Existence of Equilibrium in Minimax Inequalities, Saddle Points, Fixed Points, and Games without Convexity Sets

Rabia Nessah[*]

CNRS-LEM (UMR 8179)

IESEG School of Management

3 rue de la Digue F-59000 Lille

France

Guoqiang Tian[†]

Department of Economics

Texas A&M University

College Station, Texas 77843

USA

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Abstract

This paper characterizes the existence of equilibria in minimax inequalities without assuming any form of quasi-concavity of functions and convexity or compactness of choice sets. A new condition, called "*local dominatedness property*", is shown to be necessary and further, under some mild continuity condition, sufficient for the existence of equilibrium. We then apply the basic result obtained in the paper to generalize the existing theorems on the existence of saddle points, fixed points, and coincidence points without convexity or compactness assumptions. As an application, we also characterize the existence of pure strategy Nash equilibrium in games with discontinuous and nonquasiconcave payoff functions and nonconvex and/or noncompact strategy spaces.

Keywords: minimax inequality, saddle points, fixed points, coincidence points, discontinuity, non-quasiconcavity, non-convexity, and non-compactness.

JEL Classification: C61; C65; C72.

1 Introduction

Let X and Y be two nonempty subsets of topological spaces E and F, respectively. Let $\Psi : X \times Y \to \mathbb{R}$ be a function and $r \in \mathbb{R}$ a constant. Consider the minimax inequality problem of finding $\overline{x} \in X$ such that

$$\Psi(\overline{x}, y) \leq r, \quad \forall y \in Y. \tag{1}$$

Ky Fan [1972] introduced and studied the minimax inequality problem of finding a solution $\overline{x} \in X$ of the inequality (1) in the case where $E = F$, $X = Y$ and $r = \sup_{x \in X} \Psi(x, x)$.

The Ky Fan inequality is one of the most important tools in nonlinear analysis and in mathematical economics. Indeed, it allows one to derive many practical and theoretical results in a wide variety of fields. In many situations, the Ky Fan inequality is more flexible and adaptable than other basic theorems in nonlinear analysis, such as fixed point theorems and variational inequalities. Aubin and Ekeland [1984] remarked that it is often easier to reduce an equilibrium existence problem to a minimax inequality problem rather than to transform it into a fixed point problem. Therefore, weakening its conditions further enlarges its domain of applicability.

The many applications of Ky Fan [1972] in different areas (such as general equilibrium theory, game theory, and optimization theory) attracted researchers to weaken the conditions on the existence of its solution. Lignola [1997] relaxed the assumption on the compactness of the set X and the semicontinuity of the function $\Psi(x, y)$. Ding and Tan [1992], Tian and Zhou [1993], Georgiev and Tanaka [2000] weakened the condition of quasi-concavity of the function $\Psi(x, y)$ in y. Many other results were also obtained such as those in Aubin and Ekeland [1984], Georgiev and Tanaka [2000], Nessah and Chu [2004], Nessah and Larbani [2004], Simons [1986], Tian and Zhou [1993], Yu and Yuan [1995] and Yuan [1995]. Equilibrium problems were studied in both mathematics and economics such as those in Iusem and Soca [2003], Aubin and Ekeland [1984], Ding and Park [1998], Ding and Tan [1992], Lin [2001], Lin and Chang [1998], Lin and Park [1998], Nessah and Chu [2004] and Nessah and Larbani [2004], among which Nessah and Chu [2004] and Nessah and Larbani [2004] generalized the Ky Fan inequality to the case where two sets X and Y may be different. However, all the work mentioned above is assumed that the set X is convex. In many practical situations a choice set may not be convex and/or not compact so that the existing theorems cannot be applied.

In this paper we first provide characterization results on the existence of solution to the minimax inequality without any form of quasi-concavity of function or convexity and compactness of choice sets. We introduce a new condition, called *local-dominatedness property*. It is shown that the local dominatedness condition is necessary and further, under some mild continuity condition, sufficient for the existence of a solution to a minimax

2

inequality. We then apply the basic result to study the existence of saddle points, fixed points, and coincidence points. As an application of our basic result, we study the existence of equilibria for a noncooperative game without quasi-concavity of payoff functions and convexity or compactness of strategy spaces.

The remainder of the paper is organized as follows. In Section 2 we give basic terminologies used in our study. We introduce the concepts of α-local-dominatedness property, α-transfer quasi-concavity, and α-transfer continuity. We also provide sufficient conditions for these conditions to be true. Section 3 is dedicated to the development of existence theorems on minimax inequality for a function defined on cartesian product of two different sets without the convexity and/or compactness assumptions. In Section 4, we apply our results on the minimax inequality to offers new existence theorems on saddle points without assuming convexity and/or compactness of choice sets. In Section 5, we provide necessary and sufficient conditions for the existence of fixed points and coincidence points. We introduce the concept of f-separability that can be used to characterizes the existence of fixed point of a function without the convexity assumption. Section 6 considers the existence of coincidence points. In Section 7, we consider the existence of equilibria in games discontinuous and non-quasiconcavity of payoff functions and convexity and/or compactness of strategy spaces. Concluding remarks are offered in Section 8.

2   Notations and Definitions

Let Y be a nonempty subset of a topological space F. Denote by 2^Y be the family of all nonempty subsets of Y and $\langle Y \rangle$ the set of all finite subsets of Y. Let $S \subset Y$. Denote by int S the relative interior of S in Y and by cl S the relative closure of S in Y.

A function $f : Y \to \mathbb{R}$ is upper semicontinuous on Y if the set $\{x \in Y, f(x) \geq c\}$ is closed for all $c \in \mathbb{R}$; f is lower semicontinuous on Y if $-f$ is upper semicontinuous on Y; f is continuous on Y if f is both upper and lower semicontinuous on Y.

A function $f : Y \to \mathbb{R}$ is quasiconcave on Y if for any y_1, y_2 in Y and for any $\theta \in [0,1]$, $\min\{f(y_1), f(y_2)\} \leq f(\theta y_1 + (1-\theta)y_2)$, and f is quasiconvex on Y if $-f$ is quasiconcave on Y. A function $f : (x,y) \in Y \times Y \to \mathbb{R}$ is diagonally quasiconcave in y if for any finite points $y^1, \ldots, y^m \in Y$ and any $y \in co\{y^1, \ldots, y^m\}$, $\min_{1 \leq k \leq m} f(y, y^k) \leq f(y,y)$. A function $f : (x,y) \in Y \times Y \to \mathbb{R}$ is α-diagonally quasiconcave in y if for any finite points $y^1, \ldots, y^m \in Y$ and $y \in co\{y^1, \ldots, y^m\}$, $\min_{1 \leq k \leq m} f(y, y^k) \leq \alpha$.

DEFINITION 2.1 *(α-local-dominatedness).* Let $\alpha \in \mathbb{R}$. A function $\Psi : X \times Y \to \mathbb{R}$ is

said to be α-*locally-dominated in* y if for any $A \in \langle Y \rangle$, there exists $x \in X$ such that:

$$\max_{y \in A} \Psi(x, y) \leq \alpha.$$

The term *localness* reflects to choose finite subsets from Y. *Dominatedness* refers to the fact that $f(x, y)$ is dominated by α. α-*local-dominatedness property* says that, given any finite set $A \subset Y$, there exists a corresponding candidate point $x \in X$ such that $f(x, y)$ is dominated by α for all points in A. We will see from Theorem 3.1 below that α-local-dominatedness condition is necessary, and further under some mild condition, sufficient for the existence of solution to the minimax inequality.

REMARK 2.1 Let $H(y) = \{x \in X : \Psi(x, y) \leq \alpha\}$ for $y \in Y$. Then, $y \mapsto \Psi(x, y)$ is α-locally-dominated in y if and only if the family sets $\{H(y), \ y \in Y\}$ has the finite intersection property.

EXAMPLE 2.1 Consider the following function.

