Article ID: 0253-2778(2009)12-1248-05

Reliability analysis of varietal hypercube networks

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Abstract: As a variations of the hypercube network, the n-dimensional varietal hypercube VQ_n, proposed by Cheng and Chuang in 1994, has many desirable properties of the hypercube such as regularity and recursive structure. It was shown that the connectivity and the edge-connectivity of VQ_n are both equal to n, the restricted connectivity and the restricted edge-connectivity are both equal to $2n-2$, which implies that at least $2n-2$ vertices of VQ_n are removed to get a disconnected graph without isolated vertices.

Key words: connectivity; restricted connectivity; super connectivity; varietal hypercube; networks

CLC number: O157.5; TP302.1 **Document code**: A

AMS Subject Classification (2000): Primary 05C40; Secondary 90B10

变形超立方体网络的可靠性分析

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摘要：作为超立方体网络的变形，n 维变形超立方体 VQ_n 是 Cheng 和 Chuang 于 1994 年提出来的，它具有许多超立方体所具有的优良性质，比如正则性和递归结构. 证明了：VQ_n的连通度和边连通度都等于n，限制连通度和限制边连通度都等于$2n-2$. 这个结果意味着，为了使VQ_n 不连通且不含孤立点，至少有$2n-2$个点或者边要同时发生故障.

关键词：连通度；限制连通度；超连通度；变形超立方体；网络

0 Introduction

Throughout this paper, a graph $G=(V,E)$ always means a connected and simple graph with vertex-set V and edge-set E. We follow Ref. [1] for graph-theoretical terminology and notations not defined here. A set of vertices (resp., edges) S of G is called a vertex-cut (resp., an edge-cut) if $G-S$ disconnected. The connectivity $\kappa(G)$ (resp., the edge-connectivity $\lambda(G)$) of G is defined as the minimum cardinality of a vertex-cut (resp., an edge-cut) S. It is customary to define $\kappa(K_n)=n-1$, where K_n is a complete graph of order n.

It is well known that when the underlying topology of an interconnection network is modeled by a graph $G=(V,E)$, where V is the set of

Received: 2009-05-20; **Revised**: 2009-11-20

Foundation item: Supported by NNSF of China (10671191).

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processors and E is the set of communication links in the network, the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$ are two important parameters to measure reliability and fault tolerance of the network. The parameters, however, have an obvious deficiency, that it to tacitly assume that all elements in any subset of G can potentially fail at the same time. In other words, in the definition of $\kappa(G)$ or $\lambda(G)$, absolutely no conditions or restrictions are imposed either on the set S or on the components of $G-S$. Consequently, to compensate for these shortcomings, it would seem natural to generalize the notion of the classical connectivity by introducing some conditions or restrictions on the set S and/or the components of $G-S$.

Bauer et al[2] suggested the concept of the super connectedness. A connected graph G is said to be super vertex-connected (resp., edge-connected), if every minimum vertex-cut (resp., edge-cut) isolates a vertex of G. Esfahanian and Hakimi[3,4] introduced the concepts of the restricted cut and the restricted connectivity of a graph. A set $S \subset V(G)$ (resp., $S \subset E(G)$) is called a restricted vertex-set (resp., edge-set) if it does not contain the neighbor-set of any vertex in G as its subset. A restricted vertex-set (resp., edge-set) S is called a restricted vertex-cut (resp., edge-cut) if $G-S$ is disconnected. The restricted vertex-connectivity (resp., edge-connectivity) of G, denoted by $\kappa'(G)$ (resp., $\lambda'(G)$), is the minimum cardinality of a restricted vertex-cut (resp., edge-cut) in G. From definitions, the following proposition holds, clearly.

Proposition 0.1 Let G be a k-regular graph.

①If $\kappa'(G)$ exists and $\kappa'(G) > \kappa(G) = k$, then G must be super vertex-connected.

②If $\lambda'(G)$ exists and $\lambda'(G) > \lambda(G) = k$, then G must be super edge-connected.

Thus, as an important measurement for reliability and fault tolerance of a network, the restricted connectivity is more accurate than the classical connectivity, and has recently received much attention (see, for example, Refs. [5~12]). The restricted connectivity and the restricted edge-connectivity of many well-known graphs were determined by several researches, for example, Refs. [3,13~17].

