Pairing computation on curves with efficiently computable endomorphism and small embedding degree

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Abstract. Scott uses an efficiently computable isomorphism in order to optimize pairing computation on a particular class of curves with embedding degree 2. He pointed out that pairing implementation becomes thus faster on these curves than on their supersingular equivalent, originally recommended by Boneh and Franklin for Identity Based Encryption. We extend Scott's method to other classes of curves with small embedding degree and efficiently computable endomorphism. In particular, we optimize pairing computation on a class of curves with embedding degree 4 and discriminant 1, which are interesting for pairing based cryptography because they have a very efficient arithmetic.

1 Introduction

Pairings were first used in cryptography for attacking the discrete logarithm problem on the elliptic curve [22], but nowadays they are also used as bricks for building new cryptographic protocols such as the tripartite Diffie-Hellman protocol [15], identity-based encryption [5], short signatures [6], and others.

A cryptographic pairing is a bilinear map $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_3$, where $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ are groups of large prime order r. Known pairings on elliptic curves, i.e. the Weil, Tate pairings, map to the multiplicative group of the minimal extension of the ground field $\mathbb{F}_q$ containing the r-th roots of unity. The degree of this extension, denoted usually by k, is called the embedding degree with respect to r. The basic algorithm used in pairing computation was given by Miller and is an extension of the double-and-add method for finding a point multiple. The cost of the computation heavily depends on costs of operations in $\mathbb{F}_{q^k}$. Consequently, in practice we need curves with small embedding degree.

The reduction of the loop length in Miller's algorithm is one of the main directions taken by research in pairing computation during the past few years. These results concern only pairings on $\mathbb{G}_1 \times \mathbb{G}_2$ or $\mathbb{G}_2 \times \mathbb{G}_1$, where subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ are given by

$$\mathbb{G}_1 = E[r] \cap \mathrm{Ker}(\pi - [1]) \quad \text{and} \quad \mathbb{G}_2 = E[r] \cap \mathrm{Ker}(\pi - [q]),$$

where π is the Frobenius morphism of E, i.e. $\pi : E \to E,\ (x, y) \mapsto (x^q, y^q)$. The pairings computed by the new algorithms [2, 13] are actually powers of the Tate pairing and are called in the literature the Eta ($\mathbb{G}_1 \times \mathbb{G}_2$, $\mathbb{G}_2 \times \mathbb{G}_1$), Ate ($\mathbb{G}_2 \times \mathbb{G}_1$) and twisted Ate pairing ($\mathbb{G}_1 \times \mathbb{G}_2$).

Furthermore, Hess and, independently, Vercauteren [12, 28] showed that on some families of curves with small Frobenius trace, the complexity of Miller's algorithm is $\mathcal{O}(\frac{1}{\varphi(k)} \log r)$, where φ is the Euler totient function. However, if we want to construct a pairing $\mathbb{G}_1 \times \mathbb{G}_2$, these techniques do not represent an improvement in pairing computation on curves whose value of the trace is close to $\sqrt{q}$, like the MNT curves [21] or curves found by the Cocks-Pinch method [4].

In this paper, we propose the use of efficiently computable endomorphisms, other than the Frobenius map, to optimize pairing computation. Our method, which works on curves having a small embedding degree, applies to curves constructed by the Cocks-Pinch method.

The remainder of this paper is organized as follows. Section 2 presents background on pairings and the Cocks-Pinch method for constructing curves with complex multiplication. Section 3 presents our results which make use of endomorphisms to compute pairings. Section 4 presents an evaluation of costs of an implementation of our algorithm and compares performances to those of the Tate pairing computation. In Appendix 6 we give examples of curves constructed using the Cocks-Pinch method, with small embedding degree and endomorphism of small degree.

2 Background on pairings

In this section we give a brief overview of the definition of the Tate pairing and of Miller's algorithm [20] used in pairing computations. This algorithm heavily relies on the double-and- add method for finding a point multiple. Let E be an elliptic curve given by a Weierstrass equation:

$$y^2 = x^3 + a_1 x + a_2, \tag{1}$$

defined over a finite field $\mathbb{F}_q$, with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$. Let P_∞ denote the neutral element on the elliptic curve. Consider r a large prime dividing $\#E(\mathbb{F}_q)$ and k the embedding degree with respect to r.

