Modified Symplectic Structures in Cotangent Bundles of Lie Groups.

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In earlier work [1], we studied an extension of the canonical symplectic structure in the cotangent bundle of an affine space $Q = \mathbf{R}^N$, by additional terms implying the Poisson non-commutativity of both configuration and momentum variables. In this article, we claim that such an extension can be done consistently when Q is a Lie group G.

Keywords: Symplectic Mechanics; Noncommutative Configuration Space.

1. INTRODUCTION

As applied to physics, noncommutative geometry is understood mainly in two ways. The first one is the spectral triple approach of A.Connes [2] with the Dirac operator playing a central role in unifying, through the universal action principle, gravitation with the standard model of fundamental interactions. The second one is the quantum field theory on noncommutative spaces [3] with the Moyal product as main ingredient. Besides these, a proposition by several authors [4, 5] was made to generalise quantum mechanics in such a way that the operators corresponding to space coordinates no longer commute: $[\widehat{x}^k, \widehat{x}^\ell] \neq 0$. This was implemented by an extension of the Poisson structure on the cotangent space such that the brackets satisfy $\{x^k, x^\ell\} \neq 0$. Upon quantisation, the corresponding operators should then also be noncommutative. A particle moving in an affine space $\mathbf{A}^N$, has its configuration, in a fixed reference frame, given by an element $\{x^k\}$ of the translation group: $Q = \mathbf{R}^N$ with cotangent bundle $T^\star(Q) = \mathbf{R}^N \times \mathbf{R}^N$. In [1], we examined such an extension of the canonical symplectic two-form $\omega_0 = dx^i \wedge dp_i \to \Omega = \omega_0 + \omega_F + \omega_B$:

$$\omega_F = \frac{1}{2} F_{ij}(x)\, dx^i \wedge dx^j \ , \ \omega_B = \frac{1}{2} B^{k\ell}(p)\, dp_k \wedge dp_\ell \qquad (1.1)$$

This extension is form-invariant under a change of the reference frame lifted to the cotangent bundle:

$$T^\star(Q) \to T^\star(Q) : (x^i, p_k) \to \left(x'^i = A^i{}_j x^j + a^k \,,\, p'_k = p_\ell \,(A^{-1})^\ell{}_k \right) \qquad (1.2)$$

$$\begin{aligned} \Omega \ \to \ \Omega' &= dx'^i \wedge dp'_i + \frac{1}{2} F'_{ij}(x')\, dx'^i \wedge dx'^j + \frac{1}{2} B'^{k\ell}(p')\, dp'_k \wedge dp'_\ell \\ F'_{ij}(x') &= F_{k\ell}(x)\,(A^{-1})^k{}_i (A^{-1})^\ell{}_j \ , \ B'^{k\ell}(p') = A^k{}_i A^\ell{}_j B^{ij}(p) \end{aligned} \qquad (1.3)$$

For a general configuration space Q, a diffeomorphism $\phi : x^i \to x'^i \doteq \phi^i(x)$, when lifted to $T^\star(Q)$, becomes

$$\begin{aligned} \widetilde{\phi} &: (x^i, p_k) \to \left(x'^i = \phi^i(x), p'_k = p_\ell \frac{\partial(\phi^{-1}(x'))^\ell}{\partial x'^k} \right) \\ F'_{ij}(x') &= F_{k\ell}(x) \frac{\partial(\phi^{-1})^k(x')}{\partial x'^i} \frac{\partial(\phi^{-1})^\ell(x')}{\partial x'^j} \\ B'^{k\ell}(p', x') &= \frac{\partial \phi^k(x)}{\partial x^i} \frac{\partial \phi^\ell(x)}{\partial x^j} B^{ij}(p) \end{aligned}$$

In general $B'^{k\ell}$ is function of both variables $\{p', x'\}$ and no intrinsic meaning can be given to the particular form of the extension Ω in equation (1.1).

In this work, we show that such an extension is achieved when $Q = G$ is a Lie group. This is possible because the cotangent bundle $T^\star(G)$ has two distinguished trivialisations, the left- and right trivialisations [7] implemented respectively by the bases of the left- and right invariant differential forms.

In section **2.**, inspired by the rigid body motion, we use the left trivialisation with left invariant or *body-coordinates* and construct a left invariant two-form. In the case of constant F_{ij} and $B^{k\ell}$ fields the ω_F term arises from a symplectic one-cocycle, as introduced by Souriau [8, 9], and ω_B will be automatically left invariant. The constructed two-form Ω is obviously closed but the non degeneracy condition leads in general to a constrained Hamiltonian system. This is examined in more detail for $SU(2)$ in section **3.**. Final considerations are made in section **4.**. Some elements of Lie algebra cohomology [9, 10] are recalled in the appendix.

2. THE PHASE SPACE $\{\mathcal{M}_0 \equiv T^\star(G), \omega_0\}$

Let $\{g^\alpha, \alpha = 1, 2, \cdots, N\}$ be coordinates of a group element $g \in G$. Natural or holonomic coordinates of points $(g, \mathbf{p}_g) \in T^\star(G)$ are obtained using the basis $\{\mathbf{d}g^\mu\}$ of the cotangent space $T^\star_g(G)$. They are given by $(g^\alpha, p_\mu)_{hol}$, where $\mathbf{p}_g = p_\mu\, \mathbf{d}g^\mu$. Given a pair of dual bases $\{\mathbf{e}_\alpha\}$ of the Lie algebra $\mathcal{G} \doteq T_e(G)$ and $\{\varepsilon^\alpha\}$ of its dual $\mathcal{G}^\star$, the differential and pull-back of the left- and right translations (L_g, R_g)

define left- and right invariant vector fields and one forms: $\mathbf{e}^L_\alpha(g) \doteq L_{g*|e}\,\mathbf{e}_\alpha$, $\mathbf{e}^R_\alpha(g) \doteq R_{g*|e}\,\mathbf{e}_\alpha$, $\varepsilon^\alpha_L(g) \doteq L^*_{g^{-1}|g}\,\varepsilon^\alpha$, $\varepsilon^\alpha_R(g) \doteq R^*_{g^{-1}|g}\,\varepsilon^\alpha$. With canonical group coordinates, in terms of $L^\alpha{}_\beta(g,h) \doteq \partial(g\,h)^\alpha/\partial g^\beta$ and $R^\alpha{}_\beta(g,h) \doteq \partial(h\,g)^\alpha/\partial g^\beta$, they are explicitely given by:

$$\mathbf{e}^L_\alpha(g) = L^\mu{}_\alpha(g,e)\,\frac{\partial}{\partial g^\mu} \quad , \quad \mathbf{e}^R_\alpha(g) = R^\mu{}_\alpha(g,e)\,\frac{\partial}{\partial g^\mu}$$
$$\varepsilon^\alpha_L(g) = L^\alpha{}_\mu(g^{-1},g)\,\mathbf{d}g^\mu \quad , \quad \varepsilon^\alpha_R(g) = R^\alpha{}_\mu(g^{-1},g)\,\mathbf{d}g^\mu \tag{2.1}$$

