The Existence of Equilibria in Discontinuous and Nonconvex Games

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Abstract

This paper investigates the existence of pure strategy, dominant-strategy, and mixed strategy Nash equilibria in discontinuous and nonconvex games. We introduce a new notion of very weak continuity, called *weak transfer continuity*, which holds in a large class of discontinuous economic games and is easy to check. We show that it, together with the compactness of strategy space and the quasiconcavity of payoff functions, permits the existence of pure strategy Nash equilibria. Our equilibrium existence result neither implies nor is implied by the existing results in the literature such as those in Baye *et al.* [1993] and Reny [1999]. We provide sufficient conditions for weak transfer continuity by introducing notions of weak transfer upper continuity and weak transfer lower continuity. These conditions are satisfied in many economic games and are often quite simple to check. We also introduce the notion of weak dominant transfer upper continuity, and use it to study the existence of dominant strategy equilibria. We then generalize these results and those in Baye *et al.* [1993] and Reny [1999] without assuming any form of quasi-concavity of payoff functions or convexity of strategy spaces.

Keywords: Nash equilibrium, dominant strategy equilibrium, discontinuity, non-quasiconcavity, and nonconvexity.

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1 Introduction

Nash's concept of equilibrium in Nash (1950, 1951) is probably the most important solution in game theory. It is immune from unilateral deviations, that is, each player has no incentive to deviate from his/her strategy given that other players do not deviate from theirs. Nash [1951] proved that a finite game has a Nash equilibrium in mixed strategies. Debreu [1952] then showed that games possess a pure strategy Nash equilibrium if (1) the strategy spaces are nonempty, convex and compact, and (2) players have continuous and quasiconcave payoff functions. Game theory has then been successfully applied in many areas in economics including oligopoly theory, social choice theory, and mechanism design theory. These applications lead researchers from different areas to investigate the possibility of weakening equilibrium existence conditions to further enlarge its domain of applicability.

The uniqueness of pure strategy Nash equilibrium is established in Rosen [1965]. Nishimura and Friedman [1981] and Yao [1992] considered the existence of Nash equilibrium in games where the payoff functions are not quasi-concave (but satisfying a *strong condition*) and γ-diagonally quasiconcave, respectively. Dasgupta and Maskin [1986] established the existence of pure and mixed-strategy Nash equilibrium in games where the strategy sets are non-empty convex and compact, and payoff functions are quasiconcave, upper semicontinuous and graph continuous by using an approximation technique. Simon (1987) and Simon and Zame (1990) used a similar approach to consider the existence of mixed-strategy Nash equilibria in discontinuous games. Simon and Zame (1990) showed that if one is willing to modify the vector of payoffs at points of discontinuity so that they correspond to points in the convex hull of limits of nearby payoffs, then one can ensure a mixed-strategy equilibrium of such a suitably modified game. Vives [1990] established the existence of Nash equilibrium in games where payoffs are upper semicontinuous and satisfy certain monotonicity properties.

Baye *et al.* [1993] weakened quasiconcavity of payoffsprovided by providing necessary and sufficient conditions for the existence of pure strategy Nash equilibrium and dominant-strategy equilibrium in discontinuous games. It is shown that diagonal transfer quasiconcavity is necessary, and further, under diagonal transfer continuity and compactness, sufficient for the existence of pure strategy Nash equilibrium. Both transfer quasiconcavity and diagonal transfer continuity are very weak notions of quasiconcavity and continuity and use a basic idea of transferring nonequilibrium strategies to other nonequilibrium strategies.

Reny [1999] established the existence of Nash equilibrium in compact and quasiconcave games where the game is better-reply secure, which is a weak notion of continuity. Reny [1999] showed that better-reply security can be imposed separately as reciprocal upper semicontinuity introduced by Simon (1987) and payoff security. Bagh and Jofre [2006] further weakened reciprocal upper semicontinuity to weak reciprocal upper semicontinuity and showed that it, together with payoff security, implies better-reply security. As one shall see, both better-reply security and payoff security use a similar idea of transferring a (nonequilibrium) strategy to another strategy, and they are actually also in the forms of transfer continuity. In fact, better-reply security introduced by Reny [1999] is closely related to diagonal transfer continuity introduced by Baye *et al.* [1993] under the form of the aggregate payoff function. Nevertheless, to use the result in Reny [1999], one must analyze the closure of the graph of the vector payoff function of the game. Such an analysis involves a high dimension and is hard to check. To check the (weak) reciprocal upper semicontinuity also has a similar nature.

This paper investigates the existence of pure strategy, dominant-strategy, and mixed strategy Nash equilibria in discontinuous and nonconvex games. We introduce a very weak notion of continuity, called *weak transfer continuity*. Roughly speaking, a game is weakly transfer continuous if for every nonequilibrium strategy x, there exists some player that has a strategy yielding a strictly better payoff even if all players deviate slightly from x. Weak transfer continuity holds in many economic games and is easy to check. We provide three sets of sufficient conditions, each of which implies weak transfer continuity: (1) transfer continuity; (2) weak transfer upper continuity and payoff security,[1] and (3) upper semicontinuity and weak transfer lower continuity. These conditions are satisfied in many economic games and are often quite simple to check.

We show that weak transfer continuity, together with the compactness of strategy space and the quasiconcavity of payoff functions, guarantees the existence of pure strategy Nash equilibria. Our equilibrium existence result neither implies nor is implied by the existing results in the literature such as those in Baye *et al.* [1993] and Reny [1999]. We also introduce the notion of weak dominant transfer upper continuity, and use it to study the existence of dominant strategy equilibria. Our results permit new equilibrium existence theorems for a large class of discontinuous games. We generalize these results as well as those in Baye *et al.* [1993] and Reny [1999] without assuming any form of quasi-concavity of payoff functions or convexity of strategy spaces.

The remainder of the paper is organized as follows. Section 2 describes the notation, and provides a number of preliminary definitions. Section 3 introduces the new condition, weak transfer continuity, and provides our main pure strategy Nash equilibrium existence result and its proof. Examples illustrating the theorem are also given. We then generalize the result and those in Baye *et al.* [1993] and Reny [1999] without assuming any form of quasi-concavity of payoff functions

[1] It is worth pointing out that, while reciprocal upper semicontinuity combined with payoff security implies better-reply security, here weak transfer upper semicontinuity combined with payoff security implies weak transfer continuity.

or convexity of strategy spaces. Section 4 considers the existence of dominant strategy equilibria by introducing a similar condition, weak dominant transfer continuity. We provide a main dominant strategy Nash equilibrium existence result and its proof. We then generalize the result and the one in Baye *et al.* [1993] by relaxing the convexity of strategy spaces. Section 5 considers the existence of mixed strategy Nash equilibria by applying the main result obtained in Section 3 on the existence of pure strategy Nash equilibria. Concluding remarks are offered in Section 6.

2 Preliminaries

Consider the following noncooperative game in normal form:

$$G = (X_i,\ u_i)_{i \in I} \tag{1}$$

where $I = \{1, ..., n\}$ is the finite set of players, X_i is player i's strategy space which is a nonempty subset of a topological space E_i, and $u_i : X \longrightarrow \mathbb{R}$ is the payoff function of player i. Denote by $X = \prod\limits_{i \in I} X_i$ the set of strategy profiles of the game. For each player $i \in I$, denote by $-i = \{j \in I$ such that $j \neq i\}$ the set of all players rather than player i. Also denote by $X_{-i} = \prod\limits_{j \in -i} X_j$ the set of strategies of the players in coalition $-i$.

We say that a game $G = (X_i,\ u_i)_{i \in I}$ is compact, convex, bounded, and semi-continuous, respectively if, for all $i \in I$, X_i is compact, convex, and u_i is bounded and semi-continuous on X, respectively. We say that a game $G = (X_i,\ u_i)_{i \in I}$ is quasiconcave if, for every $i \in I$, X_i is convex and the function u_i is quasiconcave in x_i.

We say that a strategy profile $x^* \in X$ is a *Nash equilibrium* of game G if,

$$u_i(y_i, x^*_{-i}) \leq u_i(x^*)\ \forall i \in I,\ \forall y_i \in X_i.$$

We say that a strategy profile $x^* \in X$ is a *dominant-strategy equilibrium* of a game G if,

$$\forall (y_i, y_{-i}) \in X,\ u_i(y_i, y_{-i}) \leq u_i(x^*_i, y_{-i})\ \forall i \in I.$$

We review some of the basic definitions introduced in Baye *et al.* [1993], Reny [1999], Bagh and Jofre [2006] and Morgan and Scalzo [2007].

Define a function $U : X \times X \to \mathbb{R}$ by

$$U(x, y) = \sum_{i=1}^{n} U_i(y_i, x_{-i}),\ \ \forall (x, y) \in X \times X. \tag{2}$$

Definition 2.1 The function $U : X \times X \to \mathbb{R}$ is said to be diagonally transfer continuous in x if for $(x, y) \in X \times X$, $U(x, y) > U(x, x)$ implies that there exists some point $y' \in Y$ and some neighborhood $\mathcal{V}(x) \subset X$ of x such that $U(z, y^{'}) > U(z, z)$ for all $z \in \mathcal{V}(x)$. We say that a game $G = (X_i,\ u_i)_{i \in I}$ is diagonally transfer continuous if function $U : X \times X \rightarrow \mathbb{R}$ is diagonally transfer continuous in x with respect to Y.

REMARK 2.1 Since diagonal transfer continuity requires taking an open neighborhood on both sides of the inequality, it is a weak notion of continuity. It is clear that continuity implies diagonal transfer continuity.

DEFINITION 2.2 The function $U(x, y) : X \times X \rightarrow \mathbb{R}$ is said to be diagonally transfer quasiconcave in y if, for any finite subset $Y^m = \{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $X^m = \{x^1, ..., x^m\} \subset X$ such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, ..., x^{k^s}\}$ we have $\min_{1 \leq l \leq s} U(x, y^{k^l}) \leq U(x, x)$. We say that a game $G = (X_i,\ u_i)_{i \in I}$ is diagonally transfer quasiconcave if $U : X \times Y \rightarrow \mathbb{R}$ is diagonally transfer quasiconcave in y.

Theorem 1 in Baye *et al.* [1993] shows that a game that is compact, convex, diagonally transfer continuous, and diagonally transfer quasiconcave must possess a pure strategy Nash equilibrium. Note that a game is diagonally transfer quasiconcave if it is quasiconcave.

The graph of the game is $\Gamma = \{(x, u) \in X \times \mathbb{R}^n : u_i(x) = u_i,\ \forall i \in I\}$. The closure of Γ in $X \times \mathbb{R}^n$ is denoted by $\bar{\Gamma}$. The frontier of Γ, which is the set of points that are in $\bar{\Gamma}$ but not in Γ, is denoted by Fr Γ.

DEFINITION 2.3 Player i can secure a payoff of $\alpha \in \mathbb{R}$ at $x \in X$ if there exists $\overline{x}_i \in X_i$, such that $u_i(\overline{x}_i, y_{-i}) \geq \alpha$ for all y_{-i} in some open neighborhood of x_{-i}.

DEFINITION 2.4 A game $G = (X_i,\ u_i)_{i \in I}$ is payoff secure if for every $x \in X$ and any $\epsilon > 0$, every player i can secure a payoff of $u_i(x) - \epsilon$.

DEFINITION 2.5 A game $G = (X_i,\ u_i)_{i \in I}$ is better-reply secure if whenever $(x^*, u^*) \in \bar{\Gamma}$, x^* is not an equilibrium, which implies that some player i can secure a payoff strictly above u_i^* at x^*, *i.e.*, there exists $\overline{x}_i \in X_i$ such that $u_i(\overline{x}_i, y_{-i}) > u_i^*$ for all y_{-i} in some open neighborhood of x_{-i}.

