this expectation values are simple. For example, if we can expand the arbitrary state $|\psi\rangle$ in the number state basis as

$$|\psi\rangle = \sum_{j=0}^{N} C_j |j\rangle. \tag{17}$$

Then we can easily obtain,

$$\langle \psi | a^{\dagger k} a^{r-2i-k} | \psi \rangle = \sum_{j=0}^{N-Max[k,\, r-2i-k]} C^*_{j+k} C_{j+r-2i-k} \frac{1}{j!} \left((j+k+r-2i)! (j+k)! \right)^{\frac{1}{2}} \tag{18}$$

where Max yields the largest element from the list in its argument and

$$\langle a^\dagger + a \rangle = \sum_{m=0}^{N-1} \sqrt{(m+1} \left(C_m C^*_{m+1} + C^*_m C_{m+1} \right). \tag{19}$$

Therefore,

$$
\begin{aligned}
\langle (\Delta X)^n \rangle &= \sum_{r=0}^{n} \sum_{i=0}^{\frac{r}{2}} \sum_{k=0}^{r-2i} (-1)^r \frac{1}{2^{\frac{n}{2}}} t_{2i} \,^{r-2i}C_k \,^n C_r \,^r C_{2i} \\
&\times \left(\sum_{m=0}^{N-1} \sqrt{(m+1} \left(C_m C^*_{m+1} + C^*_m C_{m+1} \right) \right)^{n-r} \\
&\times \sum_{j=0}^{N-Max[k,\, r-2i-k]} C^*_{j+k} C_{j+r-2i-k} \frac{1}{j!} \left((j+k+r-2i)! (j+k)! \right)^{\frac{1}{2}}.
\end{aligned}
\tag{20}
$$

In general, if we know the effect of a^s on the state $|\Psi\rangle$ and the orthogonality conditions $\langle \Psi' | \Psi \rangle$ then we can easily find out $\langle (\Delta X)^n \rangle$. Further, since (20) is a C-number equation, analytical tools like MAPPLE and MATHEMATICA can also be used to study the possibilities of observing higher order squeezing (or higher order nonclassicality in general). This point will be more clear in section 4, where we will provide specific examples. Here we would like to note that we can normalize (15) and rewrite the condition of HOS as

$$S_{HM}(n) = \frac{\langle (\Delta X)^n - \left(\frac{1}{2} \right)_{\frac{n}{2}} \rangle}{\left(\frac{1}{2} \right)_{\frac{n}{2}}} < 0 \tag{21}$$

where the subscript HM stand for Hong Mandel.

2.1 Brandt-Greenberg operators and k-photon coherent state:

The k-photon coherent state was introduced by D'Arino and coworkers by using Brandt-Greenberg multi-photon operators [20] A_k and $A_k^\dagger$, which are defined as

$$A_k^\dagger = \left[\left[\frac{N}{k} \right] \frac{N-k}{N} \right] a^{\dagger k}, \tag{22}$$

$$A_k = (A_k^\dagger)^\dagger, \tag{23}$$

where the function [x] is defined as the greatest integer less or equal to x; $a^\dagger$ and a are the usual bosonic relation and annihilation operator and $N = a^\dagger a$ is the number operator. This particular from of Brandt-Greenberg operators is also used in the work of Buzek and Jex [21] in which they have studied the amplitude $k - th$ power squeezing of the k-photon coherent states. These operators satisfy the commutation relation analogous to (5), i.e. they satisfy,

$$[A_k, A_k^\dagger] = 1. \tag{24}$$

If any operator and its hermitian conjugate satisfies this kind of commutation relation then it has to satisfy the operator ordering theorem 1 and consequently we will be able to define Hong-Mandel squeezing in terms that particular operator (in a modified Fock space). For example, if we define to quadrature variables X_{1k} and X_{2K} as

$$
\begin{aligned}
X_{1k} &= A_k + A_k^\dagger \\
X_{2k} &= A_k - A_k^\dagger
\end{aligned}
\tag{25}
$$

then we can define the condition for nth order Hong-Mandel squeezing as

$$\sum_{r=0}^{n}\sum_{i=0}^{\frac{r}{2}}\sum_{k=0}^{r-2i}(-1)^r\frac{1}{2^{\frac{n}{2}}}t_{2i}\,^{r-2i}C_k\,^{n}C_r\,^{r}C_{2i}\langle A^\dagger + A\rangle^{n-r}\langle A^{\dagger k}A^{r-2i-k}\rangle < \left(\frac{1}{2}\right)_{\frac{n}{2}}. \tag{26}$$

This provides an extended notion of Hong-Mandel squeezing in a modified Hilbert space.