These bases implement canonical trivialisations of the tangent and cotangent bundle. For the cotangent bundle, which is the arena of symplectic or Hamiltonian formalism, we have a left and a right trivialisation:

$$\begin{aligned}\lambda \;:\; & T^\star(G) \to G\times\mathcal{G}^\star : (g, p_g = p_\mu\,\mathbf{d}g^\mu) \to \left(g, \pi^L = L^*_{g|e}\,p_g = \pi^L_\mu\,\varepsilon^\mu\right)\\ & \pi^L_\mu = \langle p_g, \mathbf{e}^L_\mu\rangle = p_\nu\,L^\nu{}_\mu(g,e)\\ \rho \;:\; & T^\star(G) \to G\times\mathcal{G}^\star : (g, p_g = p_\mu\,\mathbf{d}g^\mu) \to \left(g, \pi^R = R^*_{g|e}\,p_g = \pi^R_\mu\,\varepsilon^\mu\right)\\ & \pi^R_\mu = \langle p_g, \mathbf{e}^R_\mu\rangle = p_\nu\,R^\nu{}_\mu(g,e)\end{aligned}$$

They can be viewed as a change of coordinates of a point (g, p_g) in $T^\star(G)$:

$$(g,\mathbf{p}_g) \leftrightarrow (g^\alpha, p_\mu)_{hol} \leftrightarrow (g^\alpha, \pi^L_\mu)_\mathbf{B} \leftrightarrow (g^\alpha, \pi^R_\mu)_\mathbf{S} \tag{2.2}$$

In rigid body theory, the coordinates of the left trivialisation are the "body" coordinates, whence the subscript $(\,,)_\mathbf{B}$. The right trivialisation yields "space" coordinates with subscript $(\,,)_\mathbf{S}$. Both are related through the coadjoint representation of G in $\mathcal{G}^\star$:

$$\pi^R_\mu = \mathbf{K}_\mu{}^\nu(g)\,\pi^L_\nu = \mathbf{Ad}^\nu{}_\mu(g^{-1})\,\pi^L_\nu \tag{2.3}$$

Lifting the left multiplication in G to the cotangent bundle yields a group action: $\widetilde{L}_a : T^\star(G) \to T^\star(G) : x = (g, p_g) \to y = (ag, p'_{ag} = L^\star_{a^{-1}|ag}\,p_g)$. In body coordinates: $\left(\widetilde{L}_a\right)_\mathbf{B}$: $(g^\alpha, \pi^L_\mu)_\mathbf{B} \to ((ag)^\alpha, \pi^L_\mu)_\mathbf{B}$. The pull-back of the cotangent projection $\kappa : T^\star(G) \to G : x \doteq (g, p_g) \to g$, acting on the $\{\varepsilon^\alpha(g)\}$ yield $\widetilde{L}_a$ invariant one forms on $T^\star(G) : \langle\varepsilon^\alpha_L(x)| = \kappa^\star_x\,\varepsilon^\alpha_L(\kappa(x))$ and the differentials of the left invariant functions π^L_μ on $T^\star(G)$ also yield $\widetilde{L}_a$ invariant one forms on $T^\star(G)$. Together they provide a left invariant basis of the cotangent space at $x = (g^\alpha, \pi^L_\mu)_\mathbf{B} \in T^\star(G)$:

$$\left\{\langle\varepsilon^\alpha_L| \doteq L^\alpha{}_\mu(g^{-1},g)\,\langle\mathbf{d}g^\mu|\,,\,\langle\varepsilon^L_\mu| \doteq \langle\mathbf{d}\pi^L_\mu|\right\} \tag{2.4}$$

Its dual basis in the tangent space $T_x(T^\star(G))$ is given by

$$\left\{|\mathbf{e}^L_\alpha\rangle \doteq |\partial/\partial g^\mu\rangle\,L^\mu{}_\alpha(g,e)\,,\,|\mathbf{e}^\mu_L\rangle \doteq |\partial/\partial\pi^L_\mu\rangle\right\} \tag{2.5}$$

The canonical Liouville one-form $\langle\theta_0| = p_\alpha\,\langle dg^\alpha|$ and its associated symplectic two-form $\omega_0 = -\mathbf{d}\theta_0 = \langle\mathbf{d}g^\alpha|\wedge\langle\mathbf{d}p_\alpha|$, are obtained as:

$$\langle\theta_0| = \pi^L_\mu\,\langle\varepsilon^\mu_L|\,,\;\omega_0 = \langle\varepsilon^\mu_L|\wedge\langle\varepsilon^L_\mu| + \frac{1}{2}\,\pi^L_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}\,\langle\varepsilon^\alpha_L|\wedge\langle\varepsilon^\beta_L| \tag{2.6}$$

The Hamiltonian vector field associated to a function $A(g,\pi^L)$ on phase space $\mathcal{M}_0 \equiv T^\star(G)$, is defined by: $\imath_\mathbf{X}\,\omega_0 = \langle\mathbf{d}A|$. Its components are:

$$\begin{aligned} X^\mu &\doteq \langle\varepsilon^\mu_L|\mathbf{X}\rangle = \langle\mathbf{d}A|\mathbf{e}^\mu_L\rangle\\ X_\alpha &\doteq \langle\varepsilon^L_\alpha|\mathbf{X}\rangle = -\langle\mathbf{d}A|\mathbf{e}^L_\alpha\rangle - \pi^L_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}\,\langle\mathbf{d}A|\mathbf{e}^\beta_L\rangle \end{aligned} \tag{2.7}$$

With $\imath_\mathbf{Y}\,\omega_0 = \langle\mathbf{d}B|$, the Poisson bracket of dynamical variables: $\{A,B\}_0 \doteq \omega_0\,(\mathbf{X},\mathbf{Y})$, is obtained explicitely in (g^α, π^L_μ) variables as:

$$\{A,B\}_0 = \langle\mathbf{d}A|\mathbf{e}^L_\alpha\rangle\,\frac{\partial B}{\partial\pi^L_\alpha} - \frac{\partial A}{\partial\pi^L_\alpha}\,\langle\mathbf{d}B|\mathbf{e}^L_\alpha\rangle - \frac{\partial A}{\partial\pi^L_\alpha}\,\pi^L_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}\,\frac{\partial B}{\partial\pi^L_\beta} \tag{2.8}$$

In particular, the basic Poisson brackets are:

$$\begin{aligned} \left\{g^\alpha, g^\beta\right\}_0 = 0 \quad &, \quad \{g^\alpha, \pi^L_\nu\}_0 = L^\alpha{}_\nu(g,e)\\ \left\{\pi^L_\mu, g^\beta\right\}_0 = -L^\beta{}_\mu(g,e) \quad &, \quad \{\pi^L_\mu, \pi^L_\nu\}_0 = -\,\pi^L_\kappa\,\mathbf{f}^\kappa{}_{\mu\nu} \end{aligned} \tag{2.9}$$

The flow of a particular observable, the Hamiltonian $H(g,\pi^L)$, determines the time evolution of any observable $A(g,\pi^L)$ by the equation: $dA/dt = \{A,H\}_0$. We assume a Hamiltonian is of the form $H(g,\pi^L) = K(\pi^L) + V(g)$.