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$(x, y) \mapsto f(x, y) = x^3 - x \times y^2.$$

It is obvious if $x < 0$, the function $y \mapsto f(x, y)$ is not quasiconcave. However, it is 0-locally-dominated in y. To see this, let $\{y_1, ..., y_n\} \in \langle \mathbb{R} \rangle$. Then there exists $x = -\max_{i=1,...,n} |y_i| \in \mathbb{R}$ such that $y_i^2 \leq x^2$ for all $i \leq n$. Thus, $-xy_i^2 \leq -x^3$ for all $i \leq n$, and therefore $f(x, y_i) = x^3 - xy_i^2 \leq 0$ for all $i \leq n$.

The following definition generalizes the transfer quasiconcavity in Baye *et al.* [1993] to a function defined in the product of different sets.

DEFINITION 2.2 *(α-Transfer Quasiconcavity)* Let X be a nonempty convex subset of a vector space E and let Y be a nonempty set. A function $\Psi : X \times Y \to \mathbb{R}$ is said to be α-*transfer quasiconcave in* y if, for any finite subset $Y^m = \{y^1, ..., y^m\} \subset Y$, there exists a corresponding finite subset $X^m = \{x^1, ..., x^m\} \subset X$ such that for any subset $L \subset \{1, 2, ..., m\}$ and any $x \in co\{x^h : \ h \in L\}$, we have $\min_{h \in L} f(x, y^h) \leq \alpha$.

REMARK 2.2 When $X = Y$, a sufficient condition for a function $\Psi : X \times Y \to \mathbb{R}$ to be α-transfer quasiconcave in y is that it is α-diagonally quasiconcave in y.

The following proposition characterizes the α-local-dominatedness property if X is convex and Ψ is lower semi-continuous in x.

4

PROPOSITION 2.1 *Let X be a nonempty convex and compact subset in a topological vector space E and let Y be a nonempty set. Suppose function $\Psi(x,y)$ is lower semi-continuous in x. Then $\Psi(x,y)$ is α-locally-dominated in y if and only if it is α-transfer quasiconcave in y.*

PROOF. *Necessity* ($\Rightarrow$): Suppose that $\Psi(x,y)$ is α-locally-dominated in y. Then for each $B \in \langle Y \rangle$, there exists $\tilde{x} \in X$ such that $\sup\limits_{y \in B}\Psi(\tilde{x},y) \leq \alpha$. Now consider $Y^m = \{y^1,...,y^m\} \subset Y$. Then there exists a corresponding finite subset $X^m = \{x^1,...,x^m\} \subset X$ with $x^1 = ... = x^m = \tilde{x}$ such that for each $J \subset \{1,...,m\}$, we have $\forall x \in co\{x^j : j \in J\} = \{\tilde{x}\}$, $\inf\limits_{j \in J}\Psi(x,y^j) \leq \sup\limits_{j \in J}\Psi(x,y^j) \leq \alpha$ with $B = Y^m$.

Sufficiency ($\Leftarrow$): Suppose that $\Psi(x,y)$ is α-transfer quasiconcave in y. Note that $\Psi(x,y)$ is α-locally-dominated in y if and only if the family $\{G(y) = \{x \in X : \Psi(x,y) \leq \alpha\}, y \in Y\}$ has the finite intersection property. Suppose that this family does not have the finite intersection property. Then there exists $B = \{y^1,...,y^m\} \in \langle Y \rangle$ such that $\bigcap\limits_{y \in B} G(y) = \emptyset$, *i.e.*, for each $x \in X$, there exists $y \in B$ such that $x \notin G(y)$. Let $A = \{x^1,...,x^m\} \in \langle X \rangle$ be the corresponding points in X such that for each $M_k \subset \{1,2,...,m\}$ and any $x \in co\{x^h, h \in M_k\}$, we have $\min\limits_{h \in M_k}\Psi(x,y^h) \leq \alpha$. Let $Z = co(A)$ and $L = span(A) = span\{x^1,...,x^m\}$. Since $G(y)$ is closed, we can define a continuous function $g : Z \to [0,\infty[$ by $g(x) = \sum_{i=1}^{m} d(x,G(y^i) \cap L)$ where d is the Euclidean metric on L. Since $\bigcap\limits_{y \in B} G(y) = \emptyset$, then $g(x) > 0$ for each $x \in X$. Define another continuous function $f : Z \to Z$ by

$$f(x) = \sum_{i=1}^{m} \frac{d(x,G(y^i) \cap L)}{g(x)} x^i.$$

By Brouwer fixed point theorem, there exists $\overline{x} \in Z$ such that

$$\overline{x} = f(\overline{x}) = \sum_{i=1}^{m} \frac{d(\overline{x},G(y^i) \cap L)}{g(\overline{x})} x^i. \tag{2}$$

Let $J = \{i \in \{1,...,m\} : (\overline{x},G(y^i) \cap L) > 0\}$. Then, for each $i \in J$, $\overline{x} \notin G(y^i) \cap L$. Since $\overline{x} \in L$, $\overline{x} \notin G(y^i)$ for any $i \in J$. Thus, we have

$$\inf\limits_{i \in J}\Psi(\overline{x},y^i) > \alpha. \tag{3}$$

¿From (2), $\overline{x} \in co\{x^i : i \in J\}$, and then by α-transfer quasiconcavity, we obtain

$$\inf\limits_{i \in J}\Psi(\overline{x},y^i) \leq \alpha,$$

which contradicts (3). Therefore, the function $y \mapsto \Psi(x,y)$ is α-locally-dominated. ∎

DEFINITION 2.3 *(α-Transfer Lower continuity).* Let X be a nonempty subset of a topological space and Y be a nonempty subset. A function $f : X \times Y \to \mathbb{R}$ is said to be *α-transfer lower continuous in x with respect to Y* if for $(x, y) \in X \times Y$, $f(x, y) > \alpha$ implies that there exists some point $y^{'} \in Y$ and some neighborhood $\mathcal{V}(x) \subset X$ of x such that $f(z, y^{'}) > \alpha$ for all $z \in \mathcal{V}(x)$.

α-Transfer lower continuity in x with respect to Y says that if a point x in X is dominated by another point y in Y comparing to α, then there is an open set of points containing x, all of which can be dominated by a single point $y^{'}$. Here, transfer lower continuity in x with respect to Y refers to the fact that y may be transferred to some $y^{'}$ in order for the inequality to hold for all points in a neighborhood of x. The usual notion of lower semicontinuity would require that the first inequality hold at y for all points in a neighborhood of x. Thus, α-transfer lower continuity is weaker than the notions of continuity used in the literature.

We have the following proposition.

PROPOSITION 2.2 *Any one of the following conditions is sufficient for $f(x, y)$ to be α-transfer lower semicontinuous in x with respect to Y :*

a) $f(x, y)$ *is continuous in x;*

b) $f(x, y)$ *is lower semicontinuous in x;*

REMARK 2.3 Proposition 2.1 are still held when lower semicontinuity is weakened to transfer lower continuity.

3 Minimax Inequality without Convexity and/or Compactness

In this section we present theorems on the existence of equilibrium in the Ky Fan minimax inequality for a function defined on cartesian product of two different sets X and Y without any form of quasiconcavity of function and/or convexity and compactness of sets.

THEOREM 3.1 *Let X be a nonempty compact subset of a topological space E and Y a nonempty set. Suppose Ψ is a real-valued function on $X \times Y$ such that $\Psi(x, y)$ is α-transfer lower continuous in x with respect to Y. Then, there exists $\overline{x} \in X$ such that*

$$\Psi(\overline{x}, y) \leq \alpha \quad \forall y \in Y \tag{4}$$

if and only if Ψ is α-locally-dominated in y.

PROOF. *Necessity* ($\Rightarrow$): Let $\overline{x} \in X$ be a solution of the minimax inequality (4). Then $\Psi(\overline{x}, y) \leq \alpha$ for all $y \in Y$, and of course we have $\max\limits_{y \in A} \Psi(\overline{x}, y) \leq \alpha$ for any subset $A = \{y^1, ..., y^m\} \in \langle Y \rangle$. Hence $\Psi(x, y)$ is α-locally-dominated in y.