3   Higher order subpoissonian photon statistics

In analogy to the procedure followed to derive the Hong-Mandel higher order squeezing condition from the generalized expression (2) of higher order nonclassicality, we wish to study the nonclassicality associated with $A(a,a^\dagger) = N = a^\dagger a$. As we have already discussed, for this purpose we will require operator ordered form of N^r. Since the operator ordered expansion of N^r will not contain any off-diagonal term so it is justified to assume that the normal ordered form of $(N)^r$ can be given as

$$N^r = \sum_{i=1}^{r} C_{r,i} : N^i := \sum_{i=1}^{r} C_{r,i}a^{\dagger i}a^i. \tag{27}$$

From this equation it is clear that $C_{r,1} = C_{r,r} = 1$ and we can write N^{r+1} as

$$N^{r+1} = \sum_{i=1}^{r+1} C_{r+1,i}a^{\dagger i}a^i = \sum_{i=1}^{r} C_{r,i}a^{\dagger i}a^i a^\dagger a = \sum_{i=1}^{r}\left(C_{r,i}a^{\dagger i+1}a^{i+1} + iC_{r,i}a^{\dagger i}a^i\right) \tag{28}$$

where the operator ordering identity, $a^l a^\dagger = a^\dagger a^l + la^{l-1}$ is used. Now, we will be able to obtain closed form normal ordered expansion of N^r provided we know the solution of the recurrence relation:

$$C_{r+1,i} = iC_{r,i} + C_{r,i-1} \tag{29}$$

with $C_{r,0} = 0$ and $C_{r,1} = 1$. One can easily identify (29) as the famous recurrence relation of Stirling number of second kind [22]. Thus we can write

$$N^r = \sum_{k=1}^{r} S_2(r,k)a^{\dagger k}a^k = \sum_{k=1}^{r} S_2(r,k) : N^k := \sum_{k=1}^{r} S_2(r,k)N^{(k)}, \tag{30}$$

where $S_2(r,k)$ is the Stirling number of second kind $N^{(k)} = a^{\dagger k}a^k$ is the kth factorial moment of the number operator N. Now using (2), (3), (5) and (30) we can obtain the condition of higher order subpoissonian photon statistics as

$$\langle(\Delta N)^n\rangle = \sum_{r=0}^{n}\,^{n}C_r(-1)^r\bar{N}^r\bar{N}^{n-r} = \sum_{r=0}^{n}\sum_{k=1}^{r}S_2(r,k)\,^{n}C_r(-1)^r\langle N^{(k)}\rangle\langle N\rangle^{n-r} < \langle(\Delta N)^n\rangle_{|poissonain\rangle}$$

or,

$$d_h(n-1) = \sum_{r=0}^{n}\sum_{k=1}^{r}S_2(r,k)\,^{n}C_r(-1)^r\langle N^{(k)}\rangle\langle N\rangle^{n-r} - \sum_{r=0}^{n}\sum_{k=1}^{r}S_2(r,k)\,^{n}C_r(-1)^r\langle N\rangle^{k+n-r} < 0. \tag{31}$$

The negativity of $d_h(n-1)$ will mean $(n-1)th$ order subpoissonian photon statistics. This condition is equivalent to the condition of HOSPS obtained Mishra-Prakash [8].

3.1   Relation between the criteria of HOA and HOSPS

The criterion of HOA is expressed in terms of higher order factorial moments of number operator. There exist several criterion for the same which are essentially equivalent. Here we would like to investigate how are they related to the criterion of HOSPS. Initially, using the negativity of P function and theory of Majorization, Lee [9, 10] introduced the criterion for HOA as

Density matrix	Antibunching	SPS	HOA ($l=3$)	HOSPS ($n=4$)	Conclusions
$\frac{1}{2}(\lvert 3\rangle\langle 3\rvert + \lvert 8\rangle\langle 8\rvert)$	No	No	Yes	Yes	HOA and HOSPS can exist in absence of lower order
$\frac{1}{2}(\lvert 4\rangle\langle 4\rvert + \lvert 10\rangle\langle 10\rvert)$	No	No	No	Yes	HOA and HOSPS are different phenomenon

Table 1: HOA and HOSPS are not the manifestation of the same phenomenon and consequently it is incorrect to use the condition of HOA as a test of HOSPS.