Here, as in rigid body mechanics, the *kinetic energy* is given by

$$K \doteq \frac{1}{2}\,I^{\alpha\beta}\,\pi^L_\alpha\,\pi^L_\beta \tag{2.10}$$

where $I^{\alpha\beta}$ is the inverse of a constant, positive definite, *inertia tensor* $I_{\mu\nu}$ in the "body" frame. The *potential energy* is a function V defined on the group manifold. The Euler equations of motion read:

$$\langle \varepsilon_L^\alpha | dg/dt \rangle = L^\alpha{}_\beta(g^{-1},g)\,\frac{dg^\beta}{dt} = \frac{\partial K}{\partial \pi_\alpha^L} \tag{2.11}$$

$$\langle \varepsilon^L_\mu | d\pi^L/dt \rangle = \frac{d\pi_\mu^L}{dt} = -\frac{\partial V}{\partial g^\alpha}\,L^\alpha{}_\mu(g,e) + \frac{\partial K}{\partial \pi_\nu^L}\,\pi_\alpha^L\,\mathbf{f}^\alpha{}_{\nu\mu} \tag{2.12}$$

The first of these equations **(2.11)** relates the angular momentum π_α^L with the angular velocity in the body frame Ω_L^μ:

$$\Omega_L^\alpha \doteq L^\alpha{}_\beta(g^{-1},g)\,\frac{dg^\beta}{dt} = I^{\alpha\mu}\pi_\mu^L \;;\; \pi_\mu^L = I_{\mu\nu}\,\Omega_L^\nu \tag{2.13}$$

while the second **(2.12)** takes the classical form

$$\frac{d\pi_\mu^L}{dt} + \pi_\kappa^L\,\mathbf{f}^\kappa{}_{\mu\nu}\,\Omega_L^\nu = -\frac{\partial V}{\partial g^\alpha}\,L^\alpha{}_\mu(g,e) \tag{2.14}$$

An example of $V(g)$ is given by a *gravitational potential energy* as follows. Let $\mathbf{L} = \mathbf{e}_\alpha L^\alpha$ be a constant vector in $\mathcal{G}$ (the position of the centre of mass in the body frame) and $\gamma = \gamma_\alpha\,\varepsilon^\alpha$ a constant vector in $\mathcal{G}^\star$ (the gravitational force in the space fixed frame). The potential energy is defined as:

$$V(g) \doteq -(\gamma|\,\mathbf{Ad}(g)\,\mathbf{L}) = -\left(\mathbf{K}(g^{-1})\gamma|\,\mathbf{L}\right) \tag{2.15}$$

where $(\;|\;)$ denotes the canonical pairing between $\mathcal{G}$ and its dual $\mathcal{G}^\star$. To compute $\langle \mathbf{d}V|\mathbf{e}_\mu^L\rangle$ we use the representation of the Maurer-Cartan form:

$$D(g^{-1})\,\mathbf{d}D(g) = D'(g^{-1}\,\mathbf{d}g)$$

where D is any representation D of G, with derived representation D' of $\mathcal{G}$. In particular, $\mathbf{dAd}(g) = \mathbf{Ad}(g)\,\mathbf{ad}(\mathbf{e}_\mu)\,\varepsilon_L^\mu(g)$ and $\mathbf{dK}(g) = \mathbf{K}(g)\,\mathbf{k}(\mathbf{e}_\mu)\,\varepsilon_L^\mu(g)$. This yields:

$$\langle \mathbf{d}V|\mathbf{e}_\mu^L\rangle(g) = -\left(\mathbf{K}(g^{-1})\gamma|\,\mathbf{ad}(\mathbf{e}_\mu)\,\mathbf{L}\right) = -\left(\Gamma(g)\,|\,\mathbf{ad}(\mathbf{e}_\mu)\,\mathbf{L}\right) \tag{2.16}$$

where $\Gamma(g) \doteq \mathbf{K}(g^{-1})\,\gamma$ is the variable gravitational force in the body-fixed frame. Using the above formulae to compute $\mathbf{dK}(g^{-1})$, we obtain:

$$\frac{d\Gamma_\mu}{dt} = (\Gamma\,|\,\mathbf{ad}(\mathbf{e}_\mu)\,\Omega_L) = \Gamma_\alpha\,\mathbf{f}^\alpha{}_{\mu\beta}\,\Omega_L^\beta \tag{2.17}$$

Equation **(2.14)** reads:

$$\frac{d\pi_\mu^L}{dt} + \pi_\alpha^L\,\mathbf{f}^\alpha{}_{\mu\beta}\,\Omega_L^\beta = (\Gamma\,|\,\mathbf{ad}(\mathbf{e}_\mu)\,\mathbf{L}) = \Gamma_\alpha\,\mathbf{f}^\alpha{}_{\mu\beta}\,L^\beta \tag{2.18}$$

Together with **(2.13)**,

$$\Omega_L^\alpha \doteq L^\alpha{}_\beta(g^{-1},g)\,\frac{dg^\beta}{dt} = I^{\alpha\mu}\pi_\mu^L$$

the equations **(2.17)** and **(2.18)** form the so-called Euler-Poisson system.