Sufficiency ($\Leftarrow$): We first show that, if $\Psi(x, y)$ is α-transfer lower continuous in x with respect to Y, $\bigcap\limits_{y \in Y} H(y) = \bigcap\limits_{y \in Y} \text{cl } H(y)$ with $H(y) = \{x \in X : \Psi(x, y) \leq \alpha\}$. Indeed, let $x \in \bigcap\limits_{y \in Y} \text{cl } H(y)$ but not in $\bigcap\limits_{y \in Y} H(y)$. Then, there exists $y \in Y$ such that $x \notin H(y)$, *i.e.*, $\Psi(x, y) > \alpha$. By the α-transfer lower continuity of Ψ in x with respect to Y, there exists $y' \in Y$ and a neighborhood $\mathcal{V}(x)$ of x such that $\Psi(z, y') > \alpha$ for all $z \in \mathcal{V}(x)$. Thus, $x \notin \text{cl } H(y')$, a contradiction. The condition that $y \mapsto \Psi(x, y)$ is α-locally-dominated in y implies that $\{\text{cl } H(y) :\in Y\}$ has the finite intersection property. Since $\{\text{cl } H(y) : y \in Y\}$ is a compact family in the compact set X. Thus, $\emptyset \neq \bigcap\limits_{y \in Y} H(y)$. Hence, there exists $\overline{x} \in X$ such that $\Psi(\overline{x}, y) \leq \alpha$ for $y \in Y$. This completes the proof. ∎

REMARK 3.1 The above result generalizes the exiting results without assuming any form of quasiconcavity or $X = Y$. Note that, in Example 2.1, if the function f is defined on $[-1, 1] \times [-1, 1]$, the existing results cannot be applicable since the function $y \mapsto f(x, y)$ is not quasiconcave in y on $[-1, 1]$ for all $x \in [-1, 1]$. However, there exists a solution since Ψ is α-locally-dominated in y. The following is an example.

EXAMPLE 3.1 Let $X = Y = [1/2, 1] \cup [3/2, 2]$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, and

$$F(x, y) = x_2 y_1^2 - x_1 y_2^2 - x_2 x_1^2 + x_1 x_2^2.$$

The function F is continuous over $X \times X$. For any subset $\{(_1y_1, _2y_2), ..., (_ky_1, _ky_2)\}$ of X, let $x = (x_1, x_2) \in X$ such that $x_1 = \max\limits_{h=1,...,k} {}_iy_1$ and $x_2 = \min\limits_{h=1,...,k} {}_iy_2$. Then

$$\begin{cases} {}_iy_2^2 \geq x_2^2, & \forall i = 1, ..., k, \\ {}_iy_1^2 \leq x_1^2. \end{cases}$$

Thus,

$$\begin{cases} -x_1 \, {}_iy_2^2 \leq -x_1 \, x_2^2, & \forall i = 1, ..., k, \\ x_2 \, {}_iy_1^2 \leq x_2 \, x_1^2. \end{cases}$$

Therefore, $F(x, {}_iy) - F(x, x) \leq 0, \forall i = 1, ..., k$. The other conditions of Theorem 3.1 are obviously verified. Thus, the minimax inequality has a solution. Since X is not convex, the results in (Lignola [1997], Simons [1986], Tian and Zhou [1993], Ding and Tan [1992] and Georgiev and Tanaka [2000]) on the existence of a solution to the Ky Fan inequality are not applicable.

When X is convex, by Proposition 2.1 and Theorem 3.1, we have the following corollary.

COROLLARY 3.1 *Let X be a nonempty convex and compact subset of a topological vector space E and Y a nonempty set. Let $\Psi : X \times Y \to \mathbb{R}$ be α-transfer lower continuous in x with respect to Y. Then, the minimax inequality (4) has at least one solution if and only if $\Psi(x, y)$ is α-transfer quasiconcave in y.*

Theorem 3.1 can be generalized to the case where X is not compact.

THEOREM 3.2 *Let X be a nonempty of a topological space E and Y a nonempty set. Suppose Ψ is a real-valued function on $X \times Y$ such that:*

(a) $\Psi(x, y)$ is α-transfer lower continuous in x with respect to Y;

(b) there exists a finite subset $\{y^1, ..., y^k\} \subset Y$ such that $\bigcap_{i=1,...,k} G_\alpha(y^i)$ is compact where $G_\alpha(y) = \{x \in X : \Psi(x, y) \leq \alpha\}$.

Then, there exists $\overline{x} \in X$ such that

$$\Psi(\overline{x}, y) \leq \alpha, \quad \forall y \in Y \tag{5}$$

if and only if Ψ is α-locally-dominated in y.

PROOF. The necessity is the same as that of Theorem 3.1. We only need to prove the sufficiency. For each $y \in Y$, let $G_\alpha(y) = \{x \in X : \Psi(x, y) \leq \alpha\}$. Then, $\bigcap_{y \in Y} G_\alpha(y) = \bigcap_{y \in Y} \mathrm{cl}\, G_\alpha(y)$ by condition (a) of Theorem 3.2. The condition that $y \mapsto \Psi(x, y)$ is α-locally-dominated implies that $\{\mathrm{cl}\, G_\alpha(y) : y \in Y\}$ has the finite intersection property and therefore $\{G_\alpha(y) \cap \bigcap_{i=1,...,k} G_\alpha(y^i) : y \in Y\}$ has the finite intersection property. Since $\{G_\alpha(y) \cap \bigcap_{i=1,...,k} G_\alpha(y^i) : y \in Y\}$ is a compact family in the compact set $\bigcap_{i=1,...,k} G_\alpha(y^i)$. Thus, $\emptyset \neq \bigcap_{y \in Y} G_\alpha(y) \cap \bigcap_{i=1,...,k} G_\alpha(y^i) = \bigcap_{y \in Y} G_\alpha(y)$. Hence, there exists $\overline{x} \in X$ such that $\Psi(\overline{x}, y) \leq \alpha$ for all $y \in Y$. This completes the proof. ■

THEOREM 3.3 *Let X be a nonempty of a topological space E and Y a nonempty set. Let $\Psi(x, y)$ be a real-valued function on $X \times Y$. Then, the minimax inequality (4) has at least one solution if and only if there exists a nonempty compact subset X^0 of X such that:*

(a) $\Psi|_{X^0 \times Y}(x, y)$ is α-transfer lower continuous in x with respect to Y;

8

(b) there exists $y^0 \in Y$ such that $G(y^0)$ is compact where $G(y) = \{x \in X^0 : \Psi(x, y) \leq \alpha\}$;

(c) the function $y \mapsto \Psi|_{X^0 \times Y}(x, y)$ is α-locally-dominated in x on X^0.

PROOF. *Necessity* ($\Rightarrow$): Suppose that the minimax inequality (4) has a solution $\overline{x} \in X$. Let $X^0 = \{\overline{x}\}$. Then, the set X^0 is nonempty compact, and the restricted function $\Psi|_{X^0 \times Y}$ is α-transfer lower continuous in x with respect to Y, and the set $G(y) = \{x \in X^0 : \Psi(x, y) \leq \alpha\}$ is compact for each $y \in X$, and for each $A \in \langle Y \rangle$, $\exists x = \overline{x} \in X^0$ such that $\max_{y \in A} \Psi(x, y) \leq \max_{y \in Y} \Psi(x, y) \leq \alpha$ (because $\overline{x}$ is a solution of the minimax inequality (4)).

Sufficiency ($\Leftarrow$): For each $y \in Y$, let $G(y) = \{x \in X^0 : \Psi(x, y) \leq \alpha\}$. Then, $\bigcap_{y \in Y} G(y) = \bigcap_{y \in Y} \text{cl}\, G(y)$ by condition (1) of Theorem 3.3. The condition (3) of Theorem 3.3 implies that $\{\text{cl}\, G(y) : y \in Y\}$ has the finite intersection property and therefore $\{G(y) \cap G(y^0) : y \in Y\}$ has the finite intersection property. Since $\{G(y) \cap G(y^0) : y \in Y\}$ is a compact family in the compact set $G(y^0)$. Thus, $\emptyset \neq \bigcap_{y \in Y} G(y) \cap G(y^0) = \bigcap_{y \in Y} G(y)$. Hence, there exists $\overline{x} \in X^0$ such that each $y \in Y$, $\Psi(\overline{x}, y) \leq \alpha$. This completes the proof. ∎

4   Existence of Saddle Point

Saddle point is an important tool in variational problems and game theory. Much work has been dedicated to the problem of weakening its existence conditions. Almost all these results assume that a bifunction is defined on convex sets. In this section we present existence theorems on saddle point without any form of convexity conditions.