In the present paper, we consider the n-dimensional varietal hypercube VQ_n, as an attractive alternative to the hypercube Q_n proposed by Cheng and Chuang[18] in 1994. It was proved that VQ_n has many desirable properties of Q_n such as regularity and recursive structure. In addition, its diameter $\left\lceil \frac{2n}{3} \right\rceil$ is smaller than that of Q_n for $n \geqslant 3$. However, its reliability and fault tolerance has not been investigated in the literature so far. We show that $\kappa(VQ_n) = \lambda(VQ_n) = n$ and $\kappa'(VQ_n) = \lambda'(VQ_n) = 2n-2$. These results can provide more accurate measurements for reliability and fault tolerance of the system when VQ_n is used to model the topological structure of a large-scale parallel processing system.

1 Definitions and lemmas

The n-dimensional varietal hypercube VQ_n is the labeled graph defined recursively as follows. VQ_1 is the complete graph of two vertices labeled with 0 and 1, respectively. Assume that VQ_{n-1} has been constructed. Let VQ_{n-1}^0 (resp. VQ_{n-1}^1) be a labeled graph obtained from VQ_{n-1} by inserting a zero (resp. 1) in front of each vertex-labeling in VQ_{n-1}. For $n > 1$, VQ_n is obtained by joining vertices in VQ_{n-1}^0 and VQ_{n-1}^1, according to the rule: a vertex $x = 0x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1$ in VQ_{n-1}^0 and a vertex $y = 1y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1$ in VQ_{n-1}^1 are adjacent in VQ_n if and only if

①$x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1 = y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1$ if $n \neq 3k$, or

②$x_{n-3}\cdots x_2x_1 = y_{n-3}\cdots y_2y_1$ and $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in I$ if $n = 3k$, where

$$I = \{(00,00),(01,01),(10,11),(11,10)\}.$$

Fig. 1 shows the examples of varietal hypercubes VQ_n for $n=1$, 2, 3 and 4.

The varietal hypercube VQ_n was first proposed

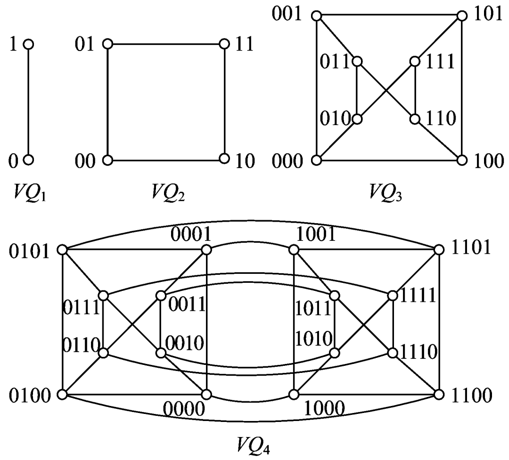

Fig. 1 The varietal hypercubes VQ_1, VQ_2, VQ_3 and VQ_4

by Cheng and Chuang[18] as an attractive alternative to the n-dimensional hypercube Q_n. Like Q_n, VQ_n is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges. In addition, VQ_n has a diameter of $\left\lceil \frac{2n}{3} \right\rceil$, smaller than that of the hypercube Q_n for $n\geqslant 3$. It was shown that VQ_n has optimal routing and broadcast algorithms which guarantee the shortest path communication. It was also shown that the ring and mesh structures can be embedded into VQ_n with dilation 1. These properties of the varietal hypercube will make it attractive for large scale interconnection networks.

From the definition, the varietal hypercube VQ_n has a simple recursive property, that is, VQ_n can be constructed from VQ^0_{n-1} and VQ^1_{n-1} by adding 2^{n-1} edges, called cross-edges, connecting all pairs of vertices between VQ^0_{n-1} and VQ^1_{n-1}. For convenience, we express VQ_n as $VQ_n = L\odot R$, where $L=VQ^0_{n-1}$ and $R=VQ^1_{n-1}$, and denote by x_Lx_R the cross-edge connecting $x_L\in L$ and $x_R\in R$. The recursive structure of VQ_n gives the following simple property.

Lemma 1.1 Let $VQ_n = L\odot R$ with $n\geqslant 2$. Then every vertex $x_L\in L$ has exactly one neighbor x_R in R connected by the cross-edge x_Lx_R.

Using this simple observation and the recursive property of VQ_n, we can also obtain the following property easily.

Lemma 1.2 VQ_n contains no triangle and any two nonadjacent vertices in VQ_n have at most two common neighbors for $n\geqslant 2$.

Proof We proceed by induction on $n\geqslant 2$. If $n=2$ there is nothing do to since both VQ^0_2 and VQ^1_2 are cycles of length four. Assume the induction hypothesis for $n-1$ when $n\geqslant 2$. Let $n\geqslant 3$ and $VQ_n=L\odot R$, where $L=VQ^0_{n-1}$ and $R=VQ^1_{n-1}$. By the induction hypothesis, neither L nor R contains triangle and any two nonadjacent vertices have at most two common neighbors for $n\geqslant 2$.