3. MODIFIED SYMPLECTIC STRUCTURE ON $T^\star(G)$

In appendix **A** it is shown that, if $\Theta = \frac{1}{2}\Theta_{\alpha\beta}\,\varepsilon^\alpha \wedge \varepsilon^\beta \in \Lambda^2(\mathcal{G}^\star)$, obeys the cocycle condition **(A.1)**, then $\Theta_L(g) \doteq (1/2)\,\Theta_{\alpha\beta}\,\varepsilon_L^\alpha(g) \wedge \varepsilon_L^\beta(g)$ is a closed left-invariant two-form on G. Including this closed two-form in the canonical two-form, one obtains another symplectic two-form on $T^\star(G)$, which, furthermore, is $\widetilde{L}_a$ invariant. So we define:

$$\omega_I = \omega_0 - \Theta_L = \langle \varepsilon_L^\mu| \wedge \langle \mathbf{d}\pi_\mu^L| + \frac{1}{2}\left(\pi_\mu^L\,\mathbf{f}^\mu{}_{\alpha\beta} - \Theta_{\alpha\beta}\right)\,\langle \varepsilon_L^\alpha| \wedge \langle \varepsilon_L^\beta| \tag{3.1}$$

The Poisson brackets are also modified and **(2.8)**, **(2.9)** become:

$$\{A,B\}_I = \frac{\partial A}{\partial g^\mu}\,L^\mu{}_\alpha(g,e)\,\frac{\partial B}{\partial \pi_\alpha^L} - \frac{\partial B}{\partial g^\mu}\,L^\mu{}_\alpha(g,e)\,\frac{\partial A}{\partial \pi_\alpha^L} - \left(\pi_\mu^L\,\mathbf{f}^\mu{}_{\alpha\beta} - \Theta_{\alpha\beta}\right)\,\frac{\partial A}{\partial \pi_\alpha^L}\,\frac{\partial B}{\partial \pi_\beta^L} \tag{3.2}$$

In particular, the fundamental brackets are:

$$\left\{g^\alpha, g^\beta\right\}_I = 0 \;,\; \left\{g^\alpha, \pi_\nu^L\right\}_I = L^\alpha{}_\nu(g,e)$$
$$\left\{\pi_\mu^L, g^\beta\right\}_I = -L^\beta{}_\mu(g,e) \;,\; \left\{\pi_\mu^L, \pi_\nu^L\right\}_I = -\left(\pi_\kappa^L\,\mathbf{f}^\kappa{}_{\mu\nu} - \Theta_{\mu\nu}\right) \tag{3.3}$$

The modified symplectic structure induces an additional interaction and the Euler equations become:

$$\Omega_L^\alpha \doteq L^\alpha{}_\beta(g^{-1},g)\,\frac{dg^\beta}{dt} = \frac{\partial K}{\partial \pi_\alpha^L} = I^{\alpha\mu}\pi_\mu^L \tag{3.4}$$

$$\frac{d\pi_\mu^L}{dt} = -\langle \mathbf{d}V|\mathbf{e}_\mu^L\rangle + \frac{\partial K}{\partial \pi_\alpha^L}\left(\pi_\kappa^L\,\mathbf{f}^\kappa{}_{\alpha\mu} - \Theta_{\alpha\mu}\right) \tag{3.5}$$

The relation between the velocity in the body frame and the angular momentum **(2.13)** is maintained: $\pi_\mu^L = I_{\mu\nu}\,\Omega_L^\nu$, while the second **(2.14)** takes the interaction into account:

$$\frac{d\pi_\mu^L}{dt} + \pi_\kappa^L\,\mathbf{f}^\kappa{}_{\mu\alpha}\,\Omega_L^\alpha = -\langle \mathbf{d}V|\mathbf{e}_\mu^L\rangle - \Omega_L^\alpha\,\Theta_{\alpha\mu} \tag{3.6}$$

For a semisimple Lie algebra $\mathcal{G}$, we have $\Theta_{\alpha\beta} = -\xi_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}$ and we may define a modified Liouville one-form:

$$\langle \theta_I| = \pi'_\mu\,\langle \varepsilon_L^\mu| \;,\; \pi'_\mu \doteq \pi_\mu^L + \xi_\mu \tag{3.7}$$

and the symplectic two-form reads

$$\omega_I = -\mathbf{d}\langle \theta_I| = \langle \varepsilon_L^\mu| \wedge \langle \mathbf{d}\pi'_\mu| + \frac{1}{2}\,\pi'_\mu\,\mathbf{f}^\mu{}_{\alpha\beta}\,\langle \varepsilon_L^\alpha| \wedge \langle \varepsilon_L^\beta| \tag{3.8}$$

This means that such that $\{g^\alpha, p'_\mu = p_\mu + \xi_\beta\,L^\beta{}_\mu(g^{-1};g)\}$ are Darboux coordinates:

$$\langle \theta_I| = p'_\mu\,\langle \mathbf{d}g^\mu| \;,\; \omega_I \doteq -\mathbf{d}\langle \theta_I| = \langle \mathbf{d}g^\mu| \wedge \langle \mathbf{d}p'_\mu| \tag{3.9}$$

In $\left(g^\alpha, \pi'_\mu\right)$ coordinates, the Hamiltonian reads

$$H' = K'(\pi') + V(g) = \frac{1}{2}\,I^{\mu\nu}\,(\pi'_\mu - \xi_\mu)\,(\pi'_\nu - \xi_\nu) + V(g) \tag{3.10}$$

and the Euler equations read:

$$L^\alpha{}_\beta(g^{-1},g)\,\frac{dg^\beta}{dt} = \frac{\partial K'}{\partial \pi'_\alpha} = I^{\alpha\mu}\,(\pi'_\mu - \xi_\mu) \tag{3.11}$$

$$\frac{d\pi'_\mu}{dt} = -\langle \mathbf{d}V|\mathbf{e}_\mu^L\rangle + \frac{\partial K'}{\partial \pi'_\alpha}\left(\pi'_\kappa\,\mathbf{f}^\kappa{}_{\alpha\mu}\right) \tag{3.12}$$

which, obviously are equivalent to **(3.4)** and **(3.12)**.

4. THE CLOSED TWO-FORM ω_L

Configuration space coordinates which do not Poisson commute, are obtained through the addition of a left-invariant and closed two-form to **(3.1)**:

$$\Upsilon^L \doteq \frac{1}{2}\Upsilon^{\mu\nu}\langle \mathbf{d}\pi^L_\mu| \wedge \langle \mathbf{d}\pi^L_\nu| \tag{4.1}$$

$$\begin{aligned}\omega_L \doteq \omega_0 - \Theta_L + \Upsilon^L &= \langle \varepsilon^\mu_L| \wedge \langle \mathbf{d}\pi^L_\mu| + \frac{1}{2}\left(\pi^L_\mu\, \mathbf{f}^\mu{}_{\alpha\beta} - \Theta_{\alpha\beta}\right)\langle \varepsilon^\alpha_L| \wedge \langle \varepsilon^\beta_L| \\ &+ \frac{1}{2}\Upsilon^{\mu\nu}\langle \mathbf{d}\pi^L_\mu| \wedge \langle \mathbf{d}\pi^L_\nu| \end{aligned}\tag{4.2}$$

With the notation $S_{\alpha\beta} \equiv \left(\pi^L_\mu\, \mathbf{f}^\mu{}_{\alpha\beta} - \Theta_{\alpha\beta}\right)$, we wite ω_L in matrix form:

$$\omega_L \equiv \frac{1}{2}\left(\langle \varepsilon^\alpha_L| \quad \langle \mathbf{d}\pi^L_\mu|\right) \wedge \begin{pmatrix} S_{\alpha\beta} & \delta_\alpha{}^\nu \\ -\delta^\mu{}_\beta & \Upsilon^{\mu\nu} \end{pmatrix} \begin{pmatrix} \langle \varepsilon^\beta_L| \\ \langle \mathbf{d}\pi^L_\nu| \end{pmatrix} \tag{4.3}$$