Consider two players, Juba and Massi, who have strategy sets X and Y, respectively. If Juba chooses a strategy $a \in X$ and Massi chooses a strategy $b \in Y$, the payoff is given by

$$f(a, b) := \text{ gain by Massi} = \text{loss by Juba}$$

(*e.g.* in euro). We allow $f(a, b)$ to be negative, and if this is the case then player Massi can obtain a negative gain, that is, a loss of $|f(a, b)|$ euro.

DEFINITION 4.1 A pair $(\overline{x}, \overline{y})$ in $X \times Y$ is called a saddle point of f in $X \times Y$, if

$$f(\overline{x}, y) \leq f(\overline{x}, \overline{y}) \leq f(x, \overline{y}) \text{ for all } x \in X \text{ and } y \in Y.$$

This definition reflects the fact that each player plays so as to maximize his or her individual interests.

Before we give our new results, we state two classical results on saddle point.

THEOREM 4.1 *(Von Neumann Theorem). Let X and Y be nonempty compact and convex subsets in a Hausdorff locally convex vector spaces E and F respectively and f a real valued function defined on $X \times Y$. Suppose (1) the function $x \mapsto f(x, y)$ is lower semicontinuous and quasiconvex on X, (2) the function $y \mapsto f(x, y)$ is upper semicontinuous and quasiconcave on Y. Then, f has a saddle point.*

THEOREM 4.2 *(Kneser Theorem) Let X be a nonempty convex subset in a Hausdorff topological vector space E and Y a nonempty compact and convex subset of a Hausdorff topological vector space F. Let f be a real valued function defined on $X \times Y$. If (1) the function $x \mapsto f(x, y)$ is concave on X, (2) the function $y \mapsto f(x, y)$ is lower semicontinuous and convex on Y, then $\min\limits_{y \in Y} \sup\limits_{x \in X} f(x, y) = \sup\limits_{x \in X} \min\limits_{y \in Y} f(x, y)$.*

By relaxing the convexity of function, we obtain the following theorem.

THEOREM 4.3 *(Saddle-Point without Convexity) Let X and Y be two nonempty compact subsets in topological spaces E and G, respectively. Let $f : X \times Y \mapsto \mathbb{R}$ be a real valued function defined on $X \times Y$ such that $f(x, y') - f(x', y)$ is 0-transfer lower continuous in (x, y) with respect to $X \times Y$. Then, the bifunction $f(x, y)$ has a saddle point if and only if for all $\{(a_i, b_i) : i = 1, ..., n\} \subset X \times Y$, there exists $(x, y) \in X \times Y$ such that $f(x, b_i) \leq f(a_i, y)$, for all $i = 1, ..., n$.*

PROOF. *Necessity* ($\Rightarrow$): Let $(\overline{x}, \overline{y}) \in X \times Y$ be a saddle point of f. Then,

$$f(\overline{x}, y) \leq f(\overline{x}, \overline{y}) \leq f(x, \overline{y}) \text{ for all } x \in X \text{ and } y \in Y. \tag{6}$$

Suppose that there exists $A = \{(a_i, b_i), i = 1, ..., n\} \subset X \times Y$, such that

$$\forall (x, y) \in X \times Y, \ \exists i = 1, ..., n \ \text{ such that } f(x, b_i) > f(a_i, y). \tag{7}$$

Let $x = \overline{x}$ and $y = \overline{y}$ in (7). Then there exists $(\tilde{a}, \tilde{b}) \in A$ such that

$$f(\overline{x}, \tilde{b}) > f(\tilde{a}, \overline{y}).$$

Now choose $x = \tilde{a}$ and $y = \tilde{b}$. Inequality (6) then becomes:

$$f(\overline{x}, \tilde{b}) \leq f(\overline{x}, \overline{y}) \leq f(\tilde{a}, \overline{y}).$$

Thus, we have $f(\overline{x}, \tilde{b}) \leq f(\tilde{a}, \overline{y}) < f(\overline{x}, \tilde{b})$, a contradiction. Therefore, for all $\{(a_i, b_i) : i = 1, ..., n\} \subset X \times Y$, there exists $(x, y) \in X \times Y$ such that $f(x, b_i) \leq f(a_i, y)$ for all $i = 1, ..., n$.

Sufficiency ($\Leftarrow$): Let $F : Z \times Z \mapsto \mathbb{R}$, where $Z = X \times Y$ and

$$F(z, t) = f(x, y') - f(x', y), \ \forall z = (x, y) \in Z \text{ and } t = (x', y') \in Z.$$

It is easy to verify that all conditions of Theorem 3.1 are satisfied for the $F(z,t)$. Then, there exists $z^0 = (x^0, y^0) \in Z$ such that

$$\max_{t \in T} F(z^0, t) \leq 0. \tag{8}$$

Now we prove that $z^0 = (x^0, y^0)$ is a saddle point of the bifunction $f(x,y)$. From (8) we get

$$\forall (x,y) \in X \times Y, \quad f(x^0, y) \leq f(x, y^0). \tag{9}$$

Letting $x = x^0$ in (9), we have $\forall y \in Y$, $f(x^0, y) \leq f(x^0, y^0)$. Letting $y = y^0$ in (9), we have $\forall x \in X$, $f(x^0, y^0) \leq f(x, y^0)$. Therefore, for all $(x,y) \in X \times Y$, we have

$$f(x^0, y) \leq f(x^0, y^0) \leq f(x, y^0),$$

i.e., $z^0 = (x^0, y^0)$ is a saddle point of the bifunction $f(x,y)$. ∎

When X is convex, by Proposition 2.1 and Theorem 4.3, we have the following corollary.

COROLLARY 4.1 *Let X and Y be two nonempty compact and convex subsets in topological spaces E and G, respectively. Let $f : X \times Y \mapsto \mathbb{R}$ be a real valued function defined on $X \times Y$ such that $f(x, y') - f(x', y)$ is 0-transfer lower continuous in (x,y) with respect to $X \times Y$. Then, the bifunction $f(x,y)$ has a saddle point if and only if $f(x, y') - f(x', y)$ is 0-transfer quasiconcave in (x', y').*

Theorem 4.3 can be generalized by relaxing the compactness of X and Y.

THEOREM 4.4 *Let X and Y be two nonempty subsets in topological spaces E and G, respectively. Let $f : X \times Y \mapsto \mathbb{R}$ such that:*

(a) $f(x, y') - f(x', y)$ is 0-transfer lower continuous in (x,y) with respect to $X \times Y$;

(b) there exists $\{(x^1, y^1), ..., (x^k, y^k)\} \subset X \times Y$ such that $\bigcap\limits_{i=1,...,k} G(x^i, y^i)$ is compact where $G(u, v) = \{(x,y) \in X \times Y : f(x, v) \leq f(u, y)\}$.

Then, the bifunction $f(x,y)$ has a saddle point if and only if for all $\{(a_i, b_i) : i = 1, ..., n\} \subset X \times Y$, there exists $(x,y) \in X \times Y$ such that $f(x, b_i) \leq f(a_i, y)$, for all $i = 1, ..., n$.

PROOF. The necessity is the same as that of Theorem 4.3 and the sufficiency is the same as that of Theorem 3.2. ∎

REMARK 4.1 The function $f(x, y') - f(x', y)$ is 0-transfer lower continuous in (x,y) with respect to $X \times Y$ if (1) $x \mapsto f(x, y)$ is lower semicontinuous function in x and (2) $y \mapsto f(x, y)$ is upper semicontinuous function in y.

5 Existence of Fixed Point

This section provides a necessary and sufficient for the existence of fixed point of a function defined on a set that may not be compact or convex.

A correspondence C defined from Y into 2^F has a fixed point $x \in Y$ if $x \in C(x)$. If C is a single-valued function, then a fixed point x of C is characterized by $x = C(x)$.