Let P be an r-torsion point and for any integer i, denote by $f_{i,P}$ a function with divisor $\mathrm{div}\,(f_{i,P}) = i(P) - (iP) - (i-1)(P_\infty)$ (see [25] for an introduction to divisors). Note that $f_{r,P}$ is such that $\mathrm{div}\,(f_{r,P}) = r(P) - r(P_\infty)$.

In order to define the Tate pairing we take Q a point in $E(\mathbb{F}_{q^k})$ representing an element of $E(\mathbb{F}_{q^k})/rE(\mathbb{F}_{q^k})$. Let T be a point on the curve such that the support of the divisor $D = (Q + T) - (T)$ is disjoint from the one of $f_{r,P}$. We then define the Tate pairing as

$$t_r(P, Q) = f_{r,P}(D). \tag{2}$$

This value is a representative of an element of $\mathbb{F}^*_{q^k}/(\mathbb{F}^*_{q^k})^r$. However for cryptographic protocols it is essential to have a unique representative so we will raise it to the $((q^k-1)/r)$-th power, obtaining an r-th root of unity. We call the resulting value the *reduced* Tate pairing

$$T_r(P,Q) = t_r(P,Q)^{\frac{q^k-1}{r}}.$$

As stated in [10], if the function $f_{r,P}$ is normalized, i.e. $(u^r_{P_\infty} f_{r,P})(P_\infty) = 1$ for some $\mathbb{F}_q$-rational uniformizer u_{P_∞} at P_∞, then one can ignore the point T and compute the pairing as

$$T_r(P,Q) = f_{r,P}(Q)^{(q^k-1)/r}.$$

In the sequel of this paper we only consider normalized functions. Before going into the details of Miller's algorithm, we recall the standard addition law on an elliptic curve in Weierstrass form. Suppose we want to compute the sum of iP and jP for $i,j \geq 1$. Let l be the line through iP and jP. Then l intersects the cubic curve E at one further point R according to Bezout's theorem (see [11]). We take v the line between R and P_∞ (which is a vertical line when R is not P_∞). Then v intersects E at one more point and we define the sum of iP and jP to be this point.

The lines l and v are functions on the curve and the corresponding divisors are

$$\begin{aligned} \operatorname{div}(l) &= (iP) + (jP) + (R) - 3(P_\infty), \\ \operatorname{div}(v) &= (R) + ((i+j)P) - 2(P_\infty). \end{aligned}$$

One can then easily check the following relation:

$$f_{i+j,P} = f_{i,P} f_{j,P} \frac{l}{v}. \tag{3}$$

Turning back to Miller's algorithm, suppose we want to compute $f_{r,P}(Q)$. We compute at each step of the algorithm on one side mP, where m is the integer with binary expansion given by the i topmost bits of the binary expansion of r, and on the other side $f_{m,P}$ evaluated at Q, by exploiting the formula above. We call the set of operations executed for each bit i of r a *Miller operation*.

Implementing pairings. In implementations, we usually prefer curves with even embedding degree. On these curves, thanks to the existence of twists, most computations in a Miller operation are done in proper subfields of $\mathbb{F}_{q^k}$. Moreover, thanks to the final exponentiation, terms contained in proper subfields of $\mathbb{F}_{q^k}$ can be ignored (see [18]). Algorithm 1 gives the pseudocode of Miller's algorithm for curves with even embedding degree. In Table 1 we give costs for the doubling and the addition step in Algorithm 1 for an implementation on $\mathbb{G}_1 \times \mathbb{G}_2$, on curves with twists[3] of degree d. We denote by $\mathbf{m}, \mathbf{s}$ the costs of multiplication and squaring in $\mathbb{F}_q$ and $\mathbf{M}, \mathbf{S}$ the costs of multiplication and squaring in $\mathbb{F}_{q^k}$.

[3] We briefly recall that a twist E' of degree d of an elliptic curve E defined over $\mathbb{F}_q$ is a curve isomorphic to E, such that the isomorphism between the two curves is minimally defined over $\mathbb{F}_{q^d}$. The reader should check [25] for more details on twists.