The degeneracy of (ω_L) is examined comsidering the equation

$$\imath_{|\mathbf{X}\rangle}\omega_L = \langle \mathbf{d}A| \tag{4.4}$$

In the bases **(2.4)**, **(2.5)**: $X^\alpha \doteq \langle \varepsilon^\alpha_L|\mathbf{X}\rangle$, $X_\mu \doteq \langle \varepsilon^L_\mu|\mathbf{X}\rangle$ and **(4.4)** reads:

$$\begin{aligned} X^\alpha \Phi_\alpha{}^\nu &= \langle \mathbf{d}A|\mathbf{e}^\nu_L\rangle + \langle \mathbf{d}A|\mathbf{e}^L_\mu\rangle \Upsilon^{\mu\nu}, \\ X_\mu \Psi^\mu{}_\beta &= -\langle \mathbf{d}A|\mathbf{e}^L_\beta\rangle + \langle \mathbf{d}A|\mathbf{e}^\alpha_L\rangle S_{\alpha\beta} \end{aligned}\tag{4.5}$$

where we introduced the matrices, linear in the momenta:

$$\Phi_\alpha{}^\nu \doteq \delta_\alpha{}^\nu + S_{\alpha\mu}\Upsilon^{\mu\nu}\ ,\ \Psi^\mu{}_\beta \doteq \delta^\mu{}_\beta + \Upsilon^{\mu\nu}S_{\nu\beta} \tag{4.6}$$

They are mutually transposed and the products $\Phi S = S\Psi$, $\Upsilon\Phi = \Psi\Upsilon$ are antisymmetric. The fundamental equation **(4.4)**, defining Hamiltonian vector fields, has a solution if Φ and Ψ have inverses, i.e. if

$$\Delta \doteq \det\Phi \equiv \det\Psi \neq 0 \tag{4.7}$$

The matrices $\Upsilon\Phi^{-1} = \Psi^{-1}\Upsilon$ and $\Phi^{-1}S = S\Psi^{-1}$ are then also antisymmetric. The Hamiltonian vector fields are obtained as:

$$\begin{aligned} X^\alpha &= (\Psi^{-1})^\alpha{}_\mu\left(\langle \mathbf{d}A|\mathbf{e}^\mu_L\rangle - \Upsilon^{\mu\nu}\langle \mathbf{d}A|\mathbf{e}^L_\nu\rangle\right) \\ &= \left(\langle \mathbf{d}A|\mathbf{e}^\nu_L\rangle + \langle \mathbf{d}A|\mathbf{e}^L_\mu\rangle \Upsilon^{\mu\nu}\right)(\Phi^{-1})_\nu{}^\alpha \\ X_\mu &= (\Phi^{-1})_\mu{}^\alpha\left(-\langle \mathbf{d}A|\mathbf{e}^L_\alpha\rangle - S_{\alpha\beta}\langle \mathbf{d}A|\mathbf{e}^\beta_L\rangle\right) \\ &= \left(-\langle \mathbf{d}A|\mathbf{e}^L_\beta\rangle + \langle \mathbf{d}A|\mathbf{e}^\alpha_L\rangle S_{\alpha\beta}\right)(\Psi^{-1})^\beta{}_\mu \end{aligned}\tag{4.8}$$

The Poisson brackets between the basic dynamical variables are:

$$\begin{aligned} \left\{g^\alpha, g^\beta\right\}_L &= -L^\alpha{}_\kappa(g,e)\, L^\beta{}_\lambda(g,e)\,\Upsilon^{\kappa\mu}(\Phi^{-1})_\mu{}^\lambda \\ \left\{g^\alpha, \pi^L_\nu\right\}_L &= L^\alpha{}_\kappa(g,e)\,(\Psi^{-1})^\kappa{}_\nu\ , \\ \left\{\pi^L_\mu, g^\beta\right\}_L &= -L^\beta{}_\kappa(g,e)\,(\Psi^{-1})^\kappa{}_\mu \\ \left\{\pi^L_\mu, \pi^L_\nu\right\}_L &= -S_{\mu\kappa}(\Psi^{-1})^\kappa{}_\nu \end{aligned}\tag{4.9}$$

For a Hamiltonian $H = K + V$, the equations of motion are:

$$\begin{aligned} \Omega^\alpha_L \doteq L^\alpha{}_\beta(g^{-1}, g)\frac{dg^\beta}{dt} &= \left(\frac{\partial K}{\partial \pi^L_\nu} + \langle \mathbf{d}V|\mathbf{e}^L_\mu\rangle \Upsilon^{\mu\nu}\right)(\Phi^{-1})_\nu{}^\alpha \\ \frac{d\pi^L_\mu}{dt} &= \left(-\langle \mathbf{d}V|\mathbf{e}^L_\beta\rangle + \frac{\partial K}{\partial \pi^L_\alpha}S_{\alpha\beta}\right)(\Psi^{-1})^\beta{}_\mu \end{aligned}$$

Since Φ, Ψ are linear in π^L, Δ is a polynomial in π^L of degree at most equal to N, the dimension of the Lie group. It defines an algebraic variety in $\mathcal{G}^\star$:

$$\Pi_1 \doteq \{(g, \pi^L) | \Delta(\pi^L) = 0\} \tag{4.10}$$

and its complement $\mathcal{V}_\Delta \doteq \mathcal{G}^\star \backslash \Pi_1$ defines a manifold

$$\mathcal{M}'_0 \doteq G \times \mathcal{V}_\Delta \tag{4.11}$$

with symplectic structure given by ω_L, restricted to $\mathcal{M}'_0$. If it happens that Π_1 itself is an algebraic manifold, an imbedded submanifold is obtained:

$$\mathcal{M}_1 \doteq G \times \Pi_1 \tag{4.12}$$

with imbedding in $\mathcal{M}_0 \doteq G \times \mathcal{G}^\star$: $j_1 : \mathcal{M}_1 \hookrightarrow \mathcal{M}_0$. The system is then constrained to $\mathcal{M}_1$ and we may look for solutions of **(4.4)** restricted to $\mathcal{M}_1$. Such solutions may exist if further conditions are imposed on the Hamiltonian. To proceed systematically, we follow the algorithm of Gotay, Nester and Hinds [11]. To keep things simple, this will be done in the next section for the semi-simple group $SU(2)$.