We start by considering the following example of a fixed point problem:

EXAMPLE 5.1 Define a function $f : X = [0,3] \to \mathbb{R}$ by

$$f(x) = \frac{x+4}{x+1}.$$

Since $\max_{x \in [0,3]} |f'(x)| = 3$, f is a 3-lipschitz. Also since $f([0,3]) = [\frac{7}{4}, 4] \not\subseteq [0,3]$, all the classical fixed point Theorems (Banach's,[1] Brouwer-Schauder-Tychonoff's,[2] Halpern-Bergman's,[3] Kakutani-Fan-Glicksberg's,[4] ...: see Aliprantis and Border [1994]) are not applicable.

Let (E, d) be a metric space. The subset $B(a, r)$ is defined by

$$B(a, r) = \{x \in E : d(x, a) < r\}$$

where $a \in X$ and $r \in \mathbb{R}_+^*$, is called open ball centered at a point a with radius r.

DEFINITION 5.1 Let X be a nonempty set in a metric space (E, d) and f be a function defined on X into E. The set X is called *f-separate* if at least one of the following conditions holds:

1) for all $A \in \langle X \rangle$, there exists $x \in X$ such that:

$$A \cap B(x, d(f(x), x)) = \emptyset;$$

2) for all $A \in \langle f(X) \rangle$, there exists $x \in X$ such that:

$$A \cap B(f(x), d(f(x), x)) = \emptyset.$$

---

[1]Banach Fixed Point Theorem: Let (K, d) be a complete metric space and let $f : K \to K$ be a d-contraction ($d \in [0, 1[$). Then, f has a unique fixed point.

[2]Brouwer-Schauder-Tychonoff Fixed Point Theorem: Let K be a nonempty compact convex subset of a locally convex Hausdorff space, and let $f : K \to K$ be a continuous function. Then the set of fixed points of f is compact and nonempty.

[3]Halpern-Bergman Fixed Point Theorem: Let K be a nonempty compact convex subset of a locally convex Hausdorff space X, and let $C : K \to 2^X$ be an inward pointing upper demicontinuous mapping with nonempty closed convex values. Then C has a fixed point.

[4]Kakutani-Fan-Glicksberg Fixed Point Theorem: Let K be a subset nonempty compact convex of a locally convex Hausdorff space, and let $C : K \to 2^K$ have closed graph and nonempty convex values. Then the set of fixed points of C is nonempty and compact.
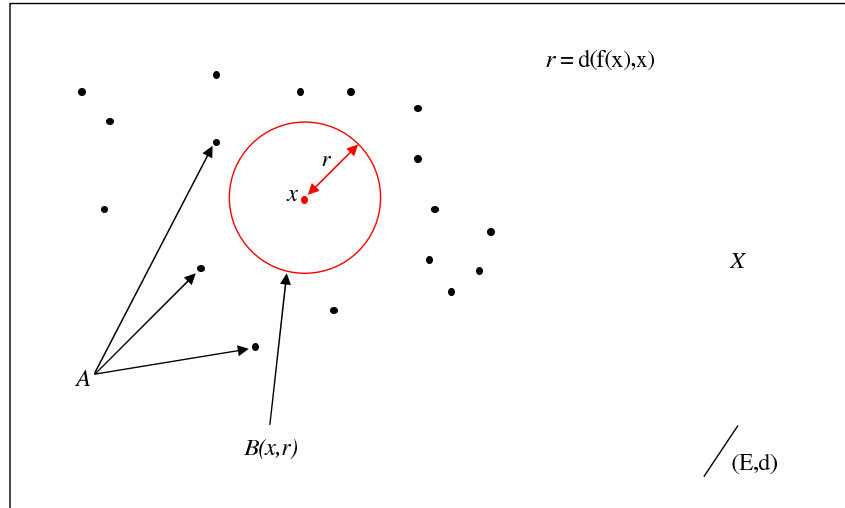
Figure 1: X is f-separate

The geometric interpretation that X is f-separate is that, for finite points in X (or in $f(X)$), one can separate these points by an open ball centered at a point x (or in $f(x)$) with radius $r(x) = d(f(x), x)$.

By relaxing the convexity of set, we have the following theorem.

THEOREM 5.1 *(Fixed-Point without Convexity Assumption) Let X be a nonempty compact subset of a metric space (E, d) and f be a continuous function over X into E. Then, f has a fixed point if and only if X is f-separate.*

PROOF. *Necessity* ($\Rightarrow$): Let $\overline{x} \in X$ be a fixed point of f. Then $f(\overline{x}) = \overline{x}$, *i.e.*, $d(f(\overline{x}), \overline{x}) = 0$. Suppose that X is not f-separate. We distinguish two cases:

1. There exists $A = \{y_1, ..., y_n\} \in \langle X \rangle$ such that $A \cap B(x, d(f(x), x)) \neq \emptyset$ for all $x \in X$. Then, $\forall x \in X$ there exists $y(x) \in A$ such that $y(x) \in B(x, d(f(x), x))$, *i.e.*, $d(f(x), x) > d(x, y(x))$.

   Let $x = \overline{x}$ in the last inequality. Then there exists $y(\overline{x}) \in A$ such that $d(f(\overline{x}), \overline{x}) > d(\overline{x}, y(\overline{x}))$. We have $d(f(\overline{x}), \overline{x}) = 0$, and therefore $0 > d(\overline{x}, y(\overline{x}))$, which is impossible. Thus, for all $A \in \langle X \rangle$, there exists $x \in X$ such that $A \cap B(x, d(f(x), x)) = \emptyset$. Hence, X is f-separate.

13

2. There exists $A = \{y_1, ..., y_n\} \in \langle f(X) \rangle$ such that for $A \cap B(f(x), d(f(x), x)) \neq \emptyset$ for all $x \in X$. Then, $\forall x \in X$ there exists $y(x) \in A$ such that $y(x) \in B(f(x), d(f(x), x))$, i.e., $d(f(x), x) > d(f(x), y(x))$.

Let $x = \overline{x}$ in the last inequality. Then there exists $y(\overline{x}) \in A$ such that $d(f(\overline{x}), \overline{x}) > d(f(\overline{x}), y(\overline{x}))$. We have $d(f(\overline{x}), \overline{x}) = 0$, and therefore $0 > d(f(\overline{x}), y(\overline{x}))$, a contradiction. Thus, for all $A \in \langle f(X) \rangle$, there exists $x \in X$ such that $A \cap B(f(x), d(f(x), x)) = \emptyset$. Hence, X is f-separate.

Sufficiency ($\Leftarrow$): Suppose that X is f-separate. Then, one of the following conditions holds:

3. For all $A \in \langle X \rangle$, there exists $x \in X$ such that $A \cap B(x, d(f(x), x)) = \emptyset$. Consider the following real-valued function φ defined on $X \times X$ and by

$$(x, y) \mapsto \varphi(x, y) = d(f(x), x) - d(x, y).$$

The function $x \mapsto \varphi(x, y)$ is then continuous over X, $\forall y \in X$. Thus, for all $A \in \langle X \rangle$, there exists $x \in X$ such that $A \cap B(x, d(f(x), x)) = \emptyset$, and then for all $y \in A$, $d(x, y) \geq d(f(x), x)$, i.e., $\max_{y \in A} \varphi(x, y) \leq 0$. By Theorem 3.1, there exists $\overline{x} \in X$ such that $\sup_{y \in X} \varphi(\overline{x}, y) \leq 0$. Then, for each $y \in X$, we have $d(\overline{x}, f(\overline{x})) \leq d(\overline{x}, y)$. Letting $y = \overline{x}$ in last inequality, we obtain $d(\overline{x}, f(\overline{x})) = 0$, which means $\overline{x} = f(\overline{x})$. Then, f has a fixed point.