$$R(l,m) = \frac{\left\langle N_x^{(l+1)}\right\rangle \left\langle N_x^{(m-1)}\right\rangle}{\left\langle N_x^{(l)}\right\rangle \left\langle N_x^{(m)}\right\rangle} - 1 < 0, \tag{32}$$

where N is the usual number operator, $\left\langle N^{(i)}\right\rangle = \left\langle N(N-1)...(N-i+1)\right\rangle$ is the ith factorial moment of number operator, l and m are integers satisfying the conditions $1 \leq m \leq l$ and the subscript x denotes a particular mode. Ba An [11] choose $m=1$ and reduced the criterion of lth order antibunching to

$$A_{x,l} = \frac{\left\langle N_x^{(l+1)}\right\rangle}{\left\langle N_x^{(l)}\right\rangle \langle N_x\rangle} - 1 < 0 \tag{33}$$

or,

$$\left\langle N_x^{(l+1)}\right\rangle < \left\langle N_x^{(l)}\right\rangle \langle N_x\rangle . \tag{34}$$

We can further simplify (34) as

$$\left\langle N_x^{(l+1)}\right\rangle < \left\langle N_x^{(l)}\right\rangle \langle N_x\rangle < \left\langle N_x^{(l-1)}\right\rangle \langle N_x\rangle^2 < \left\langle N_x^{(l-2)}\right\rangle \langle N_x\rangle^3 < ... < \langle N_x\rangle^{l+1} \tag{35}$$

and obtain the condition for $l - th$ order antibunching as

$$d(l) = \left\langle N_x^{(l+1)}\right\rangle - \langle N_x\rangle^{l+1} < 0. \tag{36}$$

This simplified criterion (36) coincides exactly with the physical criterion of HOA introduced by Pathak and Garica [12] and the criterion of Erenso, Vyas and Singh [14]. In [12] it is already shown that the depth of nonclassicality of an lth order antibunching is always more than that of $(l-1)th$ order antibunching of the same state. Consequently,

$$d(l) < d(l-1) \tag{37}$$

or,

$$\langle N^{(l+1)}\rangle\langle N\rangle^{n-r} - \langle N\rangle^{l+1+n-r} < \langle N^{(l)}\rangle\langle N\rangle^{n-r} - \langle N\rangle^{l+n-r}. \tag{38}$$

Now the condition for HOSPS, i.e. (31) can be written as

$$d_h(n-1) = \sum_{r=0}^{n}\sum_{k=1}^{r} S_2(r,k) \, {}^nC_r(-1)^r \left[\langle N^{(k)}\rangle - \langle N\rangle^k\right]\langle N\rangle^{n-r} = \sum_{r=0}^{n}\sum_{k=1}^{r} S_2(r,k) \, {}^nC_r(-1)^r d(k-1)\langle N\rangle^{n-r} < 0. \tag{39}$$

Above relation connects the condition of HOA (36) and that of HOSPS (31) but does not provide any conclusion about the mutual satisfiability. Physically it is expected from the analogy of lower order phenomenon that all states that show HOA should show HOSPS but the reverse should not be true. We have not succeed in showing that analytically but we can establish that with the help of simple density matrix of the form $\frac{1}{2}(\lvert a\rangle\langle a\rvert + \lvert b\rangle\langle b\rvert)$, where $\lvert a\rangle$ and $\lvert b\rangle$ are Fock states. The results are shown in the Table 1.

All the criterion related to HOA and HOSPS essentially lead to same kind of nonclassicality which belong to the class of strong nonclassicality according to the classification scheme of Arvind et al [23]. The Table 1 shows that HOA and HOSPS may be present in a system in absence of corresponding lower order phenomenon. It also shows that HOA and HOSPS are not the same phenomenon. To be precise, HOSPS can be present in a system even in absence of HOA. Thus it is not proper to consider the condition on HOA as the

condition of HOSPS. In [5] Duc has recently used criterion of HOA to study possibilities of observing HOSPS in photon added coherent state. Incorrect choice of criterion may yield incorrect conclusions so we need to be very careful before choosing a criterion of higher order nonclassicality. Further in section 4 we have shown that the binomial state is not always higher order squeezed but is always higher order antibunched. Thus we can conclude that although they may be derived from same generalized framework they are essentially independent criterion. This is in the sense that fulfillment of one does not mean fulfillment of the other.