Security issues. A secure pairing-based cryptosystem needs to be implemented on elliptic curve subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ such that the discrete logarithm problem is computationally difficult in $\mathbb{G}_1$, $\mathbb{G}_2$ and in $\mathbb{F}^*_{q^k}$. The best known algorithm for computing discrete logarithms on elliptic curves is the Pollard-rho method [26, 23], which has complexity $O(\sqrt{r})$, where r is the order of the groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Meanwhile, the best known algorithm for solving the discrete logarithm problem in the multiplicative group of a finite field is the index calculus algorithm, which has sub-exponential running time [17, 16]. Consequently, in order to achieve the same level of security in both the elliptic curve subgroups and in the finite field subgroup, we need to choose a q^k which is significantly larger than r. It is therefore interesting to consider the ratio of these sizes

$$\frac{k\log q}{\log r}.$$

As the efficiency of the implementation will depend critically on the so-called ρ-value

$$\rho = \frac{\log q}{\log r},$$

it is preferable to keep this value as small as possible.

Algorithm 1 Miller's algorithm

```
INPUT: An elliptic curve $E$ defined over a finite field $\mathbb{F}_q$, $P$ an $r$-torsion point on the
    curve and $Q \in E(\mathbb{F}_{q^k})$.
OUTPUT: the Tate pairing $t_r(P, Q)$.
Let $i = [\log_2(r)]$, $K \leftarrow P$,$f \leftarrow 1$
while $i \geq 1$ do
    Compute the equation of $l$ arising in the doubling of $K$
    $K \leftarrow 2K$ and $f \leftarrow f^2 l(Q)$
    if the $i$-th bit of $r$ is 1 then
        Compute the equation of $l$ arising in the addition of $K$ and $P$
        $K \leftarrow P + K$ and $f \leftarrow fl(Q)$
    end if
    Let $i \leftarrow i - 1$.
end while
return $f$.
```

The Cocks-Pinch method for constructing pairing friendly curves. Let E be an ordinary curve defined over a finite field $\mathbb{F}_q$. We denote by π the Frobenius morphism and by t its trace. Given the fact that curve must have a subgroup of large order r and that the number of points on the curve is $\#E(\mathbb{F}_q) = q + 1 - t$ we write

$$q + 1 - t = hr.$$

Table 1. Cost of one step in Miller's algorithm for even embedding degree

	Doubling	Mixed addition
$\mathcal{J}$ [1, 14]	$(1+k)\mathbf{m}+11\mathbf{s}+1\mathbf{M}+1\mathbf{S}$	$(6+k)\mathbf{m}+6\mathbf{s}+1\mathbf{M}$
$\mathcal{J}$, $y^2=x^3+b$ $d=2,6$ [7]	$(2k/d+2)\mathbf{m}+7\mathbf{s}+1\mathbf{M}+1\mathbf{S}$	$(2k/d+9)\mathbf{m}+2\mathbf{s}+1\mathbf{M}$
$\mathcal{J}$, $y^2=x^3+ax$ $d=2,4$ [7]	$(2k/d+2)\mathbf{m}+8\mathbf{s}+1\mathbf{M}+1\mathbf{S}$	$(2k/d+12)\mathbf{m}+4\mathbf{s}+1\mathbf{M}$

Furthermore, the fact that the Frobenius is an element of an order in a quadratic imaginary field $\mathbb{Q}(\sqrt{-D})$ gives

$$Dy^2 = 4q - t^2 = 4hr - (t-2)^2.$$

To sum up, in order to generate a pairing friendly curve, we are looking for q, r, k, D, t and y satisfying the following conditions

$$\begin{aligned} r &\mid Dy^2 + (t-2)^2, \\ r &\mid q^k - 1, \\ t^2 + Dy^2 &= 4q. \end{aligned}$$

Cocks and Pinch gave an algorithm (which is presented in [4]) which finds, given r and a small k, parameters q prime and t satisfying the equations above.

Algorithm 2 The Cocks-Pinch algorithm

```
INPUT: $k$,$r$ a prime number, $D$ and $k|(r-1)$.
OUTPUT: $q$, $t$ such that there is a curve with CM by $\sqrt{-D}$ over $\mathbb{F}_q$ with $q+1-t$
    points where $r|(q+1-t)$ and $r|(q^k-1)$.
Choose a primitive $k$-th root of unity $g$ in $\mathbb{F}_r$.
Choose an integer $t \leftarrow g+1 (\mathrm{mod}\ r)$.
if gcd$(t,D) \neq 1$ then
    exit (or choose another $g$).
end if
Choose an integer $y_0 = \pm(t-2)/\sqrt{-D} (\mathrm{mod}\ r)$.
$j \rightarrow 0$
repeat
    $q \leftarrow (t^2 + D(y_0 + jr)^2)/4$
    $j \leftarrow j+1$
until $q$ is prime
return $q$ and $t$
```

This method produces ordinary curves with a ρ-value approximatively 2, which is less preferred in practice. However, Vercauteren showed that for certain embedding degrees and certain values of the discriminant $-D$, there are no ordinary curves with smaller ρ-value.

