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ON THE DEFLECTION OF RECTANGULAR PLANE PLATES

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An accurate calculus is presented for the deflection coefficient of rectangular plates when they are loaded on two of their sides, for various cases of loading, from pure bending to uniform compression and for various ratios of the plates sides, as well as a comparison of these values with those calculated using the relations found in SR 1911-98.

1. Introduction

The equation of a rectangular plane plate that has been deflected after being acted upon by certain forces within its median plane (Fig. 1) has the form:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right). \quad (1)$$

where: w is the displacement of plate normal to its plane (bending deflection) due to deflection: N_x, N_{xy}, N_y – the forces acting in the median plane of the plate, that can produce its deflection (considered on unit of side length); $D = Et^3/12(1-\mu^2)$ – bending stiffness of plate for a section of unit length and thickness t.

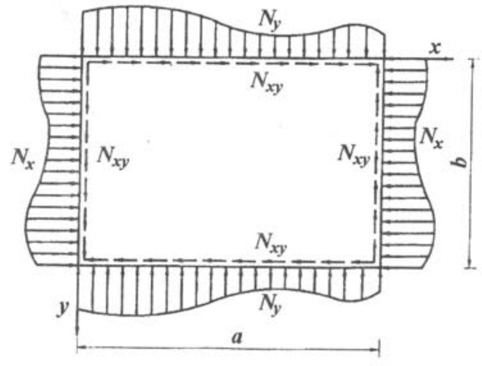

Fig. 1

The values of the forces N_x, N_{x_y}, N_y, that produce the deflection of the plate – the critical values – result from solving Eq. (1) and depend on their manner of distribution over the contour of the plate, the boundary conditions, and the dimensions of the plate. The solving of the equation will provide sets of critical values corresponding to the different manners of plate deflection, the least values set being of practical interest. The equation is difficult to solve, especially when the loads N_x, N_{xy}, N_y are variable along the sides.

2. The Deflection of Plates Loaded over Two Opposite Sides

A very interesting case is that of the plate of the plate girders loaded in bending or bending with compression over two opposite sides, case in which the Eq. (1) becomes:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} N_x \frac{\partial^2 w}{\partial x^2}, \tag{2}$$

where the load N_x is given by the relation (s. Fig. 2, as well):

$$N_x = N_0 \left(1 - \beta \frac{y}{b}\right). \tag{3}$$

The shape of N_x loading depends on the value of β coefficient (for $\beta = 0$ the load is uniform compression, while for $\beta = 2$, the load is pure bending, as between these values we have the case of a composite loading – bending with compression).

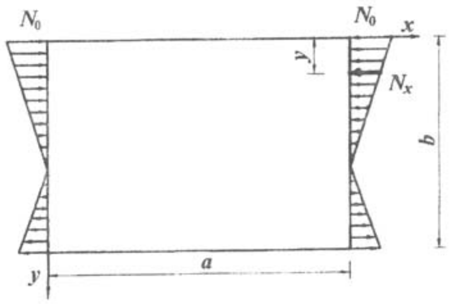

Fig. 2

The accurate solving of equation (2), with N_x given by (3), is difficult because the differential equation of the fourth order with partial derivatives has variable coefficients. For determination of the critical value of the compressive force, N_0, we can use the energetic method, according to which, at the moment when the plate was deflected, the increase in the strain energy, ΔU, is equal to the work of the exterior forces acting on the plate, ΔT, resulting an equation in $(N_0)_{cr}$.

The expressions of the two variables for a plate loaded in the median plane (Fig. 1) are:

$$\Delta U = \frac{1}{2} D \int_0^a \int_0^b \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\mu) \left[\frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} \mathrm{d}x \, \mathrm{d}y, \tag{4}$$

(5) $$\Delta T = \frac{1}{2}\int_0^a\int_0^b \left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y.$$

In the case of the plate in Fig. 2, N_x is given by the expression (3), $N_{xy} = N_y = 0$, and the plate displacement after deflection, if we consider that the plate has been articulated over its contour, can be represented by a double trigonometric series:

(6) $$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

It is necessary to observe that expression (6) meets the boundary conditions, *i.e.*, along the sides $x=0$, $x=a$, $y=0$, $y=b$ the deflections $w=0$ and the bending moments along these sides, given by the expressions

(7) $$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right), \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right),$$

are equal to zero.

In expression (6) m and n signify the number of semi-waves of plate deflection along the directions x and y, respectively.

