Turk J Phys
32 (2008) , 1 – 4.

On the First Law of Thermodynamics for (2+1) Dimensional Charged BTZ Black Hole and Charged de Sitter Space

Eduard Alexis LARRAÑAGA
National University of Colombia, National Astronomical Observatory (OAN), Bogota-COLOMBIA
e-mail: eduardalexis@gmail.com

Received 27.11.2007

Abstract

In this paper we will show that using the cosmological constant as a new thermodynamical state variable, the differential and integral mass formulas of the first law of thermodynamics for asymptotic flat spacetimes can be extended to (2+1) dimensional charged BTZ black holes and charged de Sitter space.

Key Words: General Relativity, Thermodynamics of Black Holes, Lower Dimensional Models

1. Introduction

Bekenstein and Hawking showed that black holes have non-zero entropy and they emit thermal radiation proportional to their surface gravity at the horizon. These two quantities are related to the mass via the identity

$$dM = TdS, \tag{1}$$

and is called *the first law of black hole thermodynamics* [1,2]. But when the black hole has other properties, such as angular momentum $\mathbf{J}$ and electric charge Q, the first law generalizes to

$$dM = TdS + \Omega dJ + \Phi dQ, \tag{2}$$

where $\Omega = \frac{\partial M}{\partial J}$ is the angular velocity and $\Phi = \frac{\partial M}{\partial Q}$ is the electric potential. The corresponding integral is know as the Bekenstein-Smarr mass formula and is given by the relation

$$M\frac{(D-3)}{(D-2)} = \frac{(D-3)}{(D-2)}\Phi Q + \Omega J + TS. \tag{3}$$

Gauntlett et al. [3] have proved that both the differential and integral expressions hold for asymptotically flat spacetimes with any dimension $D \geq 4$. For rotating black holes in anti-de Sitter spaces, Gibbons et al. [4] have shown that the differential expression hold for $D \geq 4$, but the integral expression is not satisfied. To rectify this situation Caldarelli et al. [5] use a cosmological constant considered as a new thermodynamical variable, and recently, Wang et al. [6] use this idea to show that differential and integral expressions are valid also in $(2+1)$ dimensions for BTZ black holes with angular momentum and Kerr-de Sitter spacetimes.

In this paper we consider the charged BTZ black hole in $(2+1)$ dimensions to show that, considering a cosmological constant as a thermodynamical state variable, both the differential and integral mass formulas of the first law can be extended.

2. The first law for the charged BTZ Black Hole

The charged BTZ black hole [7] is a solution of $(2+1)$ dimensional gravity with a negative cosmological constant $\Lambda = -\frac{1}{l^2}$. Its line element can be written as

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\varphi^2, \tag{4}$$

where the lapse function is

$$\Delta = -M + \frac{r^2}{l^2} - \frac{Q^2}{2} \ln\left(\frac{r}{l}\right). \tag{5}$$

This solution has two horizons given by the condition $\Delta = 0$. The mass of the black hole can be written in terms of the event horizon r_{H} as [8]

$$M = \frac{r_{\mathrm{H}}^2}{l^2} - \frac{Q^2}{2} \ln\left(\frac{r_{\mathrm{H}}}{l}\right). \tag{6}$$

The Bekenstein-Hawking entropy associated with the black hole is twice the perimeter of the horizon,

$$S = 4\pi r_{\mathrm{H}}, \tag{7}$$

and therefore, the mass can be written as

$$M = \frac{S^2}{16\pi^2 l^2} - \frac{Q^2}{2} \ln\left(\frac{S}{4\pi l}\right). \tag{8}$$

If we consider an invariable l, this expression lets us calculate the surface gravity, temperature and electric potential for the black hole as

$$\kappa = \frac{1}{2} \left.\frac{\partial \Delta}{\partial r}\right|_{r=r_{\mathrm{H}}} = \frac{1}{2}\left[\frac{2r_{\mathrm{H}}}{l^2} - \frac{Q^2}{2r_{\mathrm{H}}}\right] \tag{9}$$

$$T = \frac{\kappa}{2\pi} = \left.\frac{\partial M}{\partial S}\right|_{Q,l} = \frac{2S}{16\pi^2 l^2} - \frac{Q^2}{2S} \tag{10}$$

$$\Phi = \left.\frac{\partial M}{\partial Q}\right|_{S,l} = -Q \ln\left(\frac{S}{4\pi l}\right). \tag{11}$$

Varying the mass formula (8) we obtain the first law of black hole thermodynamics in differential form:

$$dM = TdS + \Phi dQ. \tag{12}$$

But if we use equations (10) and (11), the mass formula can be written as

$$M = \frac{1}{2}TS + \frac{1}{2}\Phi Q + \frac{1}{4}Q^2, \tag{13}$$

but does not correspond to the Bekenstein-Smarr formula (3) with $D = 3$. Note that the product TS in this case does not vanish,

$$TS = \frac{2S^2}{16\pi^2 l^2} - \frac{Q^2}{2} = \frac{2r_{\mathrm{H}}^2}{l^2} - \frac{Q^2}{2}, \tag{14}$$

except in the extremal case $Q = \frac{2r_{\rm H}}{l}$. Otherwise, we can try to rectify this situation by considering the effect of a varying cosmological constant.