Proposition 1. *Let E be an elliptic curve over $\mathbb{F}_q$ with a subgroup of prime order $r > 3$ and embedding degree $k > 1$ with respect to r. If E has a twist $E'/\mathbb{F}_q$ of degree k and $r \geq 4\sqrt{q}$, then E is supersingular.*

It follows that in some cases, like $k = 2$ or $k = 4$ and [4] $D = -1$, the curves produced by the Cocks-Pinch algorithm have optimal ρ-value. Moreover, with this method we may choose the value of r from the very beginning. This is an advantage, because we may choose r with low Hamming weight. On curves with such r, in Algorithm 1, we perform mostly doublings and very few additions.

The Eta pairing and its variants. To our knowledge, isogenies were proposed to speed up pairing computation for the first time by Barreto et al. [2], who introduced the Eta pairing. This idea was extended by Hess et al. [13]. We present here the main result in [13], without giving the proof.

Theorem 1. *Let E be an elliptic curve defined over $\mathbb{F}_q$, r a large prime with $r|\#E(\mathbb{F}_q)$ and k the embedding degree with respect to r. Assume that $k > 1$ and denote by t the trace of the Frobenius.*

(a) For $T = t - 1$, $Q \in \mathbb{G}_2 = E[r] \cap Ker(\pi - [q])$, $P \in \mathbb{G}_1 = E[r] \cap Ker(\pi - [1])$ we have

(i) $f_{T,Q}(P)$ defines a bilinear pairing, which we call the Ate pairing;

(ii) Let $N = gcd(T^k - 1, q^k - 1)$ and $T^k - 1 = LN$, with k the embedding degree, then

$$t_r(Q, P)^L = f_{T,Q}(P)^{c(q^k-1)/N}$$

where $c = \sum_{i=0}^{k-1} T^{k-1-i} q^i \equiv kq^{k-1} \bmod r$. For $r \nmid L$, the Ate pairing is non-degenerate.

(b) Assume E has a twist of degree d and set $m = gcd(k, d)$ and $e = k/m$. We denote by $c = \sum_{i=0}^{m-1} T^{e(m-1-i)} q^{ei} \equiv mq^{e(m-1)} \bmod r$. We have

(i) $f_{T^e,P}(Q)$ defines a bilinear pairing, which we call the twisted Eta pairing;

(ii) $t_r(P, Q)^L = f_{T^e,P}(Q)^{c(q^k-1)/N}$ and the twisted Eta pairing is non-degenerate if $r \nmid L$.

The Ate and twisted Eta pairing can be computed using Miller's algorithm with a loop length of $\log T$ and $\log T^e$, respectively. Curves constructed with the Cocks-Pinch method have ρ-value greater than 2 and $T \sim \sqrt{q}$. Consequently, the Ate and twisted Eta pairing computation will not be faster than the computation of the Tate pairing. It might even be slower because it is unlikely that T would have a small Hamming weight.

[4] Only curves with discriminant -1 have twists of degree 4.

3 Speeding up pairing computation using endomorphisms of small degree

The following result was given by Verheul [29], whose purpose was to investigate the existence of distortion maps for points of order r, i.e. maps ϕ such that for a point P, $\phi(P) \notin \langle P \rangle$. We will make use of this result to improve pairing computation.

Theorem 2. *Let E be an ordinary curve defined over $\mathbb{F}_q$ and let P be a point over E of prime order $r \neq q$. Suppose the embedding degree k is greater than 1 and denote by Q a point defined over $\mathbb{F}_{q^k}$, such that $\pi(Q) = qQ$. Then P and Q are eigenvectors of any other endomorphism of E.*

Proof. Let ϕ be an endomorphism of E. For the point P we have

$$\phi(\pi(P)) = \pi(\phi(P)) \text{ and } \phi(\pi(P)) = \phi(P). \tag{4}$$

The first equality comes from the fact that the ring $\mathrm{End}(E)$ is commutative, the second one is due to the fact that $P \in E(\mathbb{F}_q)$. It follows that $\pi(\phi(P)) = \phi(P)$, so $\phi(P)$ is an eigenvector for the eigenvalue 1 of π. This means that $\phi(P) \in \langle P \rangle$. The proof for Q is similar.