4 Examples

4.1 Binomial State

An intermediate state is a quantum state which reduces to two or more distinguishably different states (normally, distinguishable in terms of photon number distribution) in different limits. In 1985, such a state was first time introduced by Stoler *et al.* [24]. To be precise, they introduced Binomial state (BS) as a state which is intermediate between the most nonclassical number state $|n'\rangle$ and the most classical coherent state $|\alpha\rangle$. They defined BS as

$$|p, M\rangle = \sum_{n'=0}^{M} \quad B_{n'}^{M} \ |n'\rangle = \sum_{n'=0}^{M} \sqrt{{}^{M}C_{n'}p^{n'}(1-p)^{M-n'}}|n'\rangle \ \ 0 \le p \le 1. \tag{40}$$

This state[6] is called intermediate state as it reduces to number state in the limit $p \to 0$ and $p \to 1$ (as $|0, M\rangle = 0$ and $|1, M\rangle = |M\rangle$) and in the limit of $M \to \infty$, $p \to 1$, where α is a real constant, it reduces to a coherent state with real amplitude. Since the introduction of BS as an intermediate state it was always been of interest to quantum optics, nonlinear optics, atomic physics and molecular physics community. Consequently, different properties of binomial states have been studied [25-30]. In these studies it has been observed that the nonclassical phenomena (such as, antibunching, squeezing and higher order squeezing) can be seen in BS.

Using the above definition of BS we obtain

$$d(l)_{BS} = \frac{M!p^{l+1}}{(M-l-1)!} - (Mp)^{l+1}, \tag{41}$$

$$d_h(n-1)_{BS} = \sum_{r=0}^{n} \sum_{k=1}^{r} \left[S_2(r,k) \, {}^{n}C_r (-1)^r (Mp)^{n-r} \left(\frac{M!p^k}{(M-k)!} - (Mp)^k \right) \right], \tag{42}$$

and

$$
\begin{aligned}
S_{HM}(n)_{BS} \ = \ & \frac{1}{\left(\frac{1}{2}\right)_{\frac{n}{2}}} \sum_{r=0}^{n} \sum_{i=0}^{\frac{r}{2}} \sum_{k=0}^{r-2i} (-1)^r \frac{1}{2^{\frac{n}{2}}} t_{2i} \, {}^{r-2i}C_k \, {}^{n}C_r \, {}^{r}C_{2i} \langle a^\dagger + a \rangle^{n-r} \langle a^{\dagger k} a^{r-2i-k} \rangle - 1 \\
= \ & \frac{1}{\left(\frac{1}{2}\right)_{\frac{n}{2}}} \left[\sum_{r=0}^{n} \sum_{i=0}^{\frac{r}{2}} \sum_{k=0}^{r-2i} (-1)^r \frac{1}{2^{\frac{n}{2}}} t_{2i} \, {}^{r-2i}C_k \, {}^{n}C_r \, {}^{r}C_{2i} \left[2(Mp)^{1/2} \sum_{n'=0}^{M-1} B_{n'}^{M} B_{n'}^{M-1} \right]^{n-r} \right. \\
& \left. \left[\frac{M!^2 p^{r-2i}}{(M-k)!(M-r+2i+k)!} \right]^{1/2} \sum_{n'=0}^{M-Max[k,r-2i-k]} B_{n'}^{M-k} B_{n'}^{M-r+2i+k} \right] - 1
\end{aligned}
\tag{43}
$$

Above expressions are graphically represented in Fig. 1-Fig. 2. From the Fig. 1 it is clear that the BS shows HOA and HOSPS simultaneously, but they are not proportional to each other. This result is not of much interest as we have already shown in [15] that the BS is always higher order antibunched and as every higher order nonclassical state is expected to show HOSPS, independent of whether they show HOA and Hong-Mandel squeezing or not. The result of real physical relevance appears when we look at Fig. 2 which shows that the BS does not show Hong Mandel squeezing for all values of p. For example 4th order Hong Mandel squeezing vanishes for $M = 50$ and $p \ge 0.8607$. In this range the state is still nonclassical and shows HOA and HOSPS but does not show Hong Madel Squeezing. Consequently we can conclude that the HOA and Hong Mandel squeezing are two independent processes which may or may not appear together. Further it can be observed that for the same photon member (M), the region of nonclassicality decreases with the increase in order of Hong Mandel squeezing. To be precise, when $M = 50$ then $S_{HM}(4)_{BS}$ is negative till $p = 0.8607$, but $S_{HM}(6)_{BS}$ is negative till $p = 0.7943$ and $S_{HM}(8)_{BS}$ is negative till $p = 0.7343$.