By introducing expressions (3) and (6) in (4) and (5) and equalizing the last ones, we can obtain the expression of critical deflection load:

(8) $$(N_0)_{\mathrm{cr}} = \frac{\pi^4 D \sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} a_{mn}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}{\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} a_{mn}^2\frac{m^2\pi^2}{a^2} - \frac{\beta}{2}\sum\limits_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}\left[\sum\limits_{n=1}^{\infty} a_{mn}^2 - \frac{32}{\pi^2}\sum\limits_{n=1}^{\infty}\sum\limits_{i}^{\infty}\frac{a_{mn}a_{mi}ni}{(n^2-i^2)^2}\right]},$$

where, for i are considered values that render $n+i$ an odd number.

The minimum value of $(N_0)_{\mathrm{cr}}$ can be obtained from the relation:

(9) $$\frac{\partial\lfloor(N_0)_{\mathrm{cr}}\rfloor}{\partial a_{mn}} = 0.$$

A system of linear equations is obtained:

(10) $$D a_{mn}\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 = (N_0)_{\mathrm{cr}}\left\{a_{mn}\frac{m^2\pi^2}{a^2} - \frac{\beta}{2}\cdot\frac{m^2\pi^2}{a^2}\left[a_{mn} - \frac{16}{\pi^2}\sum_{i}^{\infty}\frac{a_{mi}ni}{(n^2-i^2)^2}\right]\right\}.$$

The coefficients a_{mn} are considered as unknowns in system equations (10), the system being satisfied for zero values of these coefficients (the ordinary solution), which corresponds to the unstrained plate (because w in relation (6) equals zero); for the deflected plate, *i.e.*, values different from zero of the coefficients a_{mn}, it is necessary that the system (10) determinant should be annulled. The more equations of system (10) are considered, the more accurate the critical deflection load will be.

Taking $m=1$ (the deflection of the plate along axis x is taken as a deflection semi-wave) in expression (10) and writing:

$$\sigma_{\text{cr}} = \frac{(N_0)_{\text{cr}}}{t}, \tag{11}$$

where t is plate thickness, one can write the system of linear equations:

$$a_{1n}\left[\left(1+n^2\frac{a^2}{b^2}\right)^2 - \sigma_{\text{cr}}\frac{a^2t}{\pi^2D}\left(1-\frac{\beta}{2}\right)\right] - 8\beta\sigma_{\text{cr}}\frac{a^2t}{\pi^4D}\sum_i^{\infty}\frac{a_{1i}ni}{(n^2-i^2)^2} = 0. \tag{12}$$

From system (12) can be obtained the value of σ_{cr} in various approximations, in a first approximation taking $n=1$, in a second one considering $n=1$, 2 and so on.

It has been found that sufficiently accurate values can be obtained in the third approximation ($n=1$, 2, 3), the convergence being high enough, while the fourth approximation gives values that differ from the third one in a non-significant way.

As it is common in the literature to use for σ_{cr} the expression:

$$\sigma_{\text{cr}} = k\frac{\pi^2D}{b^2t}, \tag{13}$$

the term $\frac{b^2t}{\pi^2D}\sigma_{\text{cr}}$ is made evident in equation (12) and we get:

$$a_{1n}\left[\left(1+n^2\frac{a^2}{b^2}\right)^2 - \left(1-\frac{\beta}{2}\right)\frac{a^2}{b^2}k\right] - \frac{8\beta}{\pi^2}\cdot\frac{a^2}{b^2}k\sum_i^{\infty}\frac{a_{1i}ni}{(n^2-i^2)^2} = 0. \tag{14}$$

Making in (14), in turn, $n=1$, $(i=2)$, $n=2$, $(i=1,3)$ and $n=3$, $(i=2)$, one obtains the system:

$$\begin{cases} a_{11}\left[\left(1+\frac{a^2}{b^2}\right)^2 - \left(1-\frac{\beta}{2}\right)\frac{a^2}{b^2}k\right] - a_{12}\frac{16\beta}{9\pi^2}\cdot\frac{a^2}{b^2}k = 0, \\ -a_{11}\frac{16\beta}{9\pi^2}\cdot\frac{a^2}{b^2}k + a_{12}\left[\left(1+4\frac{a^2}{b^2}\right)^2 - \left(1-\frac{\beta}{2}\right)\frac{a^2}{b^2}k\right] - a_{13}\frac{48\beta}{25\pi^2}\cdot\frac{a^2}{b^2}k = 0, \\ -a_{12}\frac{48\beta}{25\pi^2}\cdot\frac{a^2}{b^2}k + a_{13}\left[\left(1+9\frac{a^2}{b^2}\right) - \left(1-\frac{\beta}{2}\right)\frac{a^2}{b^2}k\right] = 0. \end{cases} \tag{15}$$

For various values of β (s. relation (3)) and various a/b ratios, one obtains, equalizing the system (15) determinant to zero, the values of deflection coefficient, k, shown in Table 1.