Notation 1 *In the sequel we denote the correction of two points R_1 and R_2 as follows:*

$$corr_{R_1,R_2} = \frac{l_{R_1,R_2}}{v_{R_1+R_2}}$$

where l_{R_1,R_2} is the line passing through R_1 and R_2 and $v_{R_1+R_2}$ is the vertical line through $R_1 + R_2$.

Our starting idea is a method to exploit efficiently computable endomorphisms in pairing computation suggested by Scott [24] and, later on by Zhao and Zhang [30], for a family of curves called NSS. These curves are defined over $\mathbb{F}_q$ with $q \equiv 1 \bmod 3$ and given by an equation of the form $y^2 = x^3 + B$. Since they have $k = 2$ and $\rho \sim 2$, the Eta and Ate pairings will not bring any improvement in the pairing computation. However, these curves admit an endomorphism $\phi : (x, y) \to (\beta x, y)$, where β is a non-trivial cube root of unity. Its characteristic equation is $\phi^2 + \phi + 1 = 0$. If P is an eigenvalue of ϕ such that $\phi(P) = \lambda P$, then λ verifies the equation

$$\lambda^2 + \lambda + 1 = cr.$$

We obtain

$$f^c_{r,P}(Q) = f_{\lambda^2+\lambda,P}(Q) = f_{\lambda(\lambda+1),P}(Q) = f^{\lambda+1}_{\lambda,P}(Q) \cdot f_{\lambda+1,[\lambda]P}(Q) \cdot \frac{l_{[\lambda]P,P}}{v_{[\lambda+1]P}}.$$

Since for $P = (x, y)$, λP is given by $(\beta x, y)$, we can easily compute $f_{\lambda,\lambda P}(Q)$ from $f_{\lambda,P}(Q)$ by replacing x with βx.

We apply similar techniques to curves with endomorphisms that verify a characteristic equation $x^2 + ax + b = 0$, with a, b small. In all cases, we use the Cocks-Pinch method to construct curves such that there is a $\lambda \sim \sqrt{r}$ which verifies $\lambda^2 + a\lambda + b = cr$. This will lead to a new algorithm for pairing computation, which will be more efficient than Miller's algorithm when $k \leq 4$.

Lemma 1. *Let ϕ be a separable isogeny of degree b and P, Q two points on the elliptic curve E. Then for some integer λ we have*

$$f_{\lambda,\phi(P)} \circ \phi(Q) = f^b_{\lambda,P}(Q) \left(\prod_{K \in Ker\,\phi} corr_{P,K}(Q) \right)^{\lambda} \left(\prod_{K \in Ker\,\phi} corr_{\lambda P,K}(Q) \right)^{-1}.$$

Proof. We have

$$\begin{aligned}
\phi^*(f_{\lambda,\phi(P)}) &= \lambda \sum_{K \in \mathrm{Ker}\phi} (P+K) - \sum_{K \in \mathrm{Ker}\phi} (\lambda P + K) - (\lambda - 1) \sum_{K \in \mathrm{Ker}\phi} (K) \\
&= \lambda \sum_{K \in \mathrm{Ker}\phi} ((P+K) - (K)) - \sum_{K \in \mathrm{Ker}\phi} ((\lambda P + K) - (K)) \\
&= \lambda \sum_{K \in \mathrm{Ker}\phi} ((P) - (O)) - \sum_{K \in \mathrm{Ker}\phi} (\lambda P) - (O) + \mathrm{div} \left(\left(\prod_{K \in \mathrm{Ker}\,\phi} \frac{l_{K,P}}{v_{K+P}} \right)^{\lambda} \right) \\
&\quad - \mathrm{div} \left(\prod_{K \in \mathrm{Ker}\,\phi} \frac{l_{K,\lambda P}}{v_{K+\lambda P}} \right).
\end{aligned}$$

Using the fact that $\phi^*(f_{\lambda,\phi(P)}) = f_{\lambda,\phi(P)} \circ \phi$, we obtain the equality we have announced.

In the sequel, we make use of the following relation which holds for all $m, n \in \mathbb{Z}$ and any point P on the elliptic curve

$$f_{mn,P} = f^n_{m,P} \cdot f_{n,mP}. \tag{5}$$

This equality can be easily checked using divisors.