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[6]The state is named as binomial state because the photon number distribution associated with this state $\left(i.e. |B_n^M|^2 \right)$ is simply a binomial distribution.
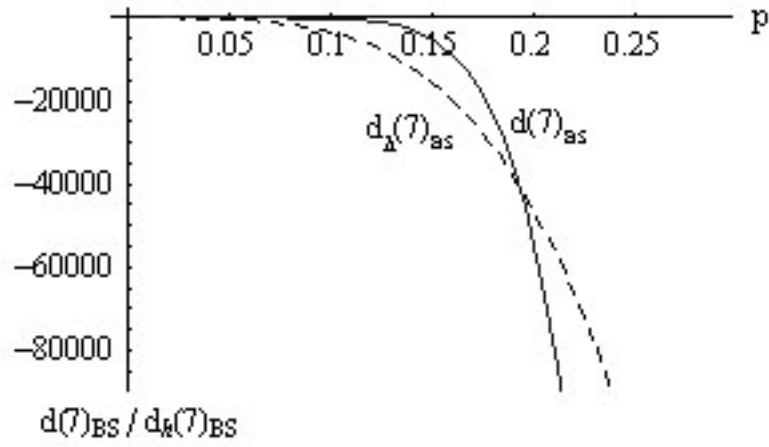
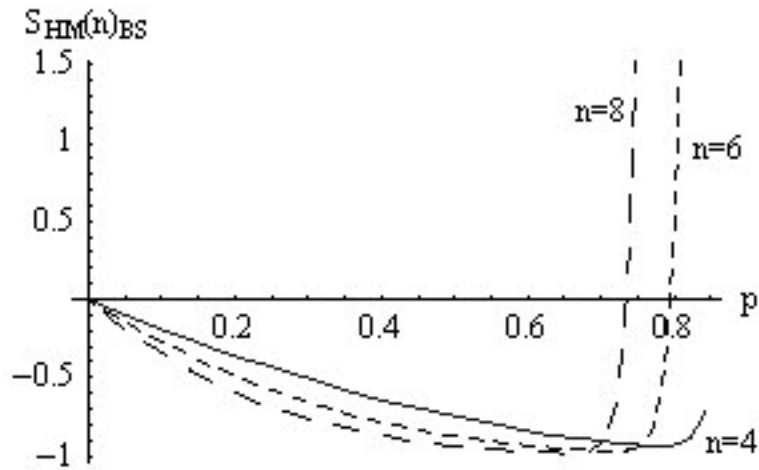Figure 1: Signature of HOA and HOSPS in Binomial state for $M = 20$.

Figure 2: Signature of Hong Mandel Squeezing in Binomial State for $M = 50$.

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4.2 Nonlinear Squeezed State

Recently Darwish [31] has introduced a class of nonlinear squeezed states. These states are named as nonlinear vacuum squeezed state (NLVSS) and nonlinear first order excited squeezed state (NLESS). Nonlinear states are expected to show nonclassical properties. Keeping that in mind, Darwish has investigated the possibilities of observing normal quadrature squeezing, amplitude squared squeezing and antibunching. To be precise the study of Darwish shows that, NLESS show subpoissonian photon statistics but does not show normal squeezing and NLVSS does not show subpoissonian statistics but shows quadrature squeezing. Here we wish to investigate the possibilities of observing HOA and HOSPS in NLESS and NLVSS. Following Darwish we can define NLVSS as

$$|\psi\rangle_V = N \sum_{n'=0}^{\infty} \frac{\sqrt{(2n')!}}{n'![f(2n')]!} \left[\frac{\xi_1}{2}\right]^{n'} |2n'\rangle, \tag{44}$$

with

$$|N|^{-2} = \sum_{n'=0}^{\infty} \frac{(2n')!}{(n'!)^2 [f(2n')!]^2} \left[\frac{\xi_1}{2}\right]^{2n'} \tag{45}$$

and NLESS as

$$|\phi\rangle_E = N' \sum_{n'=0}^{\infty} \frac{\sqrt{(2n'+1)!}}{n![f(2n'+1)]!} \left[\frac{\xi_1}{2}\right]^{n'} |2n'+1\rangle, \tag{46}$$

with

$$|N'|^{-2} = \sum_{n'=0}^{\infty} \frac{(2n'+1)!}{(n'!)^2 [f(2n'+1)!]^2} \left[\frac{\xi_1}{2}\right]^{2n'} \tag{47}$$

where $f(.)$ is a well behaved nonunitary operator valued function which is chosen in such a way that the normalization constant N and N' must be bound. The above equations define NLESS and NLVSS in general. To study the higher order nonclassical properties of NLESS and NLVSS we have chosen a particular case in which $\xi_1 = e^{i\phi} \tanh r$, $\phi = \pi$ and $f(n') = \sqrt{n'}$. This particular case is considered by Darwish [31] to study the possibility of observing squeezing, amplitude squared squeezing and quasi probability distribution nonlinear squeezed states. This particular choice of parameter yields

$$|N| = |N'| = \left[\sum_{n'=0}^{\infty} \frac{1}{(n'!)^2} \left[\frac{\tanh r}{2}\right]^{2n'}\right]^{-\frac{1}{2}}. \tag{48}$$