4. For all $C \in \langle f(X) \rangle$, there exists $x \in X$ such that $C \cap B(f(x), d(f(x), x)) = \emptyset$. Consider the following real-valued function φ defined on $X \times f(X)$ and by

$$(x, y) \mapsto \varphi(x, y) = d(f(x), x) - d(f(x), y).$$

The function $x \mapsto \varphi(x, y)$ is continuous over X, $\forall y \in f(X)$. Since for all $C \in \langle f(X) \rangle$, there exists $x \in X$ such that $C \cap B(f(x), d(f(x), x)) = \emptyset$. Thus, for all $y \in C$, $d(f(x), y) \geq d(f(x), x)$, i.e., $\max_{y \in C} \varphi(x, y) \leq 0$. By Theorem 3.1, there exists $\overline{x} \in X$ such that $\sup_{y \in f(X)} \varphi(\overline{x}, y) \leq 0$. Then, for each $y \in f(X)$, we have $d(\overline{x}, f(\overline{x})) \leq d(f(\overline{x}), y)$. Letting $y = f(\overline{x})$ in last inequality, we obtain $d(\overline{x}, f(\overline{x})) = 0$, which means $\overline{x} = f(\overline{x})$. Then, f has a fixed point.

∎

REMARK 5.1 *If X is compact in a metric space, then the conditions 1) and 2) in Definition 5.1 are equivalent.*

EXAMPLE 5.2 (*Continued*) Let us again consider Example 5.1.

$$f : X = [0, 3] \rightarrow \mathbb{R}$$
$$x \mapsto f(x) = \frac{x+4}{x+1}.$$

The point $x = 2$ is a fixed point of f in $[0, 3]$, then the set $[0, 3]$ is f-separate.

Theorem 5.1 can be generalized by relaxing the compactness of X.

THEOREM 5.2 *(Fixed-Point without Convexity or Compactness) Let X be a nonempty subset of a metric space (E, d) and f a continuous function over X into E. Suppose that there exist $\{y^1, ..., y^k\} \subset X$ such that $\bigcap\limits_{i=1,...,k} G(y^i)$ is compact where*

$$G(y) = \begin{cases} \{x \in X, \quad d(f(x), x) \leq d(x, y)\}, & \text{if condition 1) in Definition 5.1 is satisfied} \\ \{x \in X, \quad d(f(x), x) \leq d(f(x), y)\}, & \text{if condition 2) in Definition 5.1 is satisfied.} \end{cases}$$

Then, f has a fixed point if and only if X is f-separate.

PROOF. The necessity is the same as that of Theorem 5.1. We only need to prove the sufficiency in the case where condition 1) in Definition 5.1 is satisfied. For each $y \in Y$, let $G(y) = \{x \in X : \quad d(f(x), x) \leq d(x, y)\}$. Since f is continuous, $G(x)$ is a closed set. Also, the condition that X is f-separate implies that $\{G(y) : \quad y \in X\}$ has the finite intersection property and therefore $\{G(y) \cap \bigcap\limits_{i=1,...,k} G(y^i) : \quad y \in X\}$ has the finite intersection property. Since $\{G(y) \cap \bigcap\limits_{i=1,...,k} G(y^i) : \quad y \in X\}$ is a compact family in the compact set $\bigcap\limits_{i=1,...,k} G(y^i)$. Thus, $\emptyset \neq \bigcap\limits_{y \in X} G \cap \bigcap\limits_{i=1,...,k} G(y^i) = \bigcap\limits_{y \in X} G(y)$. Hence, there exists $\overline{x} \in X$ such that each $y \in X$, $d(f(\overline{x}), \overline{x}) \leq d(\overline{x}, y)$. Then, letting $y = \overline{x}$, we obtain $f(\overline{x}) = \overline{x}$. This completes the proof. ∎

The following proposition provides a sufficient condition for a set X to be f-separate.

PROPOSITION 5.1 *Let X be a nonempty, compact and convex subset of a normed space $(E, \|.\|)$ and f a continuous function over X into E such that $X \subset f(X)$ and the function $z \mapsto \|f(z) - x\|$ is quasiconvex over X, for all $x \in X$. Then, X is f-separate and thus it has a fixed point.*

PROOF. Let $A = \{y_1, y_2, ..., y_n\}$ be any set in $\langle X \rangle$. Suppose that $\forall x \in X, \exists y \in A$ such that

$$\|x - y\| < \|f(x) - x\|, \tag{10}$$

i.e., X is not f-separate. Consider the following multi-valued function:

$$C : X \rightarrow 2^X$$

15

defined by

$$x \mapsto C(x) = \{z \in X : \min_{y \in A}\|x - y\| \geq \|f(z) - x\|\}.$$

1) The condition $X \subset f(X)$ implies that for each $x \in X$, $C(x) \neq \emptyset$.

2) The continuity of f and the compactness of X imply that C is upper semicontinuous over X and for each $x \in X$, the set $C(x)$ is closed in X.

3) The quasiconvexity of function $z \mapsto \|f(z) - x\|$ implies that for each $x \in X$ so that set $C(x)$ is convex in X.

¿From 1)-3) we conclude that the function C satisfies all the conditions of Kakutani's fixed point theorem in Kakutani [1941]. Consequently, $\exists \widetilde{x} \in X$ such that $\widetilde{x} \in C(\widetilde{x})$, *i.e.*,

$$\min_{y \in A}\|\widetilde{x} - y\| \geq \|f(\widetilde{x}) - \widetilde{x}\|.$$

Let $x = \widetilde{x}$ in (10). Then there exists $y(\widetilde{x}) \in A$ such that $\|\widetilde{x} - (\widetilde{x})\| < \|f(\widetilde{x}) - \widetilde{x}\|$. Therefore, $\|f(\widetilde{x}) - \widetilde{x}\| \leq \min_{y \in A}\|\widetilde{x} - y\| \leq \|\widetilde{x} - (\widetilde{x})\| < \|f(\widetilde{x}) - \widetilde{x}\|$, a contradiction. Thus, X is f-separate. ■

EXAMPLE 5.2 Let f be the following function

$$f : X = [0, 3] \to \mathbb{R}$$
$$x \mapsto f(x) = x^2 - 2.$$

Then $\max_{x \in [0,3]}|f'(x)| = 6$, and so f is a 6-lipschitz. Also since $f([0, 3]) = [-2, 7] \not\subset [0, 3]$, all the classical fixed point Theorems (Banach's, Brouwer-Schauder-Tychonoff's, Halpern-Bergman's, Kakutani-Fan-Glicksberg's, ...: see Aliprantis and Border [1994]) are not applicable. However, since $z \mapsto |z^2 - 2 - x|$ is quasiconvex over $[0, 3]$, $\forall x \in [0, 3]$ (see Figure 2), by Proposition 5.1, f has a fixed point in $[0, 3]$. Indeed, $\overline{x} = (1 + \sqrt{5})/2$ is such a point.

In the following theorem, we show the existence of fixed point without the quasiconvexity of $z \mapsto \|f(z) - x\|$ or the convexity of X.

THEOREM 5.3 *Let $(E, \|.\|)$ be a normed space and f a function over E into E. Suppose that there exists a compact set X in E such that:*

1) The restriction of f on X is continuous;
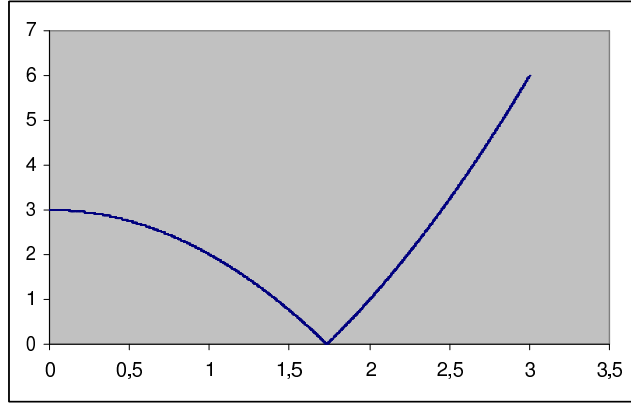
2) $f(X)$ is convex in E; and

16

Figure 2: The graph of function $z \mapsto |z^2 - 2 - x|, \forall x \in [0, 3]$

3) $X \subset f(X).$

Then, f has a fixed point.