Theorem 3. *Let E be an elliptic curve defined over a finite field and ϕ an efficiently computable endomorphism whose characteristic equation is $X^2 + aX + b = 0$. Let P be an eigenvector of ϕ such that $\phi(P) = \lambda P$, where λ satisfies $\lambda^2 + a\lambda + b = cr$, with $r \nmid c$. Then the application $a_\phi(\cdot,\cdot) : \mathbb{G}_1 \times \mathbb{G}_2 \to \mu_r$ given by*

$$\begin{aligned}
a_\phi(P,Q) = f^{\lambda+a}_{\lambda,P}(bQ) f^b_{\lambda,P}(\hat{\phi}(Q)) f_{a,\lambda P}(bQ) f_{b,P}(bQ) \left(\prod_{K \in Ker\phi} corr_{P,K}(\hat{\phi}(Q)) \right)^{\lambda} \\
\cdot \left(\prod_{K \in Ker\phi} corr_{\lambda P,K}(\hat{\phi}(Q)) \right)^{-1} corr_{\lambda^2 P, a\lambda P}(bQ)\ l_{\lambda^2 P + a\lambda P, bP}(bQ).
\end{aligned}$$

is a bilinear non-degenerate pairing.

Proof. The following equality is obtained by repeatedly applying the equality at (5)

$$f_{\lambda^2+a\lambda+b} = (f^{\lambda}_{\lambda,P}) \cdot (f_{\lambda,\lambda P}) \cdot (f^{a}_{\lambda,P}) \cdot (f_{a,\lambda P}) \cdot (f_{b,P}) \cdot corr_{\lambda^2 P,a\lambda P} \cdot l_{\lambda^2 P+a\lambda P,bP} \tag{6}$$

By applying Lemma 1, we obtain

$$f_{\lambda,\lambda P}(bQ) = f^{b}_{\lambda,P}(\hat{\phi}(Q)) \left(\prod_{K\in \mathrm{Ker}\phi} corr_{P,K}(\hat{\phi}(Q)) \right)^{\lambda} \left(\prod_{K\in \mathrm{Ker}\phi} corr_{\lambda P,K}(\hat{\phi}(Q)) \right)^{-1}$$

By replacing this term in equation (6), we derive that $a_{\phi}(P,Q)$ is a power of $t_r(P,Q)$. Hence, a_{ϕ} defined a non-degenerate pairing on $\mathbb{G}_1 \times \mathbb{G}_2$.

If the value of λ is close to $\sqrt{r}$, Theorem 3 gives an efficient algorithm to compute the Tate pairing (actually a small power of the Tate pairing). This is Algorithm 3.

Algorithm 3 Our algorithm for pairing computation for curves with an efficiently computable endomorphism

```
INPUT: An elliptic curve $E$, $P, Q$ points on $E$ and $\phi$ such that $\phi(P) = \lambda P$, $Q' = \hat{\phi}(Q)$.
OUTPUT: A power of the Tate pairing $T_r(P,Q)$.
Let $i = [\log_2(\lambda)]$, $K \leftarrow P$, $f \leftarrow 1$, $g \leftarrow 1$
while $i \geq 1$ do
    Compute equation of $l$ arising in the doubling of $K$
    $K \leftarrow 2K$ and $f \leftarrow f^2 l(bQ)$ and $g \leftarrow g^2 l(Q')$
    if the $i$-th bit of $\lambda$ is 1 then
        Compute equation of $l$ arising in the addition of $K$ and $P$
        $K \leftarrow P + K$ and $f \leftarrow f l(bQ)$ and $g \leftarrow g l(Q')$
    end if
    Let $i \leftarrow i - 1$
end while
Compute $A \leftarrow f^{\lambda+a}$
Compute $B \leftarrow g^{b}$
Compute $C \leftarrow \left(\prod_{K\in \mathrm{Ker}\phi} corr_{P,K}(Q')\right)^{\lambda} \left(\prod_{K\in \mathrm{Ker}\phi} corr_{\lambda P,K}(Q')\right)^{-1}$
Compute $D \leftarrow f_{a,\lambda P}(bQ) f_{b,P}(bQ)$
$E \leftarrow corr_{\lambda^2 P,a\lambda P}(bQ) l_{\lambda^2 P+a\lambda P,bP}(bQ)$
Return $A \cdot B \cdot C \cdot D \cdot E$
```

4 Computational costs

Suppose we use an endomorphism ϕ whose characteristic equation is

$$\phi^2 + a\phi + b = 0,$$

with a and b small. We also neglect the cost of computing the dual of ϕ at Q, $\hat{\phi}(Q)$, because $\hat{\phi}$ can be precomputed by Vélu's formulae [27] and is given by polynomials of small degree. Note that in some protocols Q is a fixed point, so all the precomputations on this point may be done before the computation of the pairing.