Using the definition of NLESS and the above choices of parameters we obtain

$$d(l)_E = |N|^2 \sum_{n'=0}^{\infty} \frac{(2n'+1)!}{(2n'-l)!} \frac{1}{(n'!)^2} \left[\frac{\tanh r}{2}\right]^{2n'} - \left[|N|^2 \sum_{n'=0}^{\infty} (2n'+1) \frac{1}{(n'!)^2} \left[\frac{\tanh r}{2}\right]^{2n'}\right]^{l+1}, \tag{49}$$

$$d_h(n-1)_E = \sum_{r=0}^{n} \sum_{k=1}^{r} \left[S_2(r,k) \, {}^nC_r (-1)^r \left(|N|^2 \sum_{n'=0}^{\infty} (2n'+1) \frac{1}{(n'!)^2} \left[\frac{\tanh r}{2}\right]^{2n'} \right)^{n-r} \right.$$
$$\left. \left(|N|^2 \sum_{n'=0}^{\infty} \frac{(2n'+1)!}{(2n'+1-k)!} \frac{1}{(n'!)^2} \left[\frac{\tanh r}{2}\right]^{2n'} - \left(|N|^2 \sum_{n'=0}^{\infty} (2n'+1) \frac{1}{(n'!)^2} \left[\frac{\tanh r}{2}\right]^{2n'} \right)^{k} \right) \right] \tag{50}$$
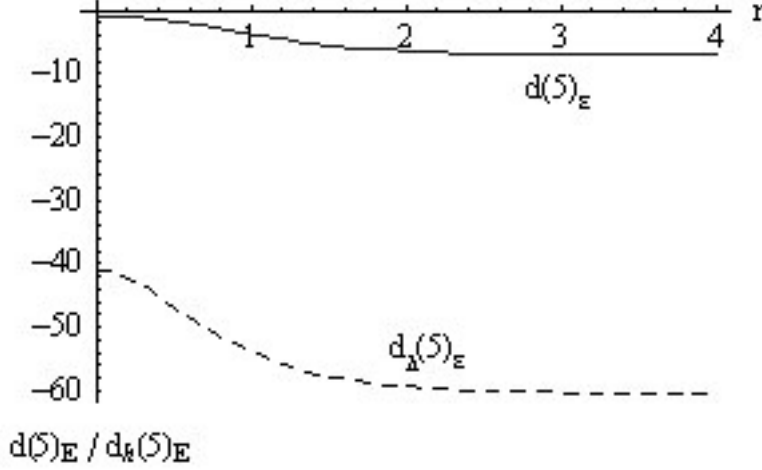
and

$$S_{HM}(n)_E = \left[\frac{1}{\left(\frac{1}{2}\right)_{\frac{n}{2}}} \sum_{i=0}^{\frac{n}{2}} \sum_{k=0}^{n-2i} \frac{1}{2^{\frac{n}{2}}} t_{2i} \, {}^{n-2i}C_k \, {}^nC_{2i} |N|^2 \right.$$
$$\left. \sum_{n'=Max\left[\frac{2k-n+2i}{2}, \frac{n-2i-2k}{2}\right]}^{\infty} \frac{\sqrt{(2n'+1)!(2n'+1+n-2i-2k)!}}{(2n'+1-k)! \, n'! \left[\frac{2n'+n-2i-2k}{2}\right]!} \left[-\frac{\tanh r}{2}\right]^{(4n'+n-2i-2k)/2} \right] - 1 \tag{51}$$

Similarly we can write

$$d(l)_V = |N|^2 \sum_{n'=0}^{\infty} \frac{(2n')!}{(2n'-l-1)!} \frac{1}{(n'!)^2} \left[\frac{\tanh r}{2}\right]^{2n'} - \left[|N|^2 \sum_{n'=0}^{\infty} (2n') \frac{1}{(n'!)^2} \left[\frac{\tanh r}{2}\right]^{2n'}\right]^{l+1}, \tag{52}$$