5. A CASE STUDY: $SU(2)$

The dynamical variables are functions on $\mathcal{M}_0 \doteq SU(2) \times su(2)^\star$. A basis $\{\mathbf{e}_\alpha\}$ of the Lie algebra $su(2)$ may be chosen such that its structure constants are the Kronecker symbols $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\mu \varepsilon^\mu{}_{\alpha\beta}$. The Killing metric $\eta_{\alpha\beta} \doteq \varepsilon^\mu{}_{\alpha\nu}\varepsilon^\nu{}_{\beta\mu} = -2\delta_{\alpha\beta}$, provides an isomorphism between $su(2)$ and $su(2)^\star$. The metric $\delta_{\alpha\beta}$ with inverse $\delta^{\mu\nu}$ will be freely used to raise or to lower indices. Θ_L is written in terms of a *magnetic field* ξ_μ as $\Theta_{\alpha\beta} = -\xi_\kappa\varepsilon^\kappa{}_{\alpha\beta}$ and any antisymmetric Υ can be written in terms of τ^λ, a *dual magnetic field in momentum space*, as $Y^{\mu\nu} = \tau^\lambda \varepsilon_\lambda{}^{\mu\nu}$. Defining $\pi'_\kappa \doteq \pi^L_\kappa + \xi_\kappa$, ω_L reads:

$$\omega_L \equiv \frac{1}{2}\left(\langle\varepsilon^\alpha_L| \quad \langle \mathbf{d}\pi^L_\mu|\right) \wedge \begin{pmatrix} \pi'_\kappa \varepsilon^\kappa{}_{\alpha\beta} & \delta_\alpha{}^\nu \\ -\delta^\mu{}_\beta & \tau^\lambda \varepsilon_\lambda{}^{\mu\nu} \end{pmatrix} \begin{pmatrix} \langle\varepsilon^\beta_L| \\ \langle \mathbf{d}\pi^L_\nu| \end{pmatrix} \tag{5.1}$$

The fundamental equation **(4.4)**: $\imath_{|\mathbf{X}\rangle}\,\omega_L = \langle \mathbf{d}H|$ becomes:

$$X^\alpha \pi'_\kappa \varepsilon^\kappa{}_{\alpha\beta} - X_\beta = H_\beta \;,\; X^\nu + X_\mu \tau^\lambda \varepsilon_\lambda{}^{\mu\nu} = H^\nu$$

where $H_\beta \doteq (\partial H/\partial g^\alpha)\, L^\alpha{}_\beta(g,e)$, $H^\nu \doteq \left(\partial H/\partial \pi^L_\nu\right)$. The matrices **(4.6)** are given explicitely by $\Phi_\alpha{}^\nu \doteq C_1\,\delta_\alpha{}^\nu + \tau_\alpha \pi'^\nu$ and $\Psi^\mu{}_\beta \doteq C_1\,\delta^\mu{}_\beta + \pi'^\mu \tau_\beta$, where $C_1 \doteq (1 - \pi'\cdot\tau)$. They obey $\Phi_\alpha{}^\nu \left(\delta_\nu{}^\beta - \tau_\nu \pi'^\beta\right) = C_1\,\delta_\alpha{}^\beta$ and $\Psi^\mu{}_\beta \left(\delta^\beta{}_\nu - \pi'^\beta \tau_\nu\right) = C_1\,\delta^\mu{}_\nu$. It follows that **(4.5)** implies:

$$X^\alpha (1-\pi'\cdot\tau) = H^\alpha - \pi'^\alpha (\tau_\beta H^\beta) - \varepsilon^{\alpha\mu}{}_\nu\, H_\mu\, \tau^\nu \tag{5.2}$$
$$X_\mu (1-\pi'\cdot\tau) = -H_\mu + \tau_\mu (\pi'^\nu H_\nu) - \varepsilon_{\mu\alpha}{}^\beta\, H^\alpha\, \pi'_\beta \tag{5.3}$$

5.1. The non degenerate case

The determinant of the matrices Φ and Ψ is given by $\Delta = (C_1)^2$. Obviously the plane $\Pi_1 \doteq \{(g,\pi^L)|(1-\pi'\cdot\tau) = 0\}$ is an algebraic manifold in $\mathcal{G}^\star$. Its complement $\mathcal{V}_\Delta \doteq \mathcal{G}^\star \backslash \Pi_1$ defines a manifold $\mathcal{M}'_0 \doteq G \times \mathcal{V}_\Delta$ with symplectic structure ω_L, retricted to $\mathcal{M}'_0$. On $\mathcal{M}'_0$, Φ and Ψ have inverses:

$$(\Psi^{-1})^\beta{}_\nu = (C_1)^{-1}\left(\delta^\beta{}_\nu - \pi'^\beta \tau_\nu\right) \;,$$
$$(\Phi^{-1})_\nu{}^\beta = (C_1)^{-1}\left(\delta_\nu{}^\beta - \tau_\nu \pi'^\beta\right) \tag{5.4}$$

For a Hamiltonian $H = K(\pi^L) + V(g)$, the Hamiltonian vector fields are read off from **(5.2)** and **(5.3)** with ensuing equations of motion:

$$\Omega^\alpha_L \doteq L^\alpha{}_\beta(g^{-1},g)\,\frac{dg^\beta}{dt} = \left(\frac{\partial K}{\partial \pi^L_\nu} + \langle \mathbf{d}V|\mathbf{e}^L_\mu\rangle\, \tau^\lambda \varepsilon_\lambda{}^{\mu\nu}\right)(\Phi^{-1})_\nu{}^\alpha$$
$$\frac{d\pi^L_\mu}{dt} = \left(-\langle \mathbf{d}V|\mathbf{e}^L_\beta\rangle + \frac{\partial K}{\partial \pi^L_\alpha}\,\pi'_\kappa \varepsilon^\kappa{}_{\alpha\beta}\right)(\Psi^{-1})^\beta{}_\mu \tag{5.5}$$

For a purely kinetic Hamiltonian, we obtain:

$$\Omega^\alpha_L = \frac{\partial K}{\partial \pi^L_\mu}\,(\Phi^{-1})_\mu{}^\alpha \;,\; \frac{d\pi^L_\mu}{dt} = \Omega^\alpha_L \pi'_\beta\, \varepsilon^\beta{}_{\alpha\mu} \tag{5.6}$$