DEFINITION 2.6 A game $G = (X_i,\ u_i)_{i \in I}$ is reciprocally upper semicontinuous if, whenever $(x, u) \in \bar{\Gamma}$ and $u_i(x) \leq u_i$ for every player i, then $u_i(x) = u_i$ for every player i.

DEFINITION 2.7 A game $G = (X_i,\ u_i)_{i \in I}$ is weakly reciprocal upper semicontinuous, if for any $(x, u) \in Fr\ \Gamma$, there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}) > u_i$.

DEFINITION 2.8 Let Z be a topological space and f be an extended real valued function defined on Z. f is said to be upper pseudocontinuous at z_0 if for all $z \in Z$ such that $f(z_0) < f(z)$, we have $\limsup_{y \to z_0} f(y) < f(z)$. f is said to be lower pseudocontinuous at z_0 if $-f$ is upper pseudocontinuous at z_0. f is said to be pseudocontinuous if it is both upper and lower pseudocontinuous.

Theorem 3.1 in Reny [1999] shows that a $G = (X_i,\ u_i)_{i \in I}$ possesses a Nash equilibrium if it is compact, bounded, quasiconcave and better-reply secure. Reny [1999] and Bagh and Jofre [2006] provided sufficient conditions for a game to be better-reply secure. Reny [1999] showed that a game $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and reciprocal upper semicontinuous. Bagh and Jofre [2006] further showed that $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and weakly reciprocal upper semicontinuous. Morgan and Scalzo [2007] showed that $G = (X_i, u_i)_{i \in I}$ is better-reply secure if u_i is pseudocontinuous, $\forall i \in I$.

REMARK 2.2 Since payoff security requires taking an open neighborhood in the upper contour set of a given level of payoff, it is a weak notion of lower semicontinuity. Since better-reply security also requires the limit payoff resulting from strategies approaching a nonequilibrium point, it is a weak notion of continuity (which displays a certain form of both lower semicontinuity and upper semicontinuity). In addition, both notions use the same idea of transferring nonequilibrium strategy to another strategy, thus they actually fall in the forms of transfer continuity.

REMARK 2.3 In fact, better-reply security is closely related to diagonal transfer continuity introduced by Baye *et al.* [1993] under the form of U. Consider the function $U : X \times X \to \mathbb{R}$ defined by (2). Suppose (x, U) is in the closure of the graph of U. Then, the fact that function U is better-reply secure in x implies that, whenever $U(x, y) > U(x, x)$, there exists $y' \in X$ and a neighborhood $\mathcal{V}(x)$ of x such that $U(z, y') > U(x, x)$ for all $z \in \mathcal{V}(x)$. Thus, aside better-reply security is required that (x, U) be in the closure of the graph of U, the only difference is that the better-reply security takes an open neighborhood in the upper contour of a given level of payoff while diagonal transfer continuity takes an open neighborhood in both upper and lower contour of a given level of payoff. Moreover, if we define a function $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \sum_{i=1}^{n} \{u_i(y_i, x_{-i}) - u_i(x)\}, \quad \forall (x, y) \in X \times X,$$

then ϕ is better reply secure in x if and only if ϕ is diagonally transfer continuous in x when (x, U) is in the closure of the graph of U.

3 Existence of Nash Equilibria

In this section we investigate the existence of pure strategy Nash equilibrium in games that may be discontinuous and may not have any form of quasiconcavities. We first provide our main result on the existence of pure strategy Nash equilibrium in discontinuous games, and then characterize the existence of pure strategy Nash equilibrium without assuming the convexity of strategy spaces or any form of quasiconcavity of payoff functions. We also generalize existence results of Baye *et al.* [1993], Reny [1999] and Bagh and Jofre [2006] by relaxing the convexity of strategy spaces and diagonal transfer quasiconcavity of payoff functions.

3.1 Nash Equilibria in Discontinuous Games

We start by introducing the notions of (weak) transfer continuity.

DEFINITION 3.1 A game $G = (X_i, u_i)_{i \in I}$ is said to be *transfer continuous* if for all player i, u_i is transfer upper continuous in x with respect to X_i, *i.e.*, if $u_i(z_i, x_{-i}) > u_i(x)$ for $z_i \in X_i$ and $x \in X$, then there is some neighborhood $\mathcal{V}(x)$ of x and $y_i \in X_i$ such that $u_i(y_i, x'_{-i}) > u_i(x')$ for all $x' \in \mathcal{V}(x)$.

DEFINITION 3.2 A game $G = (X_i, u_i)_{i \in I}$ is said to be *weakly transfer continuous* if $x \in X$ is not an equilibrium, then there exist player i, $y_i \in X_i$ and a neighborhood $\mathcal{V}(x)$ of x such that $u_i(y_i, x'_{-i}) > u_i(x')$ for all $x' \in \mathcal{V}(x)$.

Roughly speaking, a game is weakly transfer continuous if for every nonequilibrium strategy x^*, there exists some player i that has a strategy y_i yielding a strictly better payoff even if all players deviate slightly from x^*.

REMARK 3.1 Since transfer continuity and weak transfer continuity require taking an open neighborhood on both sides of the inequality, they are weak notions of continuity. It is clear that they are (weakly) transfer continuous if they are continuous. Also, weak transfer continuity only requires some player, but not all players, who can have a strategy resulting in a strictly better payoff even if all players deviate slightly from a non-Nash equilibrium. It is clear that a game G is weakly transfer continuous if it is transfer continuous. However, the following example shows the reverse may not be true.

EXAMPLE 3.1 Consider a two-player game with $X_1 = X_2 = [0, 1]$ and

$$u_1(x_1, x_2) = \begin{cases} 2 + x_1 + x_2, & \text{if } x_1 = x_2, \\ x_1 + x_2, & \text{otherwise,} \end{cases}$$

and

$$u_2(x) = x_1 + x_2.$$

One can see that the strategy $(1,0)$ is not a Nash Equilibrium and $u_1(0,0) > u_1(1,0)$ when $z_1 = 0$. Then, for all $y_1 \in [0,1]$ and any $\delta > 0$, there exists $z = (1, z_2) \in B_\delta(1,0)$ with $z_2 \neq y_1$ such that $u_1(y_1, z_2) = y_1 + z_2 \leq 1 + z_2 = u_1(1, z_2)$. Then, u_1 is not transfer continuous at (1,0) with respect to X_1.

However, since $(1,0)$ is not a Nash Equilibrium, there exists $y_2 = 1$ and a neighborhood $\mathcal{V}(1,0) \subset [0,1] \times [0,1)$ of (1,0) such that $u_2(z_1, y_2) = z_1 + y_2 > z_1 + z_2$, for each $z \in \mathcal{V}(1,0)$. Thus, we can conclude that the game is weakly transfer continuous at $(1,0)$.

Now we state our main result that is formally unrelated to Baye *et al.* [1993] and Reny [1999].

THEOREM 3.1 *Suppose $G = (X_i, u_i)_{i\in I}$ is convex, compact, bounded, and weakly transfer continuous. Then, the game G has a Nash equilibrium if and only if it is diagonally transfer quasiconcave.*

PROOF. *Sufficiency* ($\Leftarrow$). For each player $i \in I$ and every $(x_i, y) \in X_i \times X$, let

$$\varphi_i(x_i, y) = \sup_{\mathcal{V}\in\Omega(y)} \inf_{z\in\mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)]$$

where $\Omega(y)$ is the set of all open neighborhoods of y.

For each i and every $x_i \in X_i$, the function $\varphi_i(x_i, .)$ is real-valued by boundedness of payoff function. It is also lower semicontinuous over X. Indeed, for each $i \in I$, let $x_i \in X_i$ and $\mathcal{V}$ be a open neighborhood. Consider the following function

$$g^i_{\mathcal{V}}(x_i, y) = \begin{cases} \inf_{z\in\mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)], & \text{if } y \in \mathcal{V}, \\ -\infty, & \text{otherwise.} \end{cases}$$

We show that $g^i_{\mathcal{V}}(x_i, .)$ is lower semicontinuous on X. Let

$$A(x_i) = \{y \in X \text{ such that } g^i_{\mathcal{V}}(x_i, y) \leq \alpha\},\ \alpha \in \mathbb{R}.$$

Suppose that there exists $y \in X$ such that y is in the closure of $A(x_i)$, but not in $A(x_i)$. Then, there exists a sequence $\{y^p\}_{p\in\mathbb{N}} \subset A(x_i)$ converging to y. Since $y \notin A(x_i)$, $\inf_{z\in\mathcal{V}}[u_i(x_i, z_{-i}) - u_i(z)] > \alpha$. If $y \notin \mathcal{V}$, then $-\infty > \alpha$, which is impossible, and thus $y \in \mathcal{V}$ and $g^i_{\mathcal{V}}(x_i, y) > \alpha$. Otherwise, we have $\{y^p\}_{p\in\mathbb{N}} \subset A(x_i)$, and then $g^i_{\mathcal{V}}(x_i, y^p) \leq \alpha$ for every $p \in \mathbb{N}$. If there exists $\bar{p} \in \mathbb{N}$ such that $y^{\bar{p}} \in \mathcal{V}$, then $\inf_{z\in\mathcal{V}}[u_i(x_i, z_{-i}) - u_i(z)] \leq \alpha$, which contradicts the fact that

$\inf_{z \in \mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)] > \alpha$. Thus, for all $p \in \mathbb{N}$, $y^p \notin \mathcal{V}$. Since the sequence $\{y^p\}_{p \in \mathbb{N}}$ converges to $y \in \mathcal{V}$, there exists $\eta \in \mathbb{N}$ such that, for all $p \geq \eta$, $y^p \in \mathcal{V}$, which contradicts the fact that $y^p \notin \mathcal{V}$ for all $p \in \mathbb{N}$. Thus, $A(x_i)$ is closed, which means that the function $g^i_{\mathcal{V}}(x_i, .)$ is lower semicontinuous over X. Since the function $\varphi_i(x_i, .)$ is the pointwise supremum of a collection of lower semicontinuous functions on X, by Lemma 2.39, page 43 in Aliprantis and Border [1994], $\varphi_i(x_i, .)$ is lower semicontinuous on X.

Now, if there exists $\bar{x} \in X$ such that, for all $i \in I$,

$$\sup_{x_i \in X_i} \varphi_i(x_i, \bar{x}) \leq 0, \tag{3}$$

then $\bar{x}$ is a Nash equilibrium. Indeed, suppose $\bar{x}$ is not a Nash equilibrium. Since the game G is weakly transfer continuous, then there exists player i, y_i, and a neighborhood $\mathcal{V}$ of $\bar{x}$ such that $u_i(y_i, z_{-i}) > u_i(z)$, for all $z \in \mathcal{V}$. Then, $\varphi(y_i, \bar{x}) > 0$, which contradicts (3).

Consider the following collection:

$$C(y) = \{x \in X : \ \varphi_i(y_i, x) \leq 0, \ i = 1, ..., n\}, \ \ y \in X.$$

Then, for every $y \in X$, the set $C(y)$ is closed in X and by Lemma 1 in Tian [1993], the collection $\{C(y), y \in X\}$ has the finite intersection if the game G is diagonally transfer quasiconcave. Since X is compact, then $\bigcap_{y \in X} C(y) \neq \emptyset$. Hence, there exists $\bar{x} \in X$ such that for all $i \in I$, we have $\sup_{x_i \in X_i} \varphi_i(x_i, \bar{x}) \leq 0$.