Table 1

β		a/b									
		0.4	0.5	0.6	0.667	0.7	0.8	0.9	1.0	1.5	2.0
2	$m=1$	29.3635	25.636	24.178	24.000	24.142	24.627	25.598	27.132	39.559	58.368
	$m=2$	–	–	–	–	–	–	–	25.634	24.142	50.964
$\frac{4}{3}$	$m=1$	18.889	15.044	12.977	12.137	11.482	11.241	10.999	11.013	13.317	17.913
	$m=2$	–	–	–	–	–	–	–	15.047	11.467	11.013
1	$m=1$	15.157	11.624	9.744	8.977	8.367	8.132	7.869	7.812	9.249	12.35
	$m=2$	–	–	–	–	–	–	–	11.627	8.368	7.812
$\frac{4}{5}$	$m=1$	13.301	9.437	8.353	7.662	7.113	6.899	6.657	5.352	7.772	10.36
	$m=2$	–	–	–	–	–	–	–	10.063	7.113	6.595
$\frac{2}{3}$	$m=1$	12.239	9.189	7.597	6.956	6.448	6.252	6.021	5.975	7.015	9.355
	$m=2$	–	–	–	–	–	–	–	9.189	6.448	5.963
$\frac{1}{3}$	$m=1$	10.041	7.481	6.151	5.621	5.20	5.115	4.848	4.793	5.63	7.497
	$m=2$	–	–	–	–	–	–	–	7.475	5.205	4.795
1	$m=1$	8.41	6.25	5.138	4.693	4.34	4.202	4.038	4.0	4.694	6.25
	$m=2$								6.25	4.340	4.0

For long plates ($a/b > 1$), the values of coefficient k, for $m=1$, are higher than those corresponding to $m=2$, that is, the long plates get deflected by two semi-waves of deflection.

The values of coefficient k for $m=2$ have been found establishing a system of linear equations (10) in which was introduced $m=2$, and from which can be particularized, for $n=1$, 2, 3, a system similar to (15). The values of coefficient k have been calculated only for the ratios $a/b=1$; 1.5; 2 (Table 2).

3. Values of Deflection Coefficient k According to SR 1911-98

For deflection plate testing in current design activity, the deflection coefficient values for various plate loadings as well as for various ratios of their sides, are provided simplified calculus relations that, when applied for the specific cases shown in Table 1, give values as presented in Table 2.

Table 2

β	a/b									
	0.4	0.5	0.6	0.667	0.7	0.8	0.9	1.0	1.5	2.0
2	28.933	25.500	24.160	23.899	23.9	23.9	23.9	23.9	23.9	23.9
4/3	18.125	14.232	12.370	11.717	11.268	11.093	10.892	10.835	10.835	10.835
1	16.055	11.932	9.808	8.959	8.286	8.023	7.721	7.636	7.636	7.636
4/5	13.585	10.096	8.299	7.580	7.011	6.789	6.533	6.461	6.461	6.461
2/3	12.32	9.159	7.529	6.877	6.36	6.158	5.927	5.862	5.862	5.862
1/3	10.00	7.429	6.107	5.578	5.159	4.995	4.808	4.755	4.755	4.755
0	8.41	6.25	5.139	4.693	4.34	4.202	4.045	4.00	4.00	4.00

If we compare the values of the deflection coefficient, k, in the two tables we can see that they are quite close for short plates (ratio $a/b < 1$) and the type of loading that nears uniform compression ($\beta=0$). For ratios $a/b \geq 1$ and types of loading that near pure compression ($\beta = 2$), there are larger differences, *e.g.* for the ratio $a/b = 1$ and $\beta = 2$, in Table 1, $k = 50.964$, while in Table 2, $k = 23.9$, a more restrictive value if compared to the accurate one.

4. Conclusions

The calculus relations of the deflection coefficient for rectangular plates loaded along two opposite sides, as given in SR 1911-98, provide sufficiently accurate values for short plates and less than accurate ones for long plates (ratio of sides higher than 1), values that are, nonetheless, within accepted limits, as they give critical values of plate loading that are lower than the real ones.

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ASUPRA VOALĂRII PLĂCILOR PLANE DREPTUNGHIULARE

(Rezumat)

Se prezintă un calcul exact al coeficientului de voalare pentru plăci dreptunghiulare solicitate la eforturi acţionând pe două laturi ale acestora, pentru diferite cazuri de solicitare, de la încovoiere pură până la compresiune uniformă şi diferite rapoarte ale laturilor plăcilor şi compararea acestor valori cu cele calculate cu relaţiile cuprinse în SR 1911-98.