If VQ_n contains a triangle C_3, then, by the induction hypothesis, exactly two edges of C_3 are cross-edges with a common end-vertex, which, however, contradicts Lemma 1.1.

Let x and y be two nonadjacent vertices in VQ_n. By the induction hypothesis, without loss of generality, assume $x=x_L\in L$ and $y=y_R\in R$. By Lemma 1.1, x_L and y_R have at most one common neighbor $x_R\in R$ and at most one common neighbor $y_L\in L$. □

2 Connectivity of varietal hypercube VQ_n

In this section, we determine the connectivity, the super connectedness and the restricted connectivity of VQ_n. We first consider the connectivity of VQ_n.

Theorem 2.1 $\kappa(VQ_n)=\lambda(VQ_n)=n$ for $n\geqslant 1$.

Proof By Whitney's inequality (see Theorem 4.4 in Ref. [1]), we have $\kappa(VQ_n)\leqslant\lambda(VQ_n)\leqslant\delta(VQ_n)=n$. Thus, in order to prove the theorem, we only need to prove $\kappa(VQ_n)\geqslant n$. We proceed by induction on $n\geqslant 1$. The assertion is true if $n=1$ since VQ_1 is a complete graph K_2. Assume the induction hypothesis for $n-1$ when $n\geqslant 2$. To prove $\kappa(VQ_n)\geqslant n$, we only need to show that for any subset $F\subset V(VQ_n)$, if $|F|\leqslant n-1$ then VQ_n-F is connected.

Let $VQ_n = L\odot R$, and let $F_L = F\cap L$, and $F_R=F\cap R$. Then at least one of $L-F_L$ and $R-F_R$ is connected by the induction hypothesis. We can, without loss of generality, suppose that $R-F_R$ is connected. We show that any vertex u_L in $L-F_L$

can be connected to some vertex in $R-F_R$. Let u_Lu_R be the cross-edge in $VQ_n=L\odot R$. If $u_R\notin F_R$, then we are done. So we assume that $u_R\in F_R$. Consider $N=N_{VQ_n}(u_L)$, which is the neighbor-set of u_L in VQ_n. Since $|N|=n>n-1\geqslant|F|$, there is a vertex $x_L\in N$ such that both x_L and x_R are not in F. This implies that u_L in $L-F_L$ can be connected to the vertex x_R in $R-F_R$ via the cross-edge x_Lx_R. Thus, we show that $|F|\geqslant n$ for any vertex-cut F in VQ_n, that is, $\kappa(VQ_n)=|F|\geqslant n$. The theorem follows.

We now determine $\kappa'(VQ_n)$. Clearly, $\kappa'(VQ_1)$ and $\kappa'(VQ_2)$ do not exist. When $n\geqslant 3$, we have the following theorem.

Theorem 2.2 $\kappa'(VQ_n)=2n-2$ for $n\geqslant 3$.

Proof We first show $\kappa'(VQ_n)\leqslant 2n-2$ for $n\geqslant 3$. It is easy to verify that $\kappa'(VQ_3)\leqslant 4$ and $\kappa'(VQ_4)\leqslant 6$. Suppose $n\geqslant 5$ below. Let u and v be two adjacent vertices in VQ_n and let $S=N_{VQ_n}(u,v)$, which is the neighbor-set of $\{u,v\}$ in VQ_n. Then $|S|=2n-2$ since VQ_n is n-regular and contains no triangle by Lemma 1.2, and VQ_n-S is disconnected since $2^n-(2n-2)-2\geqslant 2$. Because $n\geqslant 5$ and any two distinct vertices have common neighbors at most two by Lemma 1.3, the neighbor-set $N_{VQ_n}(x)$ is not included in S for any $x\in V(VQ_n)$. This fact shows that S is a restricted vertex-cut of VQ_n. Thus $\kappa'(VQ_n)\leqslant|S|=2n-2$ for $n\geqslant 3$.

We now prove $\kappa'(VQ_n)\geqslant 2n-2$. To the end, we only need to show that for any restricted vertex-set F in VQ_n, if $|F|\leqslant 2n-3$ then VQ_n-F is connected.