Necessity (⇒): It is the same as that of Theorem 1 in Baye *et al.* [1993]. ∎

Since a game is diagonally transfer quasiconcave provided it is quasiconcave, we have the following corollary.

COROLLARY 3.1 *A game $G = (X_i, u_i)_{i \in I}$ possesses a pure strategy Nash equilibrium if it is convex, compact, bounded, weakly transfer continuous, and quasiconcave.*

REMARK 3.2 Weak transfer continuity neither implies nor is implied by better-reply security in Reny [1999] or diagonal transfer continuity in Baye *et al.* [1993].

EXAMPLE 3.2 Consider the two-player game with the following payoff functions defined on $[0, 1] \times [0.1]$.

$$u_i(x_1, x_2) = \begin{cases} \varphi_i(x_1, x_2), & \text{if } x_1 = x_2, \\ \psi_i(x_1, x_2), & \text{otherwise,} \end{cases}$$

where $\varphi_1(x) \geq 2$ and is bounded, $0 \leq \varphi_2(x) \leq 1$, and $\psi_i(x) \leq 1$, $i = 1, 2$, for all $x \in [0, 1] \times [0.1]$.

Clearly, the diagonal game G is compact and bounded. Suppose that G is also quasiconcave and satisfies the following conditions:

$$\begin{cases} \psi_1(1, x_2) > \psi_1(x_1, x_2), \ \forall x_1 < 1, \ x_1 \neq x_2, \ \forall x_2 \in [0, 1], \\ u_2(x_1, 1) > u_2(x_1, x_2), \ \forall x_2 < 1, \ \forall x_1 \in [0, 1]. \end{cases}$$

Then we can show that it is also weakly transfer continuous so that it has a Nash equilibrium by Corollary 3.1.

Indeed, suppose x is not a Nash Equilibrium. Then there exists $z \in X$ such that either $u_1(z_1, x_2) > u_1(x)$ or $u_2(x_1, z_2) > u_2(x)$.

1. $u_1(z_1, x_2) > u_1(x)$. If $x_1 = x_2$, then $u_1(z_1, x_2) > \varphi_1(x_1, x_2)$, which is impossible. Thus, $x_1 \neq x_2$. Therefore, $u_1(z_1, x_2) > \psi_1(x_1, x_2)$.

   1.1. If $x_1 < 1$, let $y_1 = 1$ and choose a neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{V}(x) \subset [0, 1) \times [0, 1]$ and for all $x' \in \mathcal{V}(x)$, $x'_1 \neq x'_2$. Thus, we have $u_1(y_1, x'_2) > \psi_1(x')$ for all $x' \in \mathcal{V}(x)$.

   1.2. If $x_1 = 1$, we must have $x_2 < 1$. Then, let $y_2 = 1$ and choose a neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{V}(x) \subset [0, 1] \times [0, 1)$ and for all $x' \in \mathcal{V}(x)$, $x'_1 \neq x'_2$. Thus, we have $u_2(x'_1, y_2) > u_2(x')$ for all $x' \in \mathcal{V}(x)$.

2. $u_2(x_1, z_2) > u_2(x)$. If $x_1 = x_2 = 1$, then $u_2(z_1, x_2) > \varphi_2(x_1, x_2)$, which is impossible by assumption. Thus, $(x_1, x_2) \neq (1, 1)$.

   2.1. If $x_1 \neq x_2$ and $x_1 < 1$, let $y_1 = 1$ and choose a neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{V}(x) \subset [0, 1) \times [0, 1]$ and for all $x' \in \mathcal{V}(x)$, $x'_1 \neq x'_2$. Thus, $u_1(y_1, x'_2) > \psi_1(x')$ for all $x' \in \mathcal{V}(x)$.

   2.2. If $x_1 = x_2$ or $x_1 = 1$, let $y_2 = 1$ and choose a neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{V}(x) \subset [0, 1] \times [0, 1)$ and for all $x' \in \mathcal{V}(x)$. Thus, we have $u_2(x'_1, y_2) > u_2(x')$ for all $x' \in \mathcal{V}(x)$.

Let $U(x, y) = u_1(y_1, x_2) + u_2(x_1, y_2)$ and let $x = (0, 0)$, $y = (0, 1)$. Then, $U(x, y) = \varphi_1(0, 0) + \psi_2(0, 1) > U(x, x) = \varphi_1(0, 0) + \varphi_2(0, 0)$. However, for all $y^{'} \in [0, 1] \times [0, 1]$ and $\delta > 0$, there exists $z \in B_\delta(x) \cap [0, 1] \times [0, 1]$ such that $z_1 = z_2 \neq y^{'}_1$. Thus, we have $U(z, y^{'}) = \psi_1(y^{'}_1, z_1) + u_2(z_1, y^{'}_2) \leq 2 \leq \varphi_1(z_1, z_2) + \varphi_2(z_1, z_1) = U(z, z)$. Thus, the game is not diagonally transfer continuous, so Theorem 1 of Baye *et al.* [1993] can not apply.

EXAMPLE 3.3 Consider the two-player game with the following payoff functions defined on $[0,1]\times[0.1]$.

$$u_1(x_1,x_2)=\begin{cases}4+x_1+x_2, & \text{if } x_1=x_2,\\ x_1+x_2, & \text{otherwise,}\end{cases}$$

$$u_2(x_1,x_2)=\begin{cases}4, & \text{if } x_1=x_2=0,\\ 6, & \text{if } (x_1,x_2)=(0,1),\\ 3(x_1+x_2), & \text{otherwise,}\end{cases}$$

Clearly, this game G is compact, bounded, and quasiconcave. We can show that it is also weakly transfer continuous so that it has a Nash equilibrium by Corollary 3.1.

To see this, suppose x is not a Nash Equilibrium. Then there exists $z\in X$ such that either $u_1(z_1,x_2)>u_1(x)$ or $u_2(x_1,z_2)>u_2(x)$.

1. $u_1(z_1,x_2)>u_1(x)$. If $x_1=x_2$, then $u_1(z_1,x_2)>4+x_1+x_2$, which is impossible. Thus, $x_1\neq x_2$. Therefore, $u_1(z_1,x_2)>x_1+x_2$.

   1.1. If $x_1<1$, let $y_1=1$ and choose a neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{V}(x)\subset[0,1)\times[0,1]$ and for all $x'\in\mathcal{V}(x)$, $x'_1\neq x'_2$. Thus, we have $u_1(y_1,x'_2)\geq 1+x'_2>x'_1+x'_2=u_1(x')$ for all $x'\in\mathcal{V}(x)$.

   1.2. If $x_1=1$, we must have $x_2<1$. Then, let $y_2=1$ and choose a neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{V}(x)\subset[0,1]\times[0,1)$ and for all $x'\in\mathcal{V}(x)$, $x'_1\neq x'_2$. Thus, we have $u_2(x'_1,y_2)=3(x'_1+1)>3(x'_1+x'_2)=u_2(x')$ for all $x'\in\mathcal{V}(x)$.

2. $u_2(x_1,z_2)>u_2(x)$. If $x_1=x_2=1$, then $u_2(z_1,x_2)>6$, which is impossible. Thus, $(x_1,x_2)\neq(1,1)$.

   2.1. If $x_1\neq x_2$ and $x_1<1$, let $y_1=1$ and choose a neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{V}(x)\subset[0,1)\times[0,1]$ and for all $x'\in\mathcal{V}(x)$, $x'_1\neq x'_2$. Thus, $u_1(y_1,x'_2)\geq 1+x'_2>x'_1+x'_2=u_1(x')$ for all $x'\in\mathcal{V}(x)$.

   2.2. If $x_1=x_2$ or $x_1=1$, let $y_2=1$ and choose a neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{V}(x)\subset[0,1]\times[0,1)$ and for all $x'\in\mathcal{V}(x)$. Thus, we have $u_2(x'_1,y_2)>u_2(x')$ for all $x'\in\mathcal{V}(x)$. Indeed, if $x'_1=0$, then $u_2(x'_1,y_2)=6>u_2(0,x'_2)=\begin{cases}4, & \text{if } x'_2=0,\\ 3x'_2, & \text{otherwise,}\end{cases}$ and if $x'_1\neq 0$, $u_2(x'_1,y_2)=3(x'_1+1)>u_2(x'_1,x'_2)$.

However, we can show the game is neither diagonally transfer continuous nor better-reply secure.

Let $U(x,y) = u_1(y_1,x_2) + u_2(x_1,y_2)$ and $x = (\epsilon,\epsilon)$, $y = (\epsilon,1)$ with $0 < \epsilon < 0$. Then, $U(x,y) = u_1(\epsilon,\epsilon) + u_2(\epsilon,1) > U(x,x) = u_1(\epsilon,\epsilon) + u_2(\epsilon,\epsilon)$. However, for all $y^{'} \in [0,1]\times[0,1]$ and $\delta > 0$, there exists $z \in B_\delta(\epsilon,\epsilon) \cap [0,1]\times[0,1]$ such that $z_1 = z_2 \notin \{0, y_1^{'}\}$. Thus, we have $U(z,y^{'}) = u_1(y_1^{'},z_1) + u_2(z_1,y_2^{'}) = y_1^{'} + z_1 + 3(z_1 + y_2^{'}) \leq 4 + 4z_1 \leq 4 + 8z_1 = U(z,z)$. Thus, this game is not diagonally transfer continuous, so Theorem 1 of Baye *et al.* [1993] can not apply.

Now, let $\overline{x} = (0,0)$ and $u = (4,4)$. Clearly (x,u) is in the closure of the graph of its vector function, and $\overline{x}$ is not a Nash equilibrium, thus, player 1 cannot obtain a payoff strictly above $u_1 = 4$. Indeed, for all $x_1 \in [0,1]$, and every $\delta > 0$, there exists $z_2 \neq x_1 \in B_\delta(0)$ such that $u_1(x_1,z_2) = x_1 + z_2 \leq u_1 = 4$. Player 2 cannot obtain a payoff strictly above $u_2 = 4$ either. To see this, for all $x_2 \in [0,1]$ and any $\delta > 0$, there exists $z_1 \neq x_1 \in B_\delta(0)$ where $z_1 = \begin{cases} \text{very small } \neq 0, & \text{if } x_2 = 1, \\ 0, & \text{if } x_2 < 1, \end{cases}$ such that $u_2(z_1,x_2) \leq u_2 = 4$. Thus, this game is not better-reply secure, so Theorem 3.1 of Reny [1999] can not apply.

While it is simple to verify weak transfer continuity, it is sometimes even simpler to verify other conditions leading to it. In addition to the fact that transfer continuity implies weak transfer continuity, weak transfer upper continuity and weak transfer lower continuity introduced below, combined respectively with payoff security and upper semicontinuity, also imply weak transfer continuity, respectively.

DEFINITION 3.3 A game $G = (X_i,\ u_i)_{i\in I}$ is said to be *upper semicontinuous* if for each player i, the payoff function u_i is upper semicontinuous over X, *i.e.* for each $x \in X$, and every $\epsilon > 0$, there exists a neighborhood $\mathcal{V}_x$ of x such that $u_i(x) \geq u_i(x') - \epsilon$, for each $x' \in \mathcal{V}(x)$.

DEFINITION 3.4 A game $G = (X_i,\ u_i)_{i\in I}$ is said to be *weakly transfer upper continuous* if $x \in X$ is not an equilibrium, then there exists player i, $\hat{x}_i \in X_i$ and a neighborhood $\mathcal{V}(x)$ of x such that $u_i(\hat{x}_i, x_{-i}) > u_i(x')$ for all $x' \in \mathcal{V}(x)$.