When considering the cosmological constant as a new state variable, the first law in differential and integral forms must be corrected to be

$$\begin{aligned} dM &= TdS + \Phi dQ + \Theta dl \\ M\frac{(D-3)}{(D-2)} &= \frac{(D-3)}{(D-2)}\Phi Q + \Omega J + TS + \frac{1}{(D-2)}\Theta l, \end{aligned}$$

where Θ is the generalized force conjugate to the parameter l. This generalized force is given by

$$\Theta = \left.\frac{\partial M}{\partial l}\right|_{S,Q} = -\frac{2S^2}{16\pi^2 l^3} + \frac{Q^2}{2l} \tag{15}$$

$$\Theta = -\frac{2r_{\rm H}^2}{l^3} + \frac{Q^2}{2l}. \tag{16}$$

All the thermodynamical quantities, including this last expression coincide with reported results [6, 8].

3. The first law for the charged de Sitter space

Now we turn our attention to the charged de Sitter space for which we have a positive cosmological constant $\Lambda = \frac{1}{l^2}$, and a line element

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\varphi^2, \tag{17}$$

where the lapse function is now

$$\Delta = M - \frac{r^2}{l^2} - \frac{Q^2}{2}\ln\left(\frac{r}{l}\right). \tag{18}$$

This time the ADM mass can be written in terms of the event horizon $r_{\rm H}$ as

$$M = \frac{r_{\rm H}^2}{l^2} + \frac{Q^2}{2}\ln\left(\frac{r_{\rm H}}{l}\right). \tag{19}$$

The Bekenstein-Hawking entropy associated with this event horizon is

$$S = 4\pi r_{\rm H}, \tag{20}$$

and therefore, the mass can be written now as

$$M = \frac{S^2}{16\pi^2 l^2} + \frac{Q^2}{2}\ln\left(\frac{S}{4\pi l}\right). \tag{21}$$

Taking first an invariable l, this expression gives the surface gravity, temperature and electric potential,

$$\kappa = \frac{1}{2}\left.\frac{\partial\Delta}{\partial r}\right|_{r=r_{\rm H}} = \frac{1}{2}\left[\frac{2r_{\rm H}}{l^2} + \frac{Q^2}{2r_{\rm H}}\right] \tag{22}$$

$$T = \frac{\kappa}{2\pi} = \left.\frac{\partial M}{\partial S}\right|_{Q,l} = \frac{2S}{16\pi^2 l^2} + \frac{Q^2}{2S} \tag{23}$$

$$\Phi = \left.\frac{\partial M}{\partial Q}\right|_{S,l} = Q\ln\left(\frac{S}{4\pi l}\right). \tag{24}$$

Varying the mass formula (21), we obtain the first law in differential form:

$$dM = TdS + \Phi dQ. \tag{25}$$

And using equations (23) and (24), we can write the mass formula as

$$M = \frac{1}{2}TS + \frac{1}{2}\Phi Q - \frac{1}{4}Q^2, \tag{26}$$

that again does not correspond to the Bekenstein-Smarr general formula (3) with $D = 3$. This time the product TS is

$$TS = \frac{2S^2}{16\pi^2 l^2} + \frac{Q^2}{2} = \frac{2r_{\mathrm{H}}^2}{l^2} + \frac{Q^2}{2}, \tag{27}$$

and it is important to note that in this case it never vanishes. However, in order to generalize the Bekenstein-Smarr formula to this case we will consider again the effect of a varying cosmological constant.

Now, the first law in differential and integral forms must be corrected to be

$$\begin{aligned} dM &= TdS + \Phi dQ + \Theta dl \\ 0 &= TS + \Theta l, \end{aligned}$$

where the generalized force Θ is given by

$$\Theta = \left.\frac{\partial M}{\partial l}\right|_{S,Q} = -\frac{2S^2}{16\pi^2 l^3} - \frac{Q^2}{2l} \tag{28}$$

$$\Theta = -\frac{2r_{\mathrm{H}}^2}{l^3} - \frac{Q^2}{2l}. \tag{29}$$

4. Conclusion

By considering the cosmological constant as a new thermodynamical state variable, we have generalized the integral Bekenstein-Smarr mass formula to the cases of $(2+1)$ dimensional charged BTZ black holes and charged de Sitter space, while the differential form of the first law is also applicable in these cases. We also obtained the generalized force associated with the cosmological term and the corresponding mass formulas for these cases. It should be noted that this results can be applicable also in higher dimensional cases.

References

[1] S. W. Hawking, *Commun. Math. Phys.,* **43**, (1975), 199.

[2] J. D. Bekenstein, *Phys. Rev.,* **D7**, (1973), 2333; *Phys. Rev.,* **D9**, (1974), 3293.

[3] J. Gauntlett, R. Myers and P. Townsend, *Class. Quantum. Grav.,* **16**, (1999), 1.

[4] G. Gibbons, M. Perry and C. N. Pope, *Class. Quantum. Grav.,* **22**, (2005), 1503.

[5] M. Caldarelli, G. Cognola and D. Klemm, *Class. Quantum. Grav.,* **17**, (2000), 399.

[6] S. Wang, S. Q. Wu, F. Xie and L. Dan, *Chyn. Phys. Lett.,* **23**, (2006), 1096.

[7] C. Martinez, C. Teitelboim and J. Zanelli, *Phys. Rev.,* **D61**, (2000), 104013

[8] A. Larranaga, **0706,1599** [gr-qc].