We also note that the endomorphism is defined over $\mathbb{F}_q$, because the curve E is ordinary. Moreover, the points in $\mathrm{Ker}\phi$ are eigenvectors for the Frobenius endomorphism. Indeed, since $\mathrm{End}(E)$ is a commutative ring, we have $\phi(\pi(K)) = \pi(\phi(K)) = O$, for all $K \in \mathrm{Ker}\phi$. It follows that $\pi(K) \in \mathrm{Ker}\phi$. Thus the points of $\mathrm{Ker}\phi$ are defined over an extension field of $\mathbb{F}_q$ of degree smaller than b. Furthermore, we have

$$\left(\prod_{K \in \mathrm{Ker}\phi} corr_{P,K}(\hat{\phi}(Q)) \right) \in \mathbb{F}_{q^k}.$$

Consequently, given that the degree of ϕ is small, we assume that the number of operations needed to compute the correction $\prod_{K \in \mathrm{Ker}\phi} corr_{P,K}(\hat{\phi}(Q))$ is negligible. Since a and b are small, we also assume that the costs of the exponentiation at line 12 and that of the computation of functions at line 14 of Algorithm 3 are negligible.

Our computations showed that our method gives better performances than Miller's algorithm in the case of ordinary curves with embedding degree 2, 3 and 4. Since in practice we usually consider curves with even embedding degree, we present only results for curves with embedding degree 2 and 4. We assume that these curves have an efficiently computable endomorphism and eigenvalues of size $\sqrt{r}$. For $k = 4$, we only considered curves with discriminant -4, since on these curves pairing implementation is very efficient due to the use of twists of degree 4. Note that for such curves, the Eta pairing algorithm (and its variants) is not faster than the Tate pairing algorithm, because $t \approx r$. In our evaluation, we only counted the number of operations performed in the doubling part of Miller's algorithm, because we suppose that λ and r have low Hamming weight (which is possible if the curve is constructed with the Cocks-Pinch method). For operations in extension fields of degree 2, we use tower fields, i.e.

$$\mathbb{F}_q \subset \mathbb{F}_{q^2} \subset \mathbb{F}_{q^4} \subset \mathbb{F}_{q^8}.$$

Note that with Karatsuba's method the cost of an operation in the extension field of degree 2 is three times the cost of the same operation in the base field. Using the formulas in Table 1 the complexity of Algorithm 3 is

$$\begin{cases} (11\mathbf{s} + (1+2k)\mathbf{m} + 2\mathbf{M} + 2\mathbf{S}) \log\lambda + \log\lambda\mathbf{M} & \text{if } D \neq -4, \\ ((2+k)\mathbf{m} + 8\mathbf{s} + 2\mathbf{M} + 2\mathbf{S}) \log\lambda + \log\lambda\mathbf{M} & \text{if } D = -4. \end{cases}$$

This is actually the cost of the doubling part and of the exponentiation at line 11 of Algorithm 3. In Table 2, we compare the performances of our method to those of Miller's algorithm, for curves with embedding degree 2 and 4, at different security levels.

Table 2. Our method versus Miller's algorithm

bit length of r	$k = 2$		$k \geq 4$ and $D = -4$	
	Miller's algorithm	This work	Miller's algorithm	This work
160 bits	3040	2400	5120	4880

As explained in [8], if we want to set up a pairing-based cryptosystem with a 160-bit elliptic curve subgroup, we may choose a MNT curve with embedding degree 6 and ρ-value close to 1 or we may take a curve with embedding degree $k \in \{2, 3, 4\}$ and ρ-value 2. Note that we may use an ordinary curve with embedding degree 2 and ρ-value approximatively 3 constructed by the Cocks-Pinch method or a supersingular curve with $k = 2$ and $\rho \sim 3$ (see Algorithm 3.3 in [8]). Table 3 presents a comparison of performances for pairing computation on curves with different embedding degrees. We assume that ordinary curves with embedding degrees 2 and 4 are constructed via the Cocks-Pinch method. On these curves we evaluate the cost of the computation performed in Algorithm 3, as explained above. For supersingular curves and MNT curves with embedding degree 6 we estimate the cost of the doubling part in Algorithm 1 using formulae in Table 1. Since on these curves, the parameter r does not necessarily have low Hamming weight, we also count the number of operations performed in the mixed addition part of the Miller operation, if a NAF representation of r is used.