5.2. The degenerate case

The equation $C_1 \equiv (1-\pi'\cdot\tau) = 0$ defines a two dimensional plane Π_1 in $su(2)^\star \cong \mathbf{R}^3$. The *primary constrained manifold*, defined by $\mathcal{M}_1 \doteq SU(2)\times\Pi_1$, is imbedded in $\mathcal{M}_0 \doteq SU(2)\times su(2)^\star$. On $\mathcal{M}_1$, the closed two-form ω_L is degenerate and the pairing of $\pi' \in su(2)^\star$ with $\tau \in su(2)$ equals 1. So $|\tau\rangle \neq 0$ and, without loss of generality, we take $\{\tau^\alpha\} = \{0,0,\tau\}$. In what follows, greek indices $\{\alpha,\beta,\mu,\nu,\cdots\}$ shall vary in $\{1,2,3\}$, while latin indices $\{a,b,m,n,\cdots\}$ assume only the values $\{1,2\}$. The imbedding is given by:

$$a\,j_1 \;:\; \mathcal{M}_1 \hookrightarrow \mathcal{M}_0 :$$
$$x_1 \equiv (g^\alpha,\pi^L_m) \to x_0 = j_1(x_1) \equiv (g^\alpha,\pi^L_m,\pi^L_3 = 1/\tau - \xi_3) \tag{5.7}$$

with its differential or push-forward:

$$j_{1\star} : T\mathcal{M}_1 \to T\mathcal{M}_0 : (x_1;X^\alpha,X_m) \to (x_0;X^\alpha,X_m,X_3=0) \tag{5.8}$$

The pull-back transforms forms on $\mathcal{M}_0$ into forms on $\mathcal{M}_1$:

$$j_1{}^\star : \bigwedge{}^\bullet(T^\star\mathcal{M}_0) \to \bigwedge{}^\bullet(T^\star\mathcal{M}_1) \tag{5.9}$$

In particular the pull-back of ω_L to the five dimensional manifold $\mathcal{M}_1$ is

$$\widetilde{\omega}_{L|1} \doteq j_1{}^\star(\omega_L) \tag{5.10}$$

The restriction of ω_L to $\mathcal{M}_1$, not to be confused with its pull-back, is denoted by $\omega_{L|1} \doteq \omega_L \circ j_1$. In matrix representation:

$$\omega_{L|1} = \frac{1}{2}\left(\langle\varepsilon^\alpha_L| \quad \langle \mathbf{d}\pi^L_\mu|\right) \wedge \begin{pmatrix} 0 & 1/\tau & -\pi'_2 & 1 & 0 & 0 \\ -1/\tau & 0 & \pi'_1 & 0 & 1 & 0 \\ \pi'_2 & -\pi'_1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \tau & 0 \\ 0 & -1 & 0 & -\tau & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle\varepsilon^\beta_L| \\ \\ \langle \mathbf{d}\pi^L_\nu| \end{pmatrix} \tag{5.11}$$

Let $(T\mathcal{M}_0)_{|1} \doteq \{(x,\mathbf{X}) \in T\mathcal{M}_0\,|\,x \in \mathcal{M}_1\}$ be the subbundle of $T\mathcal{M}_0$ restricted to $\mathcal{M}_1$. Following the GNH algorithm [11], we look for a vector field $|\mathbf{X}\rangle$ in $(T\mathcal{M}_0)_{|1}$, tangent to $\mathcal{M}_1$ and solution of $\imath_{|\mathbf{X}\rangle}\omega_{L|1} = \langle \mathbf{d}H| \circ j_1$.

Explicitely :

$$-(1/\tau)X2 + \pi'_2 X3 - X_1 = \langle \mathbf{d}V|\mathbf{e}^L_1\rangle$$
$$+(1/\tau)X1 - \pi'_1 X3 - X_2 = \langle \mathbf{d}V|\mathbf{e}^L_2\rangle$$
$$-\pi'_2 X1 + \pi'_1 X2 - X_3 = \langle \mathbf{d}V|\mathbf{e}^L_3\rangle$$

$$X1 - \tau X_2 = \partial K/\partial \pi_1^L$$
$$X2 + \tau X_1 = \partial K/\partial \pi_2^L$$
$$X3 = \partial K/\partial \pi_3^L$$

Two independent null vectors of $\omega_{L|1}$, solution of $\imath_{|\mathbf{Z}\rangle}\omega_{L|1} = 0$, are given by:

$$|\mathbf{Z}^1\rangle = |\mathbf{e}_1^L\rangle + (1/\tau)\,|\partial/\partial\pi_2^L\rangle - \pi_2'\,|\partial/\partial\pi_3^L\rangle$$
$$|\mathbf{Z}^2\rangle = |\mathbf{e}_2^L\rangle - (1/\tau)\,|\partial/\partial\pi_1^L\rangle + \pi_1'\,|\partial/\partial\pi_3^L\rangle \quad (5.12)$$

Consistency requires $\{\langle \mathbf{d}H|\mathbf{Z}^a\rangle = 0\}$ for $(a = 1,2)$ and $\pi_3' = 1/\tau$.

$$C_{21} \equiv \pi_2'\,(\partial K/\partial\pi_3^L) - \pi_3'\,(\partial K/\partial\pi_2^L) - \langle \mathbf{d}V|\mathbf{e}_1^L\rangle = 0$$
$$C_{22} \equiv \pi_3'\,(\partial K/\partial\pi_1^L) - \pi_1'\,(\partial K/\partial\pi_3^L) - \langle \mathbf{d}V|\mathbf{e}_2^L\rangle = 0 \quad (5.13)$$

These two equations define a secondary constrained manifold $\mathcal{M}_2 \subset \mathcal{M}_1$, on which a particular solution of **(??)** is

$$|\mathbf{X}_P\rangle = |\mathbf{e}_1^L\rangle\,\partial K/\partial\pi_1^L + |\mathbf{e}_2^L\rangle\,\partial K/\partial\pi_2^L + |\mathbf{e}_3^L\rangle\,\partial K/\partial\pi_3^L + |\partial/\partial\pi_3^L\rangle\, C_{23} \quad (5.14)$$

where $C_{23} \equiv \pi_1'\,(\partial K/\partial\pi_2^L) - \pi_2'\,(\partial K/\partial\pi_1^L) - \langle \mathbf{d}V|\mathbf{e}_3^L\rangle$. The general solution $|\mathbf{X}_G\rangle$ of **(??)**, on $\mathcal{M}_2$, still contains two arbitrary functions ζ_1 and ζ_2:

$$(X_G) = \zeta_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1/\tau \\ -\pi_2' \end{pmatrix} + \zeta_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1/\tau \\ 0 \\ +\pi_1' \end{pmatrix} + \begin{pmatrix} \partial K/\partial\pi_1^L \\ \partial K/\partial\pi_2^L \\ \partial K/\partial\pi_3^L \\ 0 \\ 0 \\ C_{23} \end{pmatrix} \quad (5.15)$$

This vector must be tangent to $\mathcal{M}_1$ and $\mathcal{M}_2$. This leads to three equations

$$\langle \mathbf{d}C_1\,|\mathbf{X}_G\rangle = 0;\ \langle \mathbf{d}C_{21}\,|\mathbf{X}_G\rangle = 0;\ \langle \mathbf{d}C_{22}\,|\mathbf{X}_G\rangle = 0 \quad (5.16)$$

If these three equations determine or not the two arbitrary functions ζ_1 and ζ_2, will depend on the kinetic energy $K(\pi^L)$ and on the particular form of the potential $V(g)$. If they do so, the system will have a solution. If not, they will define a tertiary constraint manifold $\mathcal{M}_3$ and the analysis must proceed.