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Figure 3: Signature of HOA and HOSPS in NLESS

$$
\begin{aligned}
d_h(n-1)_V \;=\;& \sum_{r=0}^{n}\sum_{k=1}^{r}\left[S_2(r,k)\,{}^nC_r(-1)^r\left(|N'|^2\sum_{n'=0}^{\infty}(2n')\frac{1}{(n'!)^2}\left[\frac{\tanh r}{2}\right]^{2n'}\right)^{n-r}\right.\\
& \left.\left(|N|^2\sum_{n'=0}^{\infty}\frac{(2n')!}{(2n'-k)!}\frac{1}{(n'!)^2}\left[\frac{\tanh r}{2}\right]^{2n'}-\left(|N|^2\sum_{n'=0}^{\infty}(2n')\frac{1}{(n'!)^2}\left[\frac{\tanh r}{2}\right]^{2n'}\right)^{k}\right)\right]
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
S_{HM}(n)_V \;=\;& \left[\frac{1}{\left(\frac{1}{2}\right)_{\frac{n}{2}}}\sum_{i=0}^{\frac{n}{2}}\sum_{k=0}^{n-2i}\frac{1}{2^{\frac{n}{2}}}t_{2i}\,{}^{n-2i}C_k\,{}^nC_{2i}\,|N\,|^2\right.\\
& \left.\sum_{n'=Max\left[\frac{2k-n+2i}{2},\frac{n-2i-2k}{2}\right]}^{\infty}\frac{\sqrt{(2n')!(2n'+n-2i-2k)!}}{(2n'-k)!\,n'!\left[\frac{2n'+n-2i-2k}{2}\right]!}\left[-\frac{\tanh r}{2}\right]^{(4n'+n-2i-2k)/2}\right]-1
\end{aligned}
\tag{54}
$$

Above expressions are graphically represented in Fig. 3-Fig. 7. From Fig. 3 one can easily see that the condition for fifth order antibunching (HOA) and fifth order subpoissonian photon statistics are satisfied by NLESS but the same is not satisfied by NLVSS (as shown in Fig. 4). Again Fig. 5 shows the absence of higher order Hong Mandel squeezing in NLESS. However, Fig. 6 shows the presence of higher order Hong Mandel squeezing in NLVSS. These observations (Fig.3 -Fig 6) strongly establishes the fact that the HOS and HOSPS are two independent phenomena. This observation is in accordance with the corresponding lower order observations of Darwish [31] related to these nonlinear squeezed states. Another interesting observation is that the higher order squeezing parameter $(S_{HM}(n))$ oscillates between nonclassical region and classical region in case of NLESS for $n \geq 10$. This oscillatory nature is depicted in Fig. 7

5 Conclusions

The criteria of HOSPS and Hong Mandel type of higher order squeezing are derived from a single framework. Using that framework and operator ordering theorem a simpler form of the Hong-Mandel higher order squeezing criterion is derived and generalized for the multi-photon Bose operators of Brandt and Greenberg. The relation between HOA, HOSPS and HOS is investigated in detail and certain interesting observations in this regard has been reported. For example, it is shown that the lower order antibunching, HOA and HOSPS appear in novel regimes (i.e. they may or may not appear simultaneously as shown in Table 1). But in literature HOA and HOSPS have been used as synonymous [5]. Our observations establish that it is incorrect to use the condition of HOA as a test of HOSPS. We have used binomial state, NLESS and NLVSS as examples of quantum state and have observed that BS always shows HOA and HOSPS but it does not show HOS for all values of p. So we conclude that existence of HOSPS does not guarantee the existence of HOS. This is consistent with the corresponding observations in lower order. Further, it is also observed that the NLVSS which shows higher order squeezing does not show HOSPS and HOA. The opposite is observed
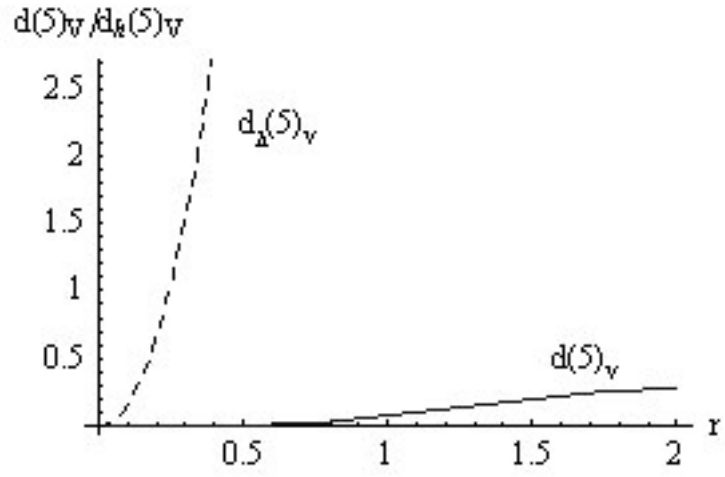
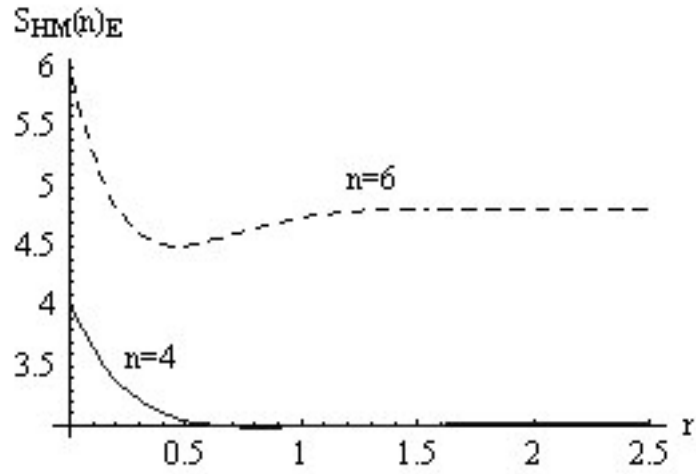Figure 4: NLVSS shows superpoissonian characteristics