REMARK 3.3 If the game G is upper semicontinuous, then G is weakly transfer upper continuous. Indeed, suppose x is not a Nash equilibrium, then there exists player i and a strategy y_i such that $u_i(y_i,x_{-i}) > u_i(x)$. Choose $\epsilon > 0$ such that $u_i(y_i,x_{-i}) - \epsilon > u_i(x)$. Since G is upper semicontinuous, then there exists a neighborhood $\mathcal{V}_x$ of x such that $u_i(y_i,x_{-i}) - \epsilon > u_i(x) \geq u_i(x') - \epsilon$, for each $x' \in \mathcal{V}(x)$.

DEFINITION 3.5 A game $G = (X_i,\ u_i)_{i\in I}$ is said to be *weakly transfer lower continuous* if x is not a Nash Equilibrium, which implies that there exists a player i, $y_i \in X_i$, and a neighborhood of $\mathcal{V}(x_{-i})$ of x_{-i} such that $u_i(y_i, x'_{-i}) > u_i(x)$ for all $x'_{-i} \in \mathcal{V}_{x_{-i}}$.

REMARK 3.4 If the game G is payoff secure, then G is weakly transfer lower continuous. To see this, suppose $x \in X$ and x is not a Nash equilibrium, then there exists a player i that has a strategy $\hat{x}_i$ such that $u_i(\hat{x}_i, x_{-i}) > u_i(x)$. Choose $\epsilon > 0$ such that $u_i(\hat{x}_i, x_{-i}) - \epsilon > u_i(x)$. Since G is payoff secure, then there exists a strategy y_i and a neighborhood $\mathcal{V}(x_{-i})$ of x_{-i} such that $u_i(y_i, x'_{-i}) \geq u_i(\hat{x}_i, x_{-i}) - \epsilon > u_i(x)$, for each $x'_{-i} \in \mathcal{V}(x_{-i})$.

We then have the following propositions that provide sufficient conditions for weak transfer continuity.

PROPOSITION 3.1 *If a game $G = (X_i,\ u_i)_{i \in I}$ is weakly transfer upper continuous and payoff secure, then it is weakly transfer continuous.*

PROOF. Suppose $\overline{x} \in X$ is not a Nash equilibrium. Then, by weak transfer upper continuity, some player i has a strategy $\hat{x}_i \in X_i$ and a neighborhood $\mathcal{V}(\overline{x})$ of $\overline{x}$ such that $u_i(\hat{x}_i, \overline{x}_{-i}) > u_i(z)$ for all $z \in \mathcal{V}(\overline{x})$. Choose $\epsilon > 0$ such that $u_i(\hat{x}_i, \overline{x}_{-i}) - \epsilon > \sup_{z \in \mathcal{V}(\overline{x})} u_i(z)$. The payoff security of G implies that there exists a strategy y_i and a neighborhood $\tilde{\mathcal{V}}(\overline{x}_{-i})$ of $\overline{x}_{-i}$ such that $u_i(y_i, z_{-i}) \geq u_i(\hat{x}_i, \overline{x}_{-i}) - \epsilon$ for all $z_{-i} \in \tilde{\mathcal{V}}(\overline{x}_{-i})$. Thus, there exists $y_i \in X_i$ and a neighborhood of $\hat{\mathcal{V}}(\overline{x})$ of $\overline{x}$ such that $u_i(y_i, z_{-i}) > u_i(z)$ for all $z \in \hat{\mathcal{V}}(\overline{x})$. ■

PROPOSITION 3.2 *If a game $G = (X_i,\ u_i)_{i \in I}$ is weakly transfer lower continuous and upper semicontinuous, then it is weakly transfer continuous.*

PROOF. Suppose $\overline{x} \in X$ is not a Nash equilibrium. Then, by weak transfer lower continuity, some player i has a strategy $\hat{x}_i \in X_i$ and a neighborhood $\mathcal{V}(\overline{x}_{-i})$ of $\overline{x}_i$ such that $u_i(\hat{x}_i, z_{-i}) > u_i(\overline{x})$ for all $z_{-i} \in \mathcal{V}(\overline{x}_{-i})$. Choose $\epsilon > 0$ such that $\inf_{z_{-i} \in \mathcal{V}(\overline{x}_{-i})} u_i(\hat{x}_i, z_{-i}) > u_i(\overline{x}) + \epsilon$. The upper semicontinuity of G implies that there exists a neighborhood $\tilde{\mathcal{V}}(\overline{x})$ of $\overline{x}$ such that $u_i(\overline{x}) + \epsilon \geq u_i(z)$ for all $z \in \tilde{\mathcal{V}}(\overline{x})$. Thus, there exists $y_i \in X_i$ and a neighborhood of $\hat{\mathcal{V}}(\overline{x})$ of $\overline{x}$ such that $u_i(y_i, z_{-i}) > u_i(z)$ for all $z \in \hat{\mathcal{V}}(\overline{x})$. ■

REMARK 3.5 It is worth to point out that, when payoffs are upper semicontinuous, payoff security implies better-reply security while weak transfer lower continuity implies weak transfer continuity.

Propositions 3.1-3.2, together with Theorem 3.1, immediately yield the following useful results.

COROLLARY 3.2 *If a game $G = (X_i, u_i)_{i \in I}$ is convex, compact, bounded, weakly transfer upper continuous and payoff secure, and quasiconcave, then it possesses a pure strategy Nash equilibrium.*

COROLLARY 3.3 *If a game $G = (X_i, u_i)_{i \in I}$ is convex, compact, bounded, weakly transfer lower continuous and upper semicontinuous, and quasiconcave, then it possesses a pure strategy Nash equilibrium.*

As an application of the above proposition, consider the following well-known noisy game.

EXAMPLE 3.4 Consider the two-player, nonzero sum, noisy games with the following payoff functions defined from $[0,1] \times [0.1]$.

$$f_i(x_i, x_{-i}) = \begin{cases} l_i(x_i), & \text{if } x_i < x_{-i}, \\ \phi_i(x_i), & \text{if } x_i = x_{-i}, \\ m_i(x_{-i}), & \text{if } x_i > x_{-i}, \end{cases}$$

where $l_i(.)$, $m_i(.)$ and $\phi_i(.)$ are upper semicontinuous over $[0,1]$, $l_i(.)$ is strictly nondecreasing on $[0,1]$ and satisfies the following conditions:

ASSUMPTION 3.1

a) $\forall x \in [0,1]$, $\forall \epsilon > 0$, there exists a neighborhood $\mathcal{V}_x$ of x such that $\phi_i(x) \geq \max(l_i(z), m_i(z)) - \epsilon$, for every $z \in \mathcal{V}_x$.

b) if $m_i(x) > \phi_i(x)$ with $x < 1$, then there exists a neighborhood $\mathcal{V}_x \subset [0,1)$ of x such that $m_i(z) > \phi_i(x)$, for every $z \in \mathcal{V}_x$.

c) if $\phi_i(x) > m_i(x)$ with $x < 1$, then there exists a neighborhood $\mathcal{V}_x \subset [0,1)$ of x such that $\phi_i(z) > m_i(x)$, for every $z \in \mathcal{V}_x$.

It is clear that this game G is compact and convex. Suppose that G is quasiconcave. When the game satisfies these conditions, one can show that it is upper semicontinuous and weakly transfer lower continuous so that it has a Nash equilibrium by Corollary 3.3.[2]

Indeed, the condition a) and the upper semicontinuity of $l_i(.)$, $m_i(.)$ and $\phi_i(.)$ over $[0,1]$, imply that the noisy game is upper semicontinuous. The conditions b) and c) imply that the game is weakly transfer lower continuous. Therefore the game processes a Nash equilibrium by Corollary 3.3.

EXAMPLE 3.5 As an application of the above example, consider the following well-known *Noisy Duel* game where $x_1, x_2 \in [0,1]$:

$$u_1(x_1, x_2) = u_2(x_1, x_2) = \begin{cases} 2x_1 - 1, & \text{if } x_1 < x_2, \\ 0, & \text{if } x_1 = x_2, \\ 2x_2 - 1, & \text{if } x_1 > x_2. \end{cases}$$

[2] As Reny [1999] showed, if $\phi_i(x) \in co\{l_i(x), m_i(x)\}$ and $l_i(x)$ is nondecreasing, then the game is quasiconcave.

Let $l_1(x) = l_2(x) = l(x) = 2x - 1$, $m_1(x) = m_2(x) = m(x) = 2x - 1$ and $\phi(x) = \phi_i(x) = 0$, $i = 1, 2$. It is clear that the functions $l(.)$, $m(.)$ and $\phi(.)$ are continuous over $[0, 1]$, and $l(.)$ is strictly nondecreasing on $[0, 1]$ and satisfies the following conditions: 1) $\forall x \in [0, 1]$, $\forall \epsilon > 0$, there exists a neighborhood $\mathcal{V}_x \subset [0, 1]$ of x such that $0 \geq \max(l(z), m(z)) - \epsilon$, for every $z \in \mathcal{V}_x$; b) if $m(x) = 2x - 1 > \phi_i(x) = 0$ with $x < 1$, then there exists a neighborhood $\mathcal{V}_x \subset (1/2, 1)$ of x such that $m(z) > \phi(x)$, for every $z \in \mathcal{V}_x$; 3) if $\phi(x) = 0 > m(x) = 2x - 1$, then there exists a neighborhood $\mathcal{V}_x \subset [0, 1/2)$ of x such that $\phi(z) > m(x)$, for every $z \in \mathcal{V}_x$. Then, according to Example 3.4, the game is upper semicontinuous and weakly transfer lower continuous and since G is compact, convex and quasiconcave, it processes a Nash equilibrium by Corollary 3.3.

3.2 Nash Equilibrium in Discontinuous and Nonconvex Games

In this subsection we characterize the existence of pure strategy Nash equilibrium in games that are both discontinuous and nonconvex. We generalize the results above as well as the existence results of Baye *et al.* [1993], Reny [1999] and Bagh and Jofre [2006] without assuming the convexity of strategy spaces or any form of quasiconcavity of payoff functions.

The following theorem generalizes Theorem 3.1 by relaxing the convexity of strategy spaces and diagonal transfer quasiconcavity of payoff functions.

THEOREM 3.2 *Suppose $G = (X_i, u_i)_{i \in I}$ is compact, bounded, and weakly transfer continuous. Then, the game G has a Nash equilibrium if and only if for all $A \in \langle X \rangle$, there exists $x \in X$ such that for all $i \in I$, $u_i(y_i, x_{-i}) \leq u_i(x),\ \forall y \in A$.*

PROOF. *Sufficiency* ($\Leftarrow$). The proof of sufficiency is the same as that of Theorem 3.1 except the last paragraph. Note that, for $C(y) = \{x \in X :\ \varphi_i(y_i, x) \leq 0,\ \forall i \in I\}$ and for $y \in X$, $C = (X_i, u_i)_{i \in I}$ has at least one pure strategy Nash equilibrium if and only if $\bigcap_{x \in X} C(x) \neq \emptyset$. Thus, it suffices to show that $\bigcap_{x \in X} C(x) \neq \emptyset$ is nonempty. Since the function $y_i \mapsto \varphi_i(x, y_i)$ is lower semicontinuous on compact set X_i, $C(x)$ is a compact subset for every $x \in X$. Thus, it suffices to show that the family $\{C(x)\}_{x \in X}$ possesses the finite intersection property. Indeed, by assumption, for every $A \in \langle X \rangle$, there exists $\hat{y} \in X$ such that $u_i(x) \leq u_i(\hat{y}_i, x_{-i})$, $\forall x \in A$ and $\forall i \in I$.