PROOF. Consider the following function:

$$\phi : X \times f(X) \to \mathbb{R}$$

defined by $(x, y) \mapsto \phi(x, y) = \|f(x) - x\| - \|x - y\|.$

- The function $x \mapsto \phi(x, y)$ is continuous over X, for each $y \in f(X)$.

- The function $y \mapsto \phi(x, y)$ is quasiconcave over $f(X)$, for each $x \in X$.

Suppose that

$$\forall x \in X, \text{ there exists } y \in f(X), \text{ such that } \phi(x, y) > 0. \tag{11}$$

Then, $f(X)$ can be covered by the sets

$$\theta_y = \{ f(x) \in f(X) : \phi(x, y) > 0 \}, \ y \in f(X).$$

Since θ_y is open in $f(X)$ and $f(X)$ is compact, it can be covered by a finite number r of subsets $\{\text{int } \theta_{y_1}, ..., \text{int } \theta_{y_r}\}$ of type θ_y. Consider a continuous partition of unity $\{h_i\}_{i=1,...,r}$ associated to this finite covering, and the following function:

$$\alpha : f(X) \to f(X), \text{ such that } \alpha(y) = \sum_{i=1}^{r} h_i(y) y_i.$$

17

Since the function α is continuous over the compact convex $f(X)$ into $f(X)$, then by Brouwer Fixed-Point Theorem, there exists $\tilde{y} = f(\tilde{x}) \in f(X)$ such that $\tilde{y} = f(\tilde{x}) = \sum_{i=1}^{r} h_i(\tilde{y}) y_i$. Let $J = \{i = 1, ..., r : h_i(\tilde{y}) > 0\}$.

The quasiconcavity of $y \mapsto \phi(x, y)$ implies that:

$$\min_{i \in J} \phi(\tilde{x}, y_i) \leq 0. \tag{12}$$

If $i \in J$, $\tilde{y} \in supp(h_i) \subset \theta_{y_i}$. Thus, $\phi(\tilde{x}, y_i) > 0$ for each $i \in J$. Therefore,

$$\min_{i \in J} \phi(\tilde{x}, y_i) > 0. \tag{13}$$

Then inequalities (12) and (13) imply $0 < \min_{i \in J} \phi(\tilde{x}, y_i) \leq 0$ which is impossible. Thus, supposition in (11) is not true, *i.e.*,

$$\exists \overline{x} \in X, \text{ such that } \forall y \in f(X), \text{ we have } \phi(\overline{x}, y) = \|f(\overline{x}) - \overline{x}\| - \|\overline{x} - y\| \leq 0.$$

Hence, $\|f(\overline{x}) - \overline{x}\| \leq \|\overline{x} - y\|$ for each $y \in f(X)$. By condition 3) of Theorem, we have $X \subset f(X)$. Thus, letting $y = \overline{x}$, we obtain $\|f(\overline{x}) - \overline{x}\| = 0$ in last inequality, which means $\overline{x}$ is a fixed point of f in X. ∎

EXAMPLE 5.3 Let f be the following function

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto f(x) = x^4 + 2x^2 - 5x + 1.$$

Let $X = [1, 2]$. We have $\max_{x \in [1,2]} |f'(x)| = 35$ so that f is a 35-lipschitz. Also, since $f([1, 2]) = [-1, 15] \not\subseteq [1, 2]$, all the classical fixed point theorems are not applicable.

However, since the restriction of f on $[1, 2]$ is continuous, $f([1, 2])$ is convex and $f([1, 2]) = [-1, 15] \supset [1, 2]$, then, by Theorem 5.3, f has a fixed point in $[1, 2]$. Indeed, $\overline{x} \simeq 1.36$ is such a point.

EXAMPLE 5.4 Let f be the following function

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto f(x) = \frac{2x^2 - 4}{x + 2}.$$

Let $X = [-1, 4]$. We then have $\max_{x \in [-1,4]} |f'(x)| = \frac{17}{9}$ so that f is a $\frac{17}{9}$-lipschitz. Also, since $f([-1, 4]) = [4\sqrt{2} - 8, 14/3] \not\subseteq [-1, 4]$, the function $z \mapsto \|f(z) - x\|$ is not quasiconvex in z and thus all the classical fixed point Theorems and Proposition 5.1 are not applicable.

However, since the restriction of f on $[-1, 4]$ is continuous, $f([-1, 4])$ is convex and $f([-1, 4]) = [4\sqrt{2} - 8, 14/3] \supset [-1, 4]$, then, by Theorem 5.3, f has a fixed point in $[-1, 4]$. Indeed, $\overline{x} = 1 + \sqrt{5}$ is such a point.

The following theorem generalizes the existence theorems on fixed point to a multi-function mapping.

THEOREM 5.4 *Let X be a nonempty compact subset of a metric space E and C a multifunction mapping defined on X into E such that the function $x \mapsto d(x, C(x))$ is lower semicontinuous over X. Then, C has a fixed point if and only if for each $A \in \langle X \rangle$, there exists $x \in X$ such that $d(x, C(x)) \leq d(x, A)$.*

PROOF. It is a consequence of Theorem 3.1 by defining $\Psi(x, y) = d(x, C(x)) - d(x, y)$. ∎

THEOREM 5.5 *Let X be a nonempty compact subset of a metric space E and Y a nonempty subset of a metric space (F, d), and let C be a multifunction mapping defined on X into Y such that the function $x \mapsto d(g(x), C(x))$ is lower semicontinuous over X, where g is a continuous function defined from X into Y. Then, C has a g-fixed point (i.e., $\exists \bar{x} \in X$ such that $g(\bar{x}) \in C(\bar{x})$) if and only if for each $A \in \langle g(X) \rangle$, there exists $x \in X$ such that $d(g(x), C(x)) \leq d(g(x), A)$.*

PROOF. It is a consequence of Theorem 3.1 by defining $\Psi(x, y) = d(g(x), C(x)) - d(g(x), y)$. ∎

THEOREM 5.6 *Let X be a nonempty compact subset of a metric space (E, d_1) and Y a nonempty subset of a metric space (F, d_2). Let C be a multifunction mapping defined on X into Y such that the function $x \mapsto d_2(g(x), C(x))$ is lower semicontinuous over X, where g is a continuous function defined from X into Y. Then, C has a g-fixed point if and only if for each $A \in \langle X \rangle$, there exists $x \in X$ such that $d_2(g(x), C(x)) \leq d_1(x, A)$.*

PROOF. It is a consequence of Theorem 3.1 by defining $\Psi(x, y) = d_2(g(x), C(x)) - d_1(x, y)$. ∎

6   Existence of Coincidence Points

This section provides a necessary and sufficient for the existence of coincidence points of two functions defined on a set that may not be compact or convex.

DEFINITION 6.1 Let X be a nonempty compact subset of a metric space (E, d_1), and let f and g be two continuous functions over X into a metric space (F, d_2). Then, f and g are said to have a coincidence point if there exists $\bar{x} \in X$ such that $f(\bar{x}) = g(\bar{x})$.