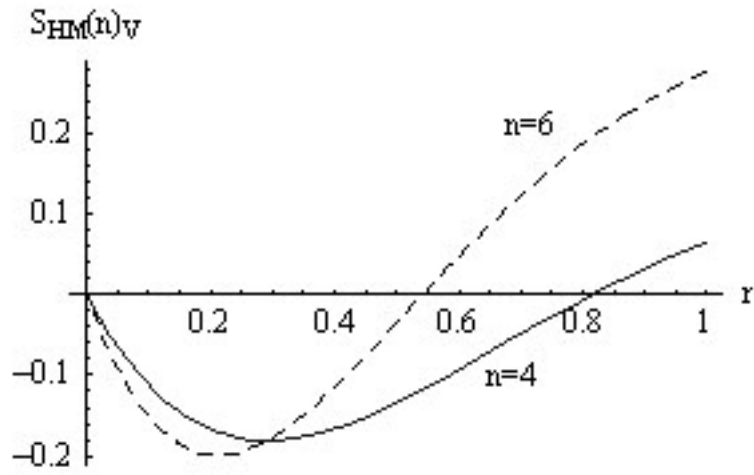Figure 5: NLESS does not exhibit signature of HOS

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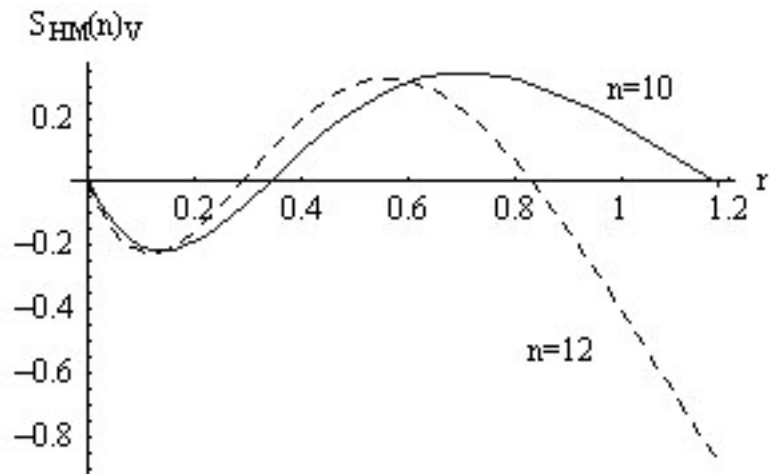Figure 6: Signature of Hong Mandel squeezing in NLVSS

Figure 7: Oscillatory nature of Hong Mandel squeezing in NLVSS

in NLESS and consequently it is established that the HOSPS and HOS are two independent signatures of higher order nonclassicality. Present study is the first one of his kind in which rigorous attempts have been made to understand the mutual relationship between different higher order nonclassical states. The effort is successful to provide an insight into the mutual relations between the well known nonclassical states and opens up a possibility of similar work in broader class of nonclassical states. The simpler framework provided for the study of possibilities of observing Hong Mandel squeezing is also expected to be useful in the future works.

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