Let $\mathcal{V} \in \mho(\hat{y}_i)$ be a neighborhood of $\hat{y}_i$. Then, $\inf_{z_i \in \mathcal{V}} [u_i(x) - u_i(z_i, x_{-i})] \leq 0$ for all $x \in A$. Thus, $\varphi_i(x, \hat{y}_i) \leq 0$, $\forall x \in A$ and $\forall i \in I$. Therefore, for every $A \in \langle X \rangle$, there exists $\hat{y} \in X$ such that $\hat{y} \in \bigcap_{x \in A} C(x)$.

Necessity (⇒): Let $x^* \in X$ be a pure strategy Nash equilibrium of the game G. Then for all $i \in I$, $u_i(y_i, x^*_{-i}) \leq u_i(x^*)$ for all $y \in X_i$, and thus we have $\max_{y \in A} u_i(y_i, x^*_{-i}) \leq u_i(x^*)$ for any subset $A = \{y^1, ..., y^m\} \in \langle Y \rangle$. ■

Theorem 1 in Baye *et al.* [1993] can also be generalized by relaxing the convexity of strategy spaces and diagonal transfer quasiconcavity of payoff functions.

THEOREM 3.3 *Suppose $G = (X_i, u_i)_{i \in I}$ is compact and diagonally transfer continuous. Then, the game G has a Nash equilibrium if and only if for all $A \in \langle X \rangle$, there exists $x \in X$ such that for all $i \in I$, $u_i(y_i, x_{-i}) \leq u_i(x)$, $\forall y \in A$.*

PROOF. The proof of necessity is the same as that of Theorem 3.2. We only need to prove sufficiency.

Let

$$F(y) = \{x \in X : \ U(x, y) \leq U(x, x), \ \forall i \in I\} \ \ y \in X.$$

It is clear that $G = (X_i, u_i)_{i \in I}$ has a pure strategy Nash equilibrium if and only if $\bigcap_{y \in X} F(y) \neq \emptyset$. Since $U(x, y)$ is diagonally transfer continuous in x, we have $\bigcap_{y \in X} F(y) = \bigcap_{y \in X} clF(y)$ by Theorem 1 in Baye *et al.* [1993]. Furthermore, by assumption, for every $A \in \langle X \rangle$, there exists $\hat{x} \in X$ such that $u_i(y_i, \hat{x}_{-i}) \leq u_i(\hat{x})$, $\forall y \in A$ and $\forall i \in I$, and thus $U(y, \hat{x}) \leq U(\hat{x})$, $\forall y \in A$, which means $F(y)$ has the finite intersection property. Since X is compact, we have $\bigcap_{y \in X} F(y) = \bigcap_{y \in X} clF(y) \neq \emptyset$. This completes the proof. ■

EXAMPLE 3.6 Consider a game with $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [1, 2] \cup [3, 4]$, $x = (x_1, x_2)$ and

$$u_1(x) = x_2 x_1^2,$$
$$u_2(x) = -x_1 x_2^2.$$

Note that, X_i is not convex for $i = 1, 2$, and the function $y_i \mapsto u_i(x_{-i}, y_i)$ is not quasiconcave for $i = 1$ so that the existing theorems on Nash equilibrium are not applicable.

However, we can show the existence of Nash equilibrium by applying Theorem 3.3. Indeed, for each $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$U(x, y) = x_2 y_1^2 - x_1 y_2^2.$$

The function U is continuous on $X \times X$. For any subset $\{({}_1y_1, {}_2y_2), ..., ({}_ky_1, {}_ky_2)\}$ of X, let $x = (x_1, x_2) \in X$ such that $x_1 = \max_{h=1,...,k} {}_iy_1$ and $x_2 = \min_{h=1,...,k} {}_iy_2$. Then, we have

$$\begin{cases} {}_iy_2^2 \geq x_2^2, \ \ \forall i = 1, ..., k, \\ {}_iy_1^2 \leq x_1^2. \end{cases}$$

Thus,

$$\begin{cases} -x_1 \ {}_iy_2^2 \leq -x_1 \ x_2^2, \ \ \forall i = 1, ..., k, \\ x_2 \ {}_iy_1^2 \leq x_2 \ x_1^2. \end{cases}$$

Therefore, $U(x, \ {}_iy) \leq U(x, x), \forall i = 1, ..., k$. According to Theorem 3.3, this game has a Nash equilibrium.

Finally we can also generalize Theorem 3.1 in Reny [1999] and the results in Bagh and Jofre [2006] by relaxing convexity of strategy spaces and quasiconcavity of payoff functions.

THEOREM 3.4 *Suppose that $G = (X_i, u_i)_{i \in I}$ is compact, bounded, and better-reply secure. Then, $G = (X_i, \ u_i)_{i \in I}$ has a Nash equilibrium if and only if for all $A \in \langle X \rangle$, there exists $x \in X$ such that for each $i \in I$, we have $u_i(y_i, x_{-i}) \leq u_i(x), \ \forall y \in A$.*

PROOF. *Necessity* ($\Rightarrow$): The proof of necessity is the same as that of Theorem 3.2. We only need to prove sufficiency.

Sufficiency ($\Leftarrow$): For each i and every $x \in X$, let

$$\underline{u}_i(x) = \sup_{\mathcal{V} \in \mho(x_{-i})} \inf_{z_{-i} \in \mathcal{V}} u_i(x_i, z_{-i}),$$

where $\mho(a)$ is the set of all open neighborhoods of a.

We have:

- $\underline{u}_i(x_i, .)$ is lower semicontinuous on X_{-i}
- If $(\overline{x}, \overline{\alpha}) \in \Gamma$ and $\forall i \in I$, $\sup_{x_i \in X_i} \underline{u}_i(x_i, \overline{x}_{-i}) \leq \overline{\alpha}_i$, then $\overline{x}$ is a Nash equilibrium (Reny [1999]).

Let $\underline{u}(x, y) = (\underline{u}(x_1, y_{-1}), ..., \underline{u}(x_I, y_{-I}))$ and $E(x) = \{(y, \alpha) \in \Gamma : \ \underline{u}(x, y) \leq \alpha\}$, where $\leq$ denotes componentwise weak order in $\mathbb{R}^I$.

Then, the game G has at least one Nash equilibrium if and only if $E = \bigcap_{x \in X} E(x) \neq \emptyset$. Thus, it suffices to show that E is nonempty. Since Γ is a compact subset of $X \times \mathbb{R}^I$, and the function $y \mapsto \underline{u}(x, y)$ is lower semicontinuous over X, for every $x \in X$, then $E(x)$ is also a compact subset of $X \times \mathbb{R}^I$. Thus, it suffices to show that the family $\{E(x)\}_{x \in X}$ possesses the finite intersection property. To see this, note that, by assumption, for every $A \in \langle X \rangle$, there exists $\hat{x} \in X$ such that $u_i(y_i, \hat{x}_{-i}) \leq u_i(\hat{x})$, $\forall y \in A$ and $\forall i \in I$.

Let $\mathcal{V} \in \mho(\hat{x}_{-i})$ be a neighborhood of $\hat{x}_{-i}$. We have $\inf_{z_{-i} \in \mathcal{V}} u_i(y_i, z_{-i}) \leq u_i(y_i, \hat{x}_{-i})$ for all $y \in A$. Then, $\underline{u}_i(y_i, \hat{x}_{-i}) \leq u_i(y_i, \hat{x}_{-i}) \leq u_i(\hat{x}) = \hat{\alpha}_i$, $\forall y \in A$ and $\forall i \in I$. Thus, for every $A \in \langle X \rangle$, there exists $(\hat{x}, \hat{\alpha}) \in \Gamma$ such that $(\hat{x}, \hat{\alpha}) \in \bigcap_{y \in A} E(y)$. This completes the proof. ■

Since weak reciprocal upper semicontinuity and payoff security imply better-reply security, we have the following corollary.

COROLLARY 3.4 *Suppose that the game $G = (X_i,\ u_i)_{i \in I}$ is compact, bounded, weakly reciprocal upper semicontinuous and payoff secure. Then, $G = (X_i,\ u_i)_{i \in I}$ has a Nash equilibrium if and only if for all $A \in \langle X \rangle$, there exists $x \in X$ such that for each $i \in I$, we have $u_i(y_i, x_{-i}) \leq u_i(x)$, $\forall y \in A$.*

4 Existence of Dominant-Strategy Equilibria

In this section we investigate the existence of dominant strategy equilibria in games that may be discontinuous and may not have any form of quasiconcavities.

We start by reviewing some of the basic definitions and results introduced and obtained in Baye *et al.* [1993].

DEFINITION 4.1 The i-th player's payoff function $u_i : X \to \mathbb{R}$ is said to be transfer upper continuous in x_i with respect to X, if for $x_i \in X_i$ and $y \in X$, $u_i(y) > u_i(x_i, y_{-i})$ implies that there exists a point $y' \in X$ and a neighborhood $\mathcal{V}(x_i)$ of x_i such that $u_i(y') > u_i(x'_i, y'_{-i})$, for all $x'_i \in \mathcal{V}(x_i)$.

DEFINITION 4.2 The i-th player's payoff function $u_i : X \to \mathbb{R}$ is said to be uniformly transfer quasiconcave on X if, for any finite subset $Y^m = \{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $\{x_i^1, ..., x_i^m\} \subset X_i$ such that for any subset $\{y_i^{k^1}, y_i^{k^2}, ..., y_i^{k^s}\}$, $1 \leq s \leq m$, and any $x_i \in co\{x_i^{k^1}, x_i^{k^2}, ..., x_i^{k^s}\}$, we have $\min_{1 \leq l \leq s} \{u_i(y^{k^l}) - u_i(x_i, y_{-i}^{k^l})\} \leq 0$.

Baye *et al.* [1993] showed that a game $G = (X_i,\ u_i)_{i \in I}$ that is convex, compact and transfer upper continuous in x_i with respect to X must possess a dominant-strategy equilibrium if and only if u_i is uniformly transfer quasiconcave on X for all $i \in I$.

In the following subsections, we first provide a new result on the existence of dominant strategy equilibrium in discontinuous games. We then characterize the existence of dominant strategy equilibrium without assuming the convexity of strategy spaces or any form of quasiconcavity of payoff functions. We also generalize existence results of Baye *et al.* [1993] by relaxing the convexity of strategy spaces and diagonal transfer quasiconcavity of payoff functions.

4.1 Dominant Strategy Equilibrium in Discontinuous Games

We start by introducing the notion of weak dominant transfer upper continuity, which permits us to get new existence results on dominant strategy equilibrium in discontinuous and nonconvex games.

DEFINITION 4.3 A game $G = (X_i, u_i)_{i \in I}$ is said to be weakly dominant transfer upper continuous if for each $\overline{x} \in X$, $\overline{x}$ is not a dominant-strategy equilibrium, then there exists player i, a strategy $y \in X$ and a neighborhood $\mathcal{V}(\overline{x}_i)$ of $\overline{x}_i$ such that $u_i(y) > u_i(z_i, y_{-i})$, for each $z_i \in \mathcal{V}(\overline{x}_i)$.

A game is weakly dominant transfer upper continuous if for every non dominant-strategy equilibrium x^*, some player i has a strategy y_i which dominates all other strategy z_i in a neighborhood of x_i^* when other players play y_{-i}.

The following theorem characterizes the existence of dominant-strategy equilibria if the game is weakly dominant transfer upper continuous and the strategy spaces of players are convex.