6. CONCLUSIONS

In this work, we analysed the consistency of a modification of the symplectic two-form on the cotangent bundle of a group manifold. This was done in order to obtain classical, i.e. Poisson, noncommuting configuration (group) coordinates. This was achieved in the non degenerate case, with the closed two-form ω_L which is then symplectic. We do not address here the general quantization problem of such a system and refer e.g. to [12] for a general review on quantization methods. It should be stressed that, whatever the quantisation scheme, any such obtained framework has little to do with *non commutative geometry*, either in the sense of A.Connes or as a quantum field theory on non-commutative spaces.

APPENDIX A: THE SYMPLECTIC ONE-COCYCLE

A one-cochain θ on $\mathcal{G}$ with values in $\mathcal{G}^\star$, on which $\mathcal{G}$ acts with the coadjoint representation $\mathbf{k}$, $\theta \in C^1(\mathcal{G},\mathcal{G}^\star,\mathbf{k})$, is a linear map $\theta : \mathcal{G} \to \mathcal{G}^\star : \mathbf{u} \to \theta(\mathbf{u})$. Its components are $\theta_{\alpha,\mu} \doteq \langle \theta(\mathbf{e}_\mu)|\mathbf{e}_\alpha\rangle$. It is a one-cocycle, $\theta \in Z^1(\mathcal{G},\mathcal{G}^\star,\mathbf{k})$, if its coboundary, $(\delta_1\theta)(\mathbf{u},\mathbf{v}) \doteq \mathbf{k}(\mathbf{u})\theta(\mathbf{v}) - \mathbf{k}(\mathbf{v})\theta(\mathbf{u}) - \theta([\mathbf{u},\mathbf{v}])$, vanishes.

$$\langle(\delta_1\theta)(\mathbf{u},\mathbf{v})|\mathbf{w}\rangle \doteq -\langle\theta(\mathbf{v})|[\mathbf{u},\mathbf{w}]\rangle + \langle\theta(\mathbf{u})|[\mathbf{v},\mathbf{w}]\rangle - \langle\theta([\mathbf{u},\mathbf{v}])|\mathbf{w}\rangle = 0$$
$$\langle(\delta_1\theta)(\mathbf{e}_\mu,\mathbf{e}_\nu)|\mathbf{e}_\alpha\rangle \doteq -\theta_{\kappa,\nu}\,\mathbf{f}^\kappa{}_{\mu\alpha} + \theta_{\kappa,\mu}\,\mathbf{f}^\kappa{}_{\nu\alpha} - \theta_{\kappa,\alpha}\,\mathbf{f}^\kappa{}_{\mu\nu} = 0$$

The one-cocycle σ is called symplectic if $\Sigma(\mathbf{u},\mathbf{v}) \doteq \langle\sigma(\mathbf{u})|\mathbf{v}\rangle$ is antisymmetric, $\Sigma(\mathbf{u},\mathbf{v}) = -\Sigma(\mathbf{v},\mathbf{u})$ or $\Sigma_{[\alpha\mu]} \doteq \sigma_{\alpha,\mu} = -\sigma_{\mu,\alpha}$. Any antisymmetric Θ defined in terms of $\theta \in C^1(\mathcal{G},\mathcal{G}^\star,\mathbf{k})$ as $\Theta_{[\alpha\beta]} = \theta_{\alpha,\beta}$ is actually a 2-cochain on $\mathcal{G}$ with values in $\mathbf{R}$ and trivial representation: $\Theta \in C^2(\mathcal{G},\mathbf{R},\mathbf{0})$. Furthermore, when $\theta \in Z^1(\mathcal{G},\mathcal{G}^\star,\mathbf{k})$, Θ is a 2-cocycle of $Z^2(\mathcal{G},\mathbf{R},\mathbf{0})$:

$$(\delta_2\Theta)(\mathbf{u},\mathbf{v},\mathbf{w}) \doteq -\Theta([\mathbf{u},\mathbf{v}],\mathbf{w}) + \Theta([\mathbf{u},\mathbf{w}],\mathbf{v}) - \Theta([\mathbf{v},\mathbf{w}],\mathbf{u}) = 0$$

$$(\delta_2\Theta)(\mathbf{e}_\alpha,\mathbf{e}_\beta,\mathbf{e}_\gamma) \doteq -\Theta_{\kappa\gamma}\mathbf{f}^\kappa{}_{\alpha\beta} + \Theta_{\kappa\beta}\mathbf{f}^\kappa{}_{\alpha\gamma} - \Theta_{\kappa\alpha}\mathbf{f}^\kappa{}_{\beta\gamma} = 0 \quad (A.1)$$

In general let $\Theta = \frac{1}{2}\,\Theta_{\alpha\beta}\,\varepsilon^\alpha \wedge \varepsilon^\beta \in \Lambda^2(\mathcal{G}^\star)$, obey the cocycle condition **(A.1)**. Acting with $L^\star{}_{g^{-1}|g}$ yields the left-invariant two form:

$$\Theta_L(g) \doteq L^\star{}_{g^{-1}|g}\,\Theta = \frac{1}{2}\,\Theta_{\alpha\beta}\,\varepsilon_L^\alpha(g) \wedge \varepsilon_L^\beta(g) \quad (A.2)$$

Using the cocycle relation and the Maurer-Cartan structure equations, it is seen that $\Theta_L(g)$ is a closed left-invariant two-form on G.

When $\mathcal{G}$ is semisimple, Θ is exact. Indeed, the Whitehead lemmas state that $H^1(\mathcal{G},\mathbf{R},\mathbf{0}) = 0$ and $H^2(\mathcal{G},\mathbf{R},\mathbf{0}) = 0$. In particular, $\Theta \in B^2(\mathcal{G},\mathbf{R},\mathbf{0})$ is a coboundary and there exists an element ξ of $C^1(\mathcal{G},\mathbf{R},\mathbf{0}) \equiv \mathcal{G}^\star$ such that $\Theta(\mathbf{u},\mathbf{v}) = (\delta_1(\xi))(\mathbf{u},\mathbf{v}) = -\xi([\mathbf{u},\mathbf{v}])$ or

$$\Theta_{\alpha\beta} = -\xi_\mu\,\mathbf{f}^\mu{}_{\alpha\beta} \quad (A.3)$$

The constant vector $\xi \in T^\star(\mathcal{G})$ is the analogue of a magnetic field in the abelian case $G \equiv \mathbf{R}^3$.

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