DEFINITION 6.2 Let X be a nonempty set in a metric space (E, d_1), and let f and g be two functions from X into a metric space (F, d_2). The set X is called fg-*separate* if one of the following conditions holds:

1) for all $A \in \langle X \rangle$, there exists $x \in X$ such that

$$A \cap B(x, d_2(f(x), g(x))) = \emptyset;$$

2) for all $A \in \langle f(X) \rangle$, there exists $x \in X$ such that

$$A \cap B(f(x), d_2(f(x), g(x))) = \emptyset.$$

3) for all $A \in \langle g(X) \rangle$, there exists $x \in X$ such that

$$A \cap B(g(x), d_2(f(x), g(x))) = \emptyset.$$

THEOREM 6.1 *(Coincidence Theorem without Convexity Assumption) Let X be a nonempty compact subset of a metric space (E, d_1), and let f and g be two continuous functions over X into a metric space (F, d_2). Then, f and g has a coincidence point if and only if X is fg-separate.*

PROOF. *Necessity* ($\Rightarrow$): It is the same as that of Theorem 5.1.
Sufficiency ($\Leftarrow$): If the first condition in Definition 6.2 is satisfied, let

$$\phi : X \times X \to \mathbb{R}, \quad (x, y) \mapsto \phi(x, y) = d_2(f(x), g(x)) - d_1(x, y).$$

If the second condition of Definition 6.2 is satisfied, let

$$\phi : X \times f(X) \to \mathbb{R}, \quad (x, y) \mapsto \phi(x, y) = d_2(f(x), g(x)) - d_2(f(x), y).$$

If the third condition of Definition 6.2 is satisfied, let

$$\phi : X \times g(X) \to \mathbb{R}, \quad (x, y) \mapsto \phi(x, y) = d_2(f(x), g(x)) - d_2(g(x), y).$$

The remaining proof of the sufficiency is the same as that in the proof of Theorem 5.1. ∎

THEOREM 6.2 *Let E be a topological space and $(F, \|.\|)$ a normed space. Let f and g be two functions over E into F. Suppose that there exists a compact set X in E such that the restriction of f and g on X are continuous and*

$$\begin{cases} 1) \ f(X) \text{ is convex in } F; \\ 2) \ g(X) \subset f(X). \end{cases} \quad \text{or} \quad \begin{cases} 1') \ g(X) \text{ is convex in } F; \\ 2') \ f(X) \subset g(X). \end{cases}$$

Then, f and g has a coincidence point.

PROOF. If Conditions 1-2 are satisfied, let

$$\phi : X \times f(X) \to \mathbb{R}, \quad (x,y) \mapsto \phi(x,y) = \|f(x) - g(x)\|_F - \|g(x) - y\|_F.$$

If Conditions $1'$-$2'$ are satisfied, let

$$\phi : X \times g(X) \to \mathbb{R}, \quad (x,y) \mapsto \phi(x,y) = \|f(x) - g(x)\|_F - \|f(x) - y\|_F.$$

The remaining proof of the sufficiency is the same as that in the proof of Theorem 5.3. ∎

7 Existence of Nash Equilibrium

As an application of our basic result on the minimax inequality, in this section we provide a result on the existence of pure strategy Nash equilibrium without assuming the convexity of strategy spaces and any form of quasiconcavity of payoff functions.

Consider the following noncooperative game in normal form:

$$G = (X_i, \ f_i)_{i \in I} \tag{14}$$

where $I = \{1, ..., n\}$ is the finite set of players, X_i is player i's strategy space which is a nonempty subset of a topological space E_i, and $f_i : X \longrightarrow \mathbb{R}$ is the payoff function of player i. Denote by $X = \prod_{i \in I} X_i$ the set of strategy profiles of the game and $f = (f_1, f_2, ..., f_n)$ the profile of payoff functions. For each player $i \in I$, denote by $-i = \{j \in I \text{ such that } j \neq i\}$ the set of all players rather than player i. Also denote by $X_{-i} = \prod_{j \in -i} X_j$ the set of strategies of the players in coalition $-i$.

DEFINITION 7.1 A strategy profile $\overline{x} \in X$ is said to be a *pure strategy Nash equilibrium* of game (14) if,

$$\forall i \in I, \ \forall y_i \in X_i, \ f_i(\overline{x}_{-i}, y_i) \leq f_i(\overline{x}).$$

The aim of each player is to choose a strategy in X_i that maximizes his payoff function.

Define a function $\Psi : X \times X \to \mathbb{R}$ by

$$\Psi(x,y) = \sum_{i=1}^{n} \{f_i(x_{-i}, y_i) - f_i(x)\}, \quad \forall (x,y) \in X \times X.$$

The following theorem generalized Theorem 1 in Baye *et al.* [1993] by relaxing the convexity of strategy spaces and 0-transfer quasiconcavity of payoff function.

THEOREM 7.1 *(Nash Equilibrium without Convexity Assumption) Let $I = \{1, ..., n\}$ be an indexed finite set, let X_i be a nonempty and compact subset of a topological space E_i. Suppose that the function $\Psi(x, y)$ is 0-transfer lower continuous in x with respect to X. Then, the game $G = (X_i, f_i)_{i \in I}$ has a Nash equilibrium if and only if, $\forall A \in \langle X \rangle$, $\exists x \in X$ such that for each $i \in I$, we have*

$$f_i(y_i, x_{-i}) \leq f_i(x), \quad \text{for each } y \in A. \tag{15}$$

PROOF. First note that the condition (15) is equivalent to the function $y \mapsto \Psi(x, y)$ is 0-locally-dominated. Then it is a straightforward consequence of Theorem 3.1 and definition of Ψ. ∎

EXAMPLE 7.1 Suppose that in game (14) $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [1, 2] \cup [3, 4]$, $x = (x_1, x_2)$ and

$$f_1(x) = x_2 x_1^2,$$
$$f_2(x) = -x_1 x_2^2.$$

In this example, X_i is not convex, $\forall i \in I$, and the function $y_i \mapsto f_i(x_{-i}, y_i)$ is not quasiconcave for $i = 1$ so that the existing theorems on Nash equilibrium such as in Nash [1951], Debreu [1952], Rosen [1965]. Nishimura and Friedman [1981], Dasgupta and Maskin [1986], Baye *et al.* [1993], Tian and Zhou [1993], Yao [1992], and Reny [1999] are not applicable.

However, we can show the existence of Nash equilibrium by applying Theorem 7.1. Indeed, for each $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$\Psi(x, y) = x_2 y_1^2 - x_1 y_2^2.$$

The function Ψ is continuous on $X \times X$. For any subset $\{(_1y_1, _2y_2), ..., (_ky_1, _ky_2)\}$ of X, let $x = (x_1, x_2) \in X$ such that $x_1 = \max\limits_{h=1,...,k} {}_iy_1$ and $x_2 = \min\limits_{h=1,...,k} {}_iy_2$. Then, we have

$$\begin{cases} {}_iy_2^2 \geq x_2^2, \quad \forall i = 1, ..., k, \\ {}_iy_1^2 \leq x_1^2. \end{cases}$$

Thus,

$$\begin{cases} -x_1 \; {}_iy_2^2 \leq -x_1 \; x_2^2, \quad \forall i = 1, ..., k, \\ x_2 \; {}_iy_1^2 \leq x_2 \; x_1^2. \end{cases}$$

Therefore, $\Psi(x, {}_iy) \leq \Psi(x, x), \forall i = 1, ..., k$. According to Theorem 7.1, this game has a Nash equilibrium.

Theorem 7.1 can be generalized by relaxing the compactness of X.

THEOREM 7.2 *(Nash Equilibrium without Convexity and Compactly Set) Let $I = \{1, ..., n\}$ be an indexed finite set and let X_i be a nonempty subset of a topological space E_i. Suppose that function $\Psi : X \times X \to \mathbb{R} \cup \{\infty\}$ satisfies the following conditions:*

(a) $\Psi(x, y)$ is 0-transfer lower continuous in x with respect to X;

(b) there exists $\{y^1, ..., y^k\} \subset X$ such that $\bigcap_{i=1,...,k} G(y^i)$ is compact where $G(y) = \{x \in X : \sum_{i=1}^{n}[f_i(x_{-i}, y_i) - f_i(x)] \leq 0\}$.

Then, the game $G = (X_i, \ f_i)_{i \in I}$ has a Nash equilibrium if and only if, $\forall A \in \langle X \rangle$, $\exists x \in X$ such that for each $i \in I$, we have $f_i(y_i, x_{-i}) \leq f_i(x), \ \forall y \in A$.

8 Conclusion

In this paper, we introduced a new condition, called "local-dominatedness property," that can be used to characterize existence of equilibria in many problems which may have non-convex and/or compact sets and have non-quasiconcave functions. We first investigated the existence of equilibrium in minimax inequalities under the local-dominatedness condition. We proved that the local-dominatedness condition is necessary, and further under mild continuity condition, sufficient for the existence of solution in minimax inequality. The basic results on the minimax inequality are then used to get new theorems on the existence of saddle points, fixed points, and coincidence points of functions. As an application of our basic result, we also characterize the existence of pure strategy Nash equilibrium in games with discontinuous and nonquasiconcave payoff functions and nonconvex and noncompact strategy spaces.

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