THEOREM 4.1 *Suppose $G = (X_i, u_i)_{i \in I}$ is compact, bounded, convex and weakly dominant transfer upper continuous. Then, the game G has a dominant-strategy equilibrium if and only if G is uniformly transfer quasiconcave.*

PROOF. *Sufficiency* ($\Leftarrow$): For each player $i \in I$ and every $(x, y_i) \in X \times X_i$, let

$$\pi_i(x, y_i) = \sup_{\mathcal{V} \in \Omega(y_i)} \inf_{z_i \in \mathcal{V}} [u_i(x) - u_i(z_i, x_{-i})]$$

where $\Omega(y_i)$ is the set of all open neighborhoods of y_i.

For each i and every $x \in X$, the function $\pi_i(x, .)$ is real-valued by bounded of payoff functions. It is also lower semicontinuous over X_i. To see this, for $i \in I$, let $x \in X$ and $\mathcal{V}$ be a open neighborhood in X_i. Consider the following function

$$h^i_{\mathcal{V}}(x, y_i) = \begin{cases} \inf_{z_i \in \mathcal{V}} [u_i(x) - u_i(z_i, x_{-i})], & \text{if } y_i \in \mathcal{V}, \\ -\infty, & \text{otherwise.} \end{cases}$$

We show that $h^i_{\mathcal{V}}(x, .)$ is lower semicontinuous on X_i. Let

$$A(x) = \{y_i \in X_i : \ h^i_{\mathcal{V}}(x, y_i) \leq \alpha\}, \ \alpha \in \mathbb{R}.$$

Suppose that there exists $y_i \in X_i$ such that y_i is in the closure of $A(x)$, but not in $A(x)$. Then, there exists a sequence $\{y_i^p\}_{p\in\mathbb{N}} \subset A(x)$ that converges to y_i. Since $y_i \notin A(x)$, $\inf\limits_{z_i\in\mathcal{V}}[u_i(x) - u_i(z_i, x_{-i})] > \alpha$. If $y_i \notin \mathcal{V}$, then $-\infty > \alpha$, which is impossible, and thus $y_i \in \mathcal{V}$. Also, we have $h_{\mathcal{V}}^i(x, y_i) > \alpha$. Otherwise, we have $\{y_i^p\}_{p\in\mathbb{N}} \subset A(x)$, and then $h_{\mathcal{V}}^i(x, y_i^p) \leq \alpha$, for every $p \in \mathbb{N}$. If there exists $\bar{p} \in \mathbb{N}$ such that $y_i^{\bar{p}} \in \mathcal{V}$, then $\inf\limits_{z_i\in\mathcal{V}}[u_i(x) - u_i(z_i, x_{-i})] \leq \alpha$, which contradicts $\inf\limits_{z_i\in\mathcal{V}}[u_i(x) - u_i(z_i, x_{-i})] > \alpha$. Then, for all $p \in \mathbb{N}$, $y_i^p \notin \mathcal{V}$. Since the sequence $\{y_i^p\}_{p\in\mathbb{N}}$ converges to y_i and $y_i \in \mathcal{V}$, then there exists $\eta \in \mathbb{N}$ such that for all $p \geq \eta$, $y_i^p \in \mathcal{V}$, which contradicts the fact that for all $p \in \mathbb{N}$, $y_i^p \notin \mathcal{V}$. Thus, the set $A(x)$ is closed, *i.e.* the function $h_{\mathcal{V}}^i(x, .)$ is lower semicontinuous over X_i. Since the function $\pi_i(x, .)$ is the pointwise supremum of a collection of lower semicontinuous functions on X_i, then according to Lemma 2.39, page 43 in Aliprantis and Border [1994], $\pi_i(x, .)$ is lower semicontinuous on X_i.

Note that, if there exists $\bar{y} \in X$ such that for all $i \in I$,

$$\sup_{x\in X} \pi_i(x, \bar{y}_i) \leq 0, \tag{4}$$

then $\bar{y}$ is a dominant-strategy equilibrium. To see this, suppose $\bar{y}$ is not a dominant-strategy equilibrium. Since the game G is weakly dominant transfer upper continuous, then there exists player i, a strategy $x \in X$, and a neighborhood $\mathcal{V}$ of $\bar{y}_i$ such that $u_i(x) > u_i(z_i, x_{-i})$, for all $z_i \in \mathcal{V}$. Then, $\pi(x, \bar{y}_i) > 0$ which contradicts (4).

Consider the following collection:

$$H(x) = \{y \in X : \ \pi_i(x, y_i) \leq 0, \ i = 1, ..., n\}, \ \ x \in X.$$

Then, by lower semicontinuity of π, $H(x)$ is closed for all $x \in X$, and by Lemma 1 in Tian [1993], the collection $\{H(x), x \in X\}$ has the finite intersection if the game G is uniformly transfer quasiconcave. Since X is compact, then $\bigcap\limits_{x\in X} H(x) \neq \emptyset$. Hence, there exists $\bar{y} \in X$ such that for all $i \in I$, we have $\sup\limits_{x\in X} \pi_i(x, \bar{y}_i) \leq 0$.

Necessity ($\Rightarrow$): Suppose that the game G has a dominant-strategy equilibrium $\bar{y} \in X$. We need to show that G is uniformly transfer quasiconcave. For any finite subset $A = \{x^1, ..., x^m\} \subset X$, let the corresponding finite subset $B = \{y^1, ..., y^m\} = \{\bar{y}\}$. Then, for any subset $J \subset \{1, 2, ..., m\}$, for any $y \in co\{y^h, \ h \in J\} = \{\bar{y}\}$ and any $i \in I$ we have $\min\limits_{h\in J}[u_i(x^h) - u_i(y_i, x_{-i}^h)] \leq u_i(x^h) - u_i(y_i, x_{-i}^h) \leq 0$. Hence, G is uniformly transfer quasiconcave. ■

While it is also simple to verify weak dominant transfer upper continuity, it is sometimes even simpler to verify other conditions leading to it. The following proposition provides sufficient conditions for a game to be weakly dominant transfer upper continuous.

PROPOSITION 4.1 *Any of the following conditions implies that the game $G = (X_i, u_i)_{i \in I}$ is weakly dominant transfer upper continuous.*

(a) u_i is continuous in x_i.

(b) u_i is upper semi-continuous in x_i.

(c) u_i is transfer upper continuous in x_i.

4.2 Dominant Strategy Equilibrium in Discontinuous and Nonconvex Games

In this subsection we characterize the existence of dominant strategy equilibrium in games that are both discontinuous and nonconvex. We generalize the results above as well as the existence results of Baye *et al.* [1993] without assuming the convexity of strategy spaces or any form of quasiconcavity of payoff functions.

The following theorem generalizes Theorem 4.1 by relaxing the convexity of strategy spaces and diagonal transfer quasiconcavity of payoff functions.

THEOREM 4.2 *Suppose $G = (X_i, u_i)_{i \in I}$ is compact, bounded, and weakly dominant transfer upper continuous. Then, the game G has a dominant-strategy equilibrium if and only if for all $A \in \langle X \rangle$, there exists $x \in X$ such that $u_i(x) \leq u_i(y_i, x_{-i})$, for each $x \in A$.*

PROOF. *Sufficiency* ($\Leftarrow$): For each player $i \in I$ and every $(x, y_i) \in X \times X_i$, let

$$\pi_i(x, y_i) = \sup_{\mathcal{V} \in \Omega(y_i)} \inf_{z_i \in \mathcal{V}} [u_i(x) - u_i(z_i, x_{-i})]$$

where $\Omega(y_i)$ is the set of all open neighborhoods of y_i.

For each i and every $x \in X$, the function $\pi_i(x, .)$ is both real-valued and lower semicontinuous over X_i (see the sufficiency proof of Theorem 4.1).

If there exists $\bar{y} \in X$ such that for all $i \in I$,

$$\sup_{x \in X} \pi_i(x, \bar{y}_i) \leq 0,$$

then $\bar{y}$ is a dominant-strategy equilibrium.

Consider the following collection:

$$H(x) = \{y \in X : \ \pi_i(x, y_i) \leq 0, \ i \in I\}, \ \ x \in X.$$

Then, the game G has at least one dominant-strategy equilibrium if and only if $H = \bigcap_{x \in X} H(x) \neq \emptyset$. Thus, it suffices to show that H is nonempty. Since the function $y_i \mapsto \pi_i(x, y_i)$ is lower semicontinuous on X_i, for every $x \in X$, then $H(x)$ is also a compact subset of X. Thus, it suffices to show that the family $\{H(x)\}_{x\in X}$ possesses the finite intersection property. To see this, we have $\forall A \in \langle X\rangle$ and $\forall i \in I$, $\exists y_i \in X_i$ such that

$$u_i(x) \leq u_i(y_i, x_{-i}), \quad \text{for each } x \in A.$$

Then, for every $A \in \langle X\rangle$, there exists $\hat{y} \in X$ such that $u_i(x) \leq u_i(\hat{y}_i, x_{-i})$, $\forall x \in A$ and $\forall i \in I$.

Let $U \in \mho(\hat{y}_i)$ be a neighborhood of $\hat{y}_i$. Then, $\inf\limits_{z_i \in U}[u_i(x) - u_i(z_i, x_{-i})] \leq 0$ for all $x \in A$. Thus, $\pi_i(x, \hat{y}_i) \leq 0$, $\forall x \in A$ and $\forall i \in I$. Therefore, for every $A \in \langle X\rangle$, there exists $\hat{y} \in X$ such that $\hat{y} \in \bigcap\limits_{x\in A} H(x)$.

Necessity ($\Rightarrow$): It is the same as that of Theorem 3.2, so it is omitted here. ■

The existence theorem on dominant strategy in Baye *et al.* [1993] can be also generalized by relaxing the convexity of strategy spaces and the uniform transfer quasiconcavity of payoff functions. We first introduce the following weak notion of continuity.

DEFINITION 4.4 Let X be a nonempty subset of a topological space and Y be a nonempty subset. A function $f : X \times Y \to \mathbb{R}$ is said to be α-transfer lower continuous in x with respect to Y if for $(x, y) \in X \times Y$, $f(x, y) > \alpha$ implies that there exists some point $y' \in Y$ and some neighborhood $\mathcal{V}(x) \subset X$ of x such that $f(z, y') > \alpha$ for all $z \in \mathcal{V}(x)$.

Let $\hat{X} = \prod\limits_{i\in I} X = X^n$. A generic element of $\hat{X}$ is denoted by $\hat{y} = (y^1, \ldots, y^I)$. Define a function $\phi : X \times \hat{X} \to \mathbb{R}$ by

$$\phi(x, \hat{y}) = \sum_{i=1}^{n}\{u_i(y^i) - u_i(x_i, y^i_{-i})\}, \;\; \forall (x, \hat{y}) \in X \times \hat{X}.$$

Assume that for each i, X_i is a nonempty and compact subset of a topological space E_i and u_i is continuous on X. Then, for all $x \in X$, the maximum of $\phi(x, .)$ over $\hat{X}$ and $\min\limits_{x\in X} \max\limits_{\hat{y}\in\hat{X}} \phi(x, \hat{y})$ exists.

Note that, by the definition of ϕ, we have

$$\forall x \in X, \;\; \max_{\hat{y}\in\hat{X}} \phi(x, \hat{y}) \geq 0. \tag{5}$$

The following Lemma shows the relationship between the solution of ϕ and dominant strategy equilibrium for $G = (X_i, u_i)_{i\in I}$.