Table 3. Pairing computation at 80 bits security level

bit length of $\mathbb{F}_{q^k}$	value of k and ρ	doubling step	mixed addition
960	supersingular curves $k = 2$ and $\rho \sim 3$	3040	-
960	ordinary curves $k = 2$ and $\rho \sim 3$	2400	-
960	MNT curves $k = 6$ and $\rho \sim 1$	7680	1760
1280	$k = 4$ and $\rho \sim 2$	4720	-

Table 4 presents total costs for the Miller loop and for the final exponentiation, in terms of number of operations in $\mathbb{F}_q$, for different types of curves and embedding degrees. In the final exponentiation, we assume that applying the Frobenius operator can be done for free and we estimate only the cost of $\Phi_r(q)/r$. The last column presents global costs of pairing computation. Note that the size of q is different for these families of curves. We have therefore taken into account costs of integer multiplication for different bit lengths (see [3] for GMP benchmarks). We conclude by giving in Appendix 6 examples of curves constructed with the Cocks-Pinch method, with endomorphism verifying an equation of the form $X^2 + aX + b = 0$ and roots $\lambda \sim \sqrt{q}$. We also note that on these curves, the GLV method [9] can be used to speed up scalar multiplication in the implementation of a pairing-based protocol. We therefore believe that these curves offer a good choice for pairing-based cryptography at 80 bits security level.

Table 4. Pairing computation at 80 bits security level

value of k and ρ	Miller loop	final exponentiation	total cost
supersingular curves $k = 2$ and $\rho \sim 3$	12160	5760	17920
ordinary curves $k = 2$ and $\rho \sim 3$	9920	5760	15680
MNT curves $k = 6$ and $\rho \sim 1$	9440	7200	16640
$k = 4$ and $\rho \sim 2$	11566	4705	16271

5 Conclusion

We have given a new algorithm for pairing computation on curves with endomorphisms of small degree. Our method applies to curves constructed with the Cocks-Pinch method and is more efficient than Miller's algorithm on curves with embedding degree 2 and 4.

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6 Appendix 1

In order to display the equations of the endomorphism easily, we give first a small example.

Example 1. A toy example

We take $D = -4 \cdot 2$ and we want a curve with an endomorphism whose characteristic equation will be

$$X^2 + 2 = 0.$$

We choose $\lambda = 66543$ verifying the equation $\lambda^2 + 2 = r$, with

$$r = 4427970851.$$

Our implementation of the Cocks-Pinch method in MAGMA [19] found the following curve

$$y^2 = x^3 + 4976887516324122696283x + 2211950007255165642796$$

over the prime field $\mathbb{F}_q$, with

$$q = 14930662548972368088859.$$

This curve has $k = 2$ with respect to r. The endomorphism corresponding to $\alpha = \sqrt{-2}$ in $\mathbb{Z}[\sqrt{-2}]$ is

$$[\alpha](x, y) = \left(7465331274486184044429 \frac{x^2 + 4976887516324122696285x + 2}{x + 4976887516324122696285}\right.,$$
$$\left. 11197940817690300409659 \frac{x^2 + 9953775032648245392570x + 8294812527206871160477}{(x + 4976887516324122696285)^2} y\right).$$

As observed in [9], computing this endomorphism is slightly harder than doubling. The equations of the dual of α are similar.

Example 2. Consider $D = -3$ and an endomorphism with characteristic equation given by $X^2{+}2X{+}4 = 0$. We found $\lambda = 2^{40}{+}2^{29}{+}1$ verifying $\lambda^2{+}2{*}\lambda{+}4 = r$ where r is given by $r = 1210106699470122931716103$. We have

$$q = 126422926680861157408034773355095519523073976963357.$$

The curve E given by the equation $y^2 = x^3 + 1$ has embedding 2 with respect to r.

Example 3. Consider $D = -4$ and an endomorphism with characteristic equation given by $X^2{+}2X{+}2 = 0$. We found $\lambda = 2^{40}{+}2^{25}{+}1$ verifying $\lambda^2{+}2{*}\lambda{+}2 = r$ where r is given by $r = 1208999607721222100484101$. We have

$$q = 19838652498577664643118213771277780493813406356549557.$$

The curve E given by the equation $y^2 = x^3 + x$ has embedding 4 with respect to r.