LEMMA 4.1 *A strategy profile $\overline{x} \in X$ is a dominant-strategy equilibrium for $G = (X_i,\ u_i)_{i\in I}$ if and only if $\max\limits_{\hat{y}\in\hat{X}} \phi(\overline{x}, \hat{y}) = 0$.*

PROOF. *Necessity* ($\Rightarrow$): Let $\overline{x} \in X$ be a dominant-strategy equilibrium for $G = (X_i,\ u_i)_{i \in I}$. Then, $u_i(\overline{x}_i, y^i_{-i}) \geq u_i(y^i)$, $\forall y^i \in X$, $\forall i \in I$. Hence, $\phi(\overline{x}, \hat{y}) = \sum_{i=1}^{n}\{u_i(y^i) - u_i(\overline{x}_i, y^i_{-i})\} \leq 0$, $\forall \hat{y} \in \hat{X}$, *i.e.,* $\max_{\hat{y} \in \hat{X}} \phi(\overline{x}, \hat{y}) \leq 0$. Combining this inequality with inequality (5), we have $\max_{\hat{y} \in \hat{X}} \phi(\overline{x}, \hat{y}) = 0$.

Sufficiency ($\Leftarrow$): Let $\overline{x} \in X$ be a strategy profile such that $\max_{\hat{y} \in \hat{X}} \phi(\overline{x}, \hat{y}) = 0$. This equality implies $\forall \hat{y} \in \hat{X}$, $\phi(\overline{x}, \hat{y}) = \sum_{i=1}^{n}\{u_i(y^i) - u_i(\overline{x}_i, y^i_{-i})\} \leq 0$. For each $i \in I$, we write $\phi(\overline{x}, \hat{y}) = u_i(y^i) - u_i(\overline{x}_i, y^i_{-i}) + \sum_{j=1, j\neq i}^{n}\{u_j(y^j) - u_j(\overline{x}_j, y^j_{-j})\} \leq 0\ \forall \hat{y} \in \hat{X}$. Letting $\hat{y} = (\overline{x}, ..., \overline{x}, y^i, \overline{x}, ..., \overline{x}) \in \hat{X}$ with y^i arbitrarily chosen in X, we have $\sum_{j=1, j\neq i}^{n}\{u_j(\overline{x}) - u_j(\overline{x}_j, \bar{x}_{-j})\} = 0$, and thus $u_i(y^i) \leq u_i(\overline{x}_i, y^i_{-i})$ $\forall y^i \in X$, $i = 1, \ldots, I$. Thus, $\overline{x}$ is a dominant-strategy equilibrium for the game $G = (X_i,\ u_i)_{i \in I}$. ■

By the inequality (5) and Lemma 4.1, we have the following proposition.

PROPOSITION 4.2 *Suppose that X is compact and u_i is continuous on X. Let*

$$\alpha = \min_{x \in X} \max_{\hat{y} \in \hat{X}} \phi(x, \hat{y}). \tag{6}$$

Then, the game $G = (X_i,\ u_i)_{i \in I}$ has at least one dominant strategy equilibrium if and only if $\alpha = 0$.

DEFINITION 4.5 $G = (X_i, u_i)_{i \in I}$ is 0-transfer lower continuous if ϕ is 0-transfer lower continuous in x with respect to $\hat{X}$.

We then have the following result.

THEOREM 4.3 *Suppose $G = (X_i, u_i)_{i \in I}$ is compact and 0-transfer lower continuous in x with respect to $\hat{X}$. Then, $G = (X_i,\ u_i)_{i \in I}$ has a dominant-strategy equilibrium if and only if for all $A \in \langle X \rangle$, there exists $x \in X$ such that $u_i(x) \leq u_i(y_i, x_{-i})$, for each $x \in A$.*

PROOF. *Sufficiency* ($\Leftarrow$): Let $D(y) = \{x \in X : \phi(x, y) \leq 0\}$ for $y \in Y$. Since G is 0-transfer lower continuous, $\bigcap_{y \in Y} D(y) = \bigcap_{y \in Y} \text{cl}\, D(y)$. To see this, let $x \in \bigcap_{y \in Y} \text{cl}\, D(y)$ but not in $\bigcap_{y \in Y} D(y)$. Then, there exists $y \in Y$ such that $x \notin D(y)$, *i.e.*, $\phi(x, y) > 0$. By 0-transfer lower continuity of ϕ in x with respect to X, there exists $y^{'} \in X$ and a neighborhood $\mathcal{V}(x)$ of x such that $\phi(z, y^{'}) > 0$ for all $z \in \mathcal{V}(x)$. Thus, $x \notin \text{cl}\, D(y^{'})$, a contradiction. Since by assumption, for all $A \in \langle X \rangle$, there exists $x \in X$ such that $u_i(x) \leq u_i(y_i, x_{-i})$, for each $x \in A$, we know that $\{\text{cl}\, D(y) :\in Y\}$ has the finite intersection property. Also, $\{\text{cl}\, D(y) : y \in Y\}$ is a compact family in the compact

X. Thus, $\emptyset \neq \bigcap_{y \in X} D(y)$. Hence, there exists $\overline{x} \in X$ such that $\phi(x^*, y) \leq 0 = \phi(x^*, x^*)$ for $y \in X$, and thus, by Lemma 4.1, x^* is a dominant strategy equilibrium.

Necessity ($\Rightarrow$): It is the same as that of Theorem 3.2, so it is omitted here. ■

COROLLARY 4.1 *Suppose that the game* (1) *is partially separable*[3], X_i *is a nonempty and compact subset of a topological space* E_i, *and* $h_i(x_i)$ *is upper semicontinuous over* X_i, $\forall i \in I$. *Then, the game* $G = (X_i,\ u_i)_{i \in I}$ *has a dominant-strategy equilibrium.*

EXAMPLE 4.1 Again consider Example 3.6.

$$u_1(x) = x_2 x_1^2,$$
$$u_2(x) = -x_1 x_2^2.$$

Since X_i is not convex $\forall i \in I$, Theorem 4 in Baye *et al.* [1993] is not applicable.

For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we have

$$\Phi(x, (y, z)) = y_2 y_1^2 - x_1^2 y_2 - z_1 z_2^2 + z_1 x_2^2.$$

Note that Φ is continuous on $X \times \hat{X}$. For any subset $\{(({}_1y_1,{}_2\,y_2), ({}_1z_1,{}_2\,z_2)), ..., (({}_ky_1,{}_k\,y_2), ({}_kz_1,{}_k\,z_2))\}$ of $\hat{X}$, let $x = (x_1, x_2) \in X$ such that $x_1 = \max_{h=1,\ldots,k}\ {}_iy_1$ and $x_2 = \min_{h=1,\ldots,k}\ {}_iz_2$. Then

$$\begin{cases} {}_iz_2^2 \geq x_2^2, & \forall i = 1, ..., k, \\ {}_iy_1^2 \leq x_1^2. \end{cases}$$

Thus,

$$\begin{cases} {}_iz_1\ {}_iz_2^2 \geq_i z_1\ x_2^2, & \forall i = 1, ..., k, \\ {}_iy_2\ {}_iy_1^2 \leq_i y_2\ x_1^2. \end{cases} \quad \text{and then} \quad \begin{cases} -\ {}_iz_1\ {}_iz_2^2 +\ {}_iz_1\ x_2^2 \leq 0, & \forall i = 1, ..., k, \\ {}_iy_2\ {}_iy_1^2 -\ {}_iy_2\ x_1^2 \leq 0. \end{cases}$$

Therefore, $\Phi(x, (\ {}_iy,\ \ {}_iz)) \leq 0$, $\forall i = 1, ..., k$. According to Theorem 4.3, this game has a dominant-strategy equilibrium. Indeed, $\overline{x} = (4, 1)$ is such a point.

5 Nash Equilibrium in Mixed Strategies

In this section, we consider the existence of mixed strategy Nash equilibria by applying the pure strategy existence results derived in the previous sections. Assume that each X_i is a compact

[3] A game $G = (I, (X)_{i \in I}, (f)_{i \in I})$ is partially separable if for each $i \in I$ there exist two functions $h_i : X_i \to \mathbb{R}$ and $g_{-i} : X_{-i} \to \mathbb{R}$ such that $u_i(x) = h_i(x_i) + g_{-i}(x_{-i})$ for all $x \in X$.

Hausdorff space. Let u_i be bounded and measurable for all $i \in I$ and let M_i be the regular, countably additive probability measures on the *Borel* subsets of X_i, M_i is compact in the weak* topology. Let us consider U_i be the extend of u_i to $M = \prod_{i \in I} M_i$ by defining $U_i(\mu) = \int_X u_i(x) d\mu(x)$ for all $\mu \in M$ with $d\mu(x) = d\mu_1(x_1) \times d\mu_2(x_2) \times ... \times d\mu_n(x_n)$, and let $\overline{G} = (M_i, U_i)_{i \in I}$ denote the mixed extension of G.

DEFINITION 5.1 A mixed strategy Nash equilibrium of the game G is an n-tuple of probability measures $(\mu_1^*, ..., \mu_n^*) \in M$ such that for all $i \in I$

$$U_i(\mu^*) = \int_X u_i(x) d\mu^*(x) \geq \max_{\mu_i \in M_i} \int_X u_i(x) d\mu_1^*(x_1) \times ... \times d\mu_i(x_i) \times .... \times d\mu_n^*(x_n).$$

The definitions of weak transfer continuity, weak transfer upper continuity, weak transfer lower continuity, upper semicontinuity, payoff security, etc. given in Subsection 3.1 apply in obvious ways to the mixed extension $\overline{G}$ by replacing X_i with M_i in each definition. However, it may be noted that weak transfer continuity (resp., weak transfer upper continuity, weak transfer lower continuity, payoff security) of $\overline{G}$ neither implies nor is implied by weak transfer continuity (resp., weak transfer upper continuity, weak transfer lower continuity, payoff security) of G.

LEMMA 5.1 *If G is upper semicontinuous, then the mixed extension of G is also upper semicontinuous.*

PROOF. See the proof of Proposition 5.1 in Reny [1999] page 1052. ■

Nash [1950] and Glicksberg [1952] show that a game that is compact, Hausdorff, and continuous possesses mixed strategy Nash equilibria. Robson [1994] proves that in a compact game with metric strategy spaces, if each player's payoff is u.s.c. in all players' strategies, and continuous in the other players' strategies, then the game possesses a mixed strategy Nash equilibrium.

The following theorem generalizes the mixed strategy Nash equilibrium existence of Nash [1950], Glicksberg [1952], and Robson [1994] by weakening continuity condition.

THEOREM 5.1 *Suppose that $G = (X_i, u_i)_{i \in I}$ is a compact, Hausdorff game. Then G has a mixed strategy Nash equilibrium if its mixed extension $\overline{G}$ is weakly transfer continuous. Moreover, $\overline{G}$ is weakly transfer continuous if it is 1) weakly transfer upper continuous and payoff secure, or 2) weakly transfer lower continuous and upper semicontinuous.*

The following example illustrates an application of Theorem 5.1.

EXAMPLE 5.1 Consider the following *silent duel* game studied by Dasgupta and Maskin [1986], Karlin [1959]:

$$u_i(x_1, x_2) = \begin{cases} x_1 - x_2 + x_1 x_2, & \text{if } x_1 < x_2, \\ 0, \ \ \text{if } x_1 = x_2, & \\ x_1 - x_2 - x_1 x_2, & \text{otherwise,} \end{cases} \quad , \quad i = 1, 2.$$

Note that the extended game $\overline{G}$ is weakly transfer lower continuous and the game G is upper semicontinuous. Then, according to Lemma 5.1, the extended game $\overline{G}$ is also upper semicontinuous. Thus, Theorem 5.1 implies that the game G has a mixed-strategy Nash equilibrium.

Monteiro and Page [2007] introduce the concept of uniformly payoff security for games that are compact, Hausdorff, bounded and measurable. They show that if a game is compact and uniformly payoff secure, then its mixed extension $\bar{G}$ is payoff secure, but the reverse may not be true, as shown by an example in Carmona [2005].

DEFINITION 5.2 The game G is uniformly payoff secure if for every $x_i \in X_i$, and every $\epsilon > 0$, there is a strategy $\overline{x}_i \in X_i$ such that for every $y_{-i} \in X_{-i}$ there exists a neighborhood $\mathcal{V}(y_{-i})$ of y_{-i} such that $u_i(\overline{x}_i, z_{-i}) \geq u_i(x_i, y_{-i}) - \epsilon$, for all $z_{-i} \in \mathcal{V}(y_{-i})$.

DEFINITION 5.3 The game G is said to be *uniformly transfer continuous* if for every $x_i \in X_i$, and every $\epsilon > 0$, there is a strategy $\overline{x}_i \in X_i$ such that for every $y_{-i} \in X_{-i}$ there exists a neighborhood $\mathcal{V}(x_i, y_{-i})$ of (x_i, y_{-i}) such that

$$u_i(\overline{x}_i, z_{-i}) + \epsilon \geq u_i(x_i, y_{-i}) \geq u_i(z) - \epsilon, \text{ for all } z \in \mathcal{V}(x_i, y_{-i}).$$

Thus, a game G is uniformly transfer continuous if for any strategy $x_i \in X_i$, player i can choose a strategy $\overline{x}_i \in X_i$ to secure a payoff of $u_i(x_i, y_{-i}) - \epsilon$ against deviations by other players in some neighborhood of $y_{-i} \in X_{-i}$, and would be better off at (x_i, y_{-i}) even if all players deviate slightly from (x_i, y_{-i}) for all strategy profiles $y_{-i} \in X_{-i}$.

PROPOSITION 5.1 *If a game $G = (X_i,\ u_i)_{i \in I}$ is uniformly transfer continuous, then the mixed extension $\overline{G}$ is weakly transfer continuous.*

PROOF. Suppose $\overline{\mu} \in X$ is not a mixed strategy Nash equilibrium. Then, there exists a player i, a measure $\mu_i^* \in M_i$ and a $\epsilon > 0$ such that

$$U_i(\mu_i^*, \overline{\mu}_{-i}) - \epsilon = \int_X u_i(x) d\mu_i^*(x_i) d\overline{\mu}_{-i}(x_{-i}) - \epsilon > U_i(\overline{\mu}) = \int_X u_i(x) d\overline{\mu}(x). \tag{7}$$

Since the game G is uniformly transfer continuous, then the function u_i is upper semicontinuous over X and uniformly payoff secure. According to Proposition 5.1 of Reny [1999], the function $\int_X u_i(x)d\overline{\mu}(x)$ is upper semicontinuous in μ. Thus, there exists $\mathcal{V}_1(\overline{\mu})$ such that:

$$\int_X u_i(x)d\overline{\mu}(x) \geq \int_X u_i(x)d\mu(x) - \epsilon/2, \text{ for all } \mu \in \mathcal{V}_1(\overline{\mu}). \tag{8}$$

Also, according to the proof of Theorem 1 in Monteiro and Page [2007], there exists a measure $\tilde{\mu}_i \in M_i$ and a neighborhood $\mathcal{V}_2(\overline{\mu}_{-i})$ of $\overline{\mu}_{-i}$ such that

$$\int_X u_i(x)d\tilde{\mu}_i(x_i)d\mu_{-i}(x_{-i}) \geq \int_X u_i(x)d\mu_i^*(x_i)d\overline{\mu}_{-i}(x_{-i}) - \epsilon/2, \text{ for all } \mu_{-i} \in \mathcal{V}_2(\overline{\mu}_{-i}). \tag{9}$$

Combining (7), (8) and (9), then we conclude: there exists a measure $\tilde{\mu}_i \in M_i$ and a neighborhood $\mathcal{V}(\overline{\mu})$ of $\overline{\mu}$ such that for all $\mu \in \mathcal{V}(\overline{\mu})$, we have

$$\begin{aligned} \int_X u_i(x)d\tilde{\mu}_i(x_i)d\mu_{-i}(x_{-i}) + \epsilon/2 &\geq \int_X u_i(x)d\mu_i^*(x_i)d\overline{\mu}_{-i}(x_{-i}) \\ &> \int_X u_i(x)d\overline{\mu}(x) + \epsilon \\ &\geq \int_X u_i(x)d\mu(x) + \epsilon/2 \end{aligned}$$

Thus, the mixed game $\overline{G}$ is weakly transfer continuous. ■

Proposition 5.1, together with Corollary 5.1, immediately yields the following useful result.

COROLLARY 5.1 *If a game $G = (X_i, u_i)_{i \in I}$ is compact, bounded, Hausdorff, and uniformly transfer continuous, then it possesses a mixed strategy Nash equilibrium.*

As an application of the above proposition, consider the following well-known concession game.

EXAMPLE 5.2 Let us consider $i = 1, 2$ and $x_1, x_2 \in [0, 1]$ with:

$$u_i(x_i, x_{-i}) = \begin{cases} l_i(x_i), & \text{if } x_i < x_{-i}, \\ \phi_i(x_i), & \text{if } x_i = x_{-i}, \\ m_i(x_i), & \text{if } x_i > x_{-i}. \end{cases}$$

We make the following assumption on u_i:

ASSUMPTION 5.1

a) $\forall x \in [0, 1]$, $\forall \epsilon > 0$, there exists a neighborhood $\mathcal{V}(x)$ of x such that $\phi_i(x) \geq \max(m_i(z), l_i(z)) - \epsilon$, for every $z \in \mathcal{V}(x)$.

b) $\forall x \in [0,1]$, $\forall \epsilon > 0$, there exists $y \in [0,1]$ such that $\min\{\phi_i(y), m_i(y), l_i(y)\} \geq \max\{\phi_i(x), m_i(x), l_i(x)\} - \epsilon$.

Then we have the following result.

PROPOSITION 5.2 *Suppose the concession game satisfies Assumption 5.1, and the functions* $l_i(.)$*,* $m_i(.)$ *and* $\phi_i(.)$ *are upper semicontinuous on* $[0,1]$*. Then, the game has a mixed-strategy Nash equilibrium.*

PROOF. Upper semicontinuity of $l_i(.)$, $m_i(.)$ and $\phi_i(.)$, together with condition a) in Assumption 5.1, implies that the concession game is upper semicontinuous. Condition b) implies that for each $x_i \in X_i$, and $\epsilon > 0$ there exists a strategy $\overline{x}_i \in X_i$ such that for every $y_i \in X_{-i}$ there exists a neighborhood $\mathcal{V}(y_i)$ of y_i such that $u_i(\overline{x}_i, z_i) \geq u_i(x_i, y_i) - \epsilon$, for all $z_i \in \mathcal{V}(y_i)$. Then, it is uniformly transfer continuous. It is clear that this game G is compact, then by Corollary 5.1, we conclude that the game has a mixed strategy Nash equilibrium. ■

6 Conclusion

In this paper, we characterize the existence of equilibria in games which may have nonconvex strategy spaces and non-quasiconcave payoff functions. We first offer new Nash equilibrium existence results for a large class of discontinuous games, which rely on (weak) transfer upper continuities. We then characterize the existence of pure-strategy, dominant-strategy, and mixed strategy Nash equilibria in noncooperative games which may not have convex strategy spaces or non-quasiconcave payoff functions.

These results permit us to significantly weaken the key assumptions, such as continuity, convexity, and quasi-concavity on the existence of Nash equilibria. We also provide examples where our general results are applicable, but the existing theorems for pure strategy, dominant-strategy, and mixed strategy Nash equilibria fail to hold. These new results help us understand the existence or non-existence of pure strategy, dominant-strategy, and mixed strategy Nash equilibria in discontinuous and non-concave games.

References

Aliprantis, C.B., Border, K.C. (1994): *Infinite Dimensional Analysis*. Springer-Verlag, New York.

Athey, S. (2001): Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information. *Econometrica*, **69**, 861–889.

Bagh, A., Jofre, A. (2006): Reciprocal Upper Semicontinuous and Better Reply Secure Games: A Comment. *Econometrica*, **74**, 1715–1721.

Baye, M.R., Tian, G., Zhou, J. (1993): Characterizations of the Existence of Equilibria in Games with Discontinuous and Non-Quasiconcave Payoffs. *The Review of Economic Studies*, **60**, 935–948.

Carmona, G. (2005): On the Existence of Equilibria in Discontinuous Games: Three Counterexamples. *International Journal of Game Theory*, **33**, 181–187.

Dasgupta, P., Maskin, E. (1986): The Existence of Equilibrium in Discontinuous Economic Games, I: Theory. *The Review of Economic Studies*, **53**, 1–26.

Debreu, G. (1952): A Social Equilibrium Existence Theorem. *Proceedings of the National Academy of Sciences of the U. S. A.*, 38.

Gatti, R.J. (2005): A Note on the Existence of Nash Equilibrium in Games with Discontinuous Payoffs. *Cambridge Economics Working Paper No. CWPE 0510.* Available at SSRN: http://ssrn.com/abstract=678701.

Glicksberg, I.L. (1952): A Further Generalization of the Kakutani Fixed Point Theorem. *Proceedings of the American Mathematical Society*, **3**, 170–174.

Karlin, S. (1959): *Mathematical Methods and Theory in Games, Programming and Economics*, Vol. II (London: Pregamon Press).

Monteiro, P.K., Page, F.H.Jr. (2007): Uniform Payoff Security and Nash Equilibrium in Compact Games. *Journal of Economic Theory*, **134**, 566–575.

Morgan, J., Scalzo, V. (2007): Pseudocontinuous Functions and Existence of Nash Equilibria. *Journal of Mathematical Economics*, **43**, 174–183.

Nash, J. (1950): Equilibrium Points in n-Person Games. *Proceedings of the National Academy of Sciences*, **36**, 48–49.

Nash, J.F. (1951): Noncooperative Games. *Annals of Maths*, **54**, 286–295.

Nishimura, K., Friedman, J. (1981): Existence of Nash Equilibriumin n-Person Games without Quasi-Concavity. *International Economic Review*, **22**, 637–648.

Reny, J.P. (1999): On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games. *Econometrica*, **67**, 1029–1056.

Robson, A. J. (1994): An informationally Robust's Equilibrium in Two-Person Nonzero-Sum Games. *Games and Economic Behavior*, **2**, 233-245.

Rosen, J.B. (1965): Existence and Uniqueness of Equilibrium Point for Concave n-Person Games. *Econometrica*, **33**, 520–534.

Simon, L. (1987): Games with Discontinuous Payoffs. *Review of Economic Studies*, **54**, 569–597.

Simon, L., W. Zame. (1990): Discontinuous Games and Endogenous Sharing Rules. *Econometrica*, **58**, 861–872.

Tian, G. (1993): Necessary and Sufficent Conditions for Maximization of a Clasee of Preference Relations. *Review of Economic Studies*, **60**, 949–958

Tian, G., Zhou, Z. (1995): Transfer Continuities, Generalizations of the Weierstrass Theorem and Maximum Theorem: A Full Characterization. *Journal of Mathematical Economics*, **24**, 281–303.

Vives, X. (1990): Nash Equilibrium with Strategic Complementarities. *Journal of Mathematical Economics*, **19**, 305–321.

Yao, J.C. (1992): Nash Equilibria in n-Person Games without Convexity. *Applied Mathematics letters*, **5**, 67–69.