Let $VQ_n=L\odot R$, and let $F_L=F\cap L$, and $F_R=F\cap R$. Obviously, $F_L\cap F_R=\emptyset$. Thus, either $|F_L|\leqslant n-2$ or $|F_R|\leqslant n-2$. We can, without loss of generality, suppose that $|F_R|\leqslant n-2$. Then $R-F_R$ is connected since $\kappa(R)=\kappa(VQ_{n-1})=n-1$. We show that any vertex u_L in $L-F_L$ can be connected to some vertex in $R-F_R$. Let u_Lu_R be the cross-edge in $VQ_n=L\odot R$. If $u_R\notin F_R$, then we are done. So we assume that $u_R\in F_R$. Since F is a restricted vertex-set, there exists a vertex v_L adjacent to u_L in $L-F_L$. Consider $N=N_{VQ_n}(u_L,v_L)$, the neighbor-set of $\{u_L,v_L\}$ in VQ_n. Since $|N|=2n-2>2n-3$, there is a vertex $x_L\in N$ such that neither x_L or x_R is in F. This implies that u_L in $L-F_L$ can be connected to the vertex x_R in $R-F_R$ via the cross-edge x_Lx_R.

Thus, we show that $|S|\geqslant 2n-2$ for any restricted vertex-cut S in VQ_n, that is, $\kappa'(VQ_n)=|S|\geqslant 2n-2$. The theorem follows. □

Corollary 2.3 VQ_n is super vertex-connected for any $n\geqslant 1$.

Proof Since VQ_1 is a complete graphs K_2 and VQ_2 is a cycle of length four, they are super vertex-connected clearly. By Theorem 2.1 and Theorem 2.2, we have $\kappa'(VQ_n)=2n-2>n=\kappa(VQ_n)$ for $n\geqslant 3$. By Proposition 0.1, VQ_n is also super vertex-connected for $n\geqslant 3$. □

Theorem 2.4 $\lambda'(VQ_n)=2n-2$ for $n\geqslant 2$.

Proof Consider an edge xy in VQ_n and the set of its adjacent edges $E(xy)=\{e\in E(VQ_n)\setminus\{xy\}: e=xu \text{ or } e=yu\in E(VQ_n), u\in V(VQ_n)\}$. Clearly, $VQ_n-E(xy)$ is disconnected and $|E(xy)|=2n-2$ since VQ_n is n-regular. Since VQ_n contains no triangles, $VQ_n-E(xy)$ contains no isolated vertex, thus, $E(xy)$ is a restricted edge-cut of VQ_n. This gives $\lambda'(VQ_n)\leqslant|E(xy)|=2n-2$.

We now show $\lambda'(VQ_n)\geqslant 2n-2$. Clearly, $\lambda'(VQ_2)\geqslant 2$. We assume that $n\geqslant 3$ and F is a restricted edge-set of VQ_n. We need to prove that if $|F|\leqslant 2n-3$ then VQ_n-F is connected. Let $VQ_n=L\odot R$. Then at least one of two $(n-1)$-dimensional varietal hypercubes L and R contains elements in F at most $\left\lfloor\frac{2n-3}{2}\right\rfloor\leqslant n-2$. Without loss of generality, assume that L contains elements in F at most $n-2$. Then, by Theorem 2.1, $L-F$ is a connected spanning subgraph of L. In order to prove that VQ_n-F is connected, we only need to show that any vertex x_R in R can be connected to some vertex in $L-F\cap E(L)$.

If the cross-edge x_Lx_R is not in F, then there is noting to do. Suppose that $x_Lx_R\in F$. Since F is a restricted edge-set, there exists an edge x_Ry_R in

R such that $x_R y_R \notin F$. Since VQ_n contains no triangle by Lemma 1.2, $|N_{VQ_n}(x_R, y_R)| = 2n-2 > 2n-3$. If $y_L y_R \notin F$, we are done. If $y_L y_R \in F$, then there exists at least one vertex $u_R \in N_{VQ_n}(x_R, y_R)$ such that the cross-edge $u_L u_R$ is not F and either $x_R u_R$ or $y_R u_R$ is not in F. Thus, x_R can be connected to the vertex u_L in L via the vertex u_R and the cross-edge $u_L u_R$.

Thus, we show that $|S| \geqslant 2n-2$ for any restricted edge-cut S in VQ_n, that is, $\lambda'(VQ_n) = |S| \geqslant 2n-2$. The theorem follows. □

Corollary 2.5 VQ_n is super edge-connected if $n \neq 2$.

Proof Since it is a complete graph K_2, VQ_1 is super edge-connected clearly. VQ_2 is a cycle of length four, it is not super edge-connected. By Theorem 2.1 and Theorem 2.4, we have

$$\lambda'(VQ_n) = 2n-2 > n = \lambda(VQ_n)$$

for $n \geqslant 3$. By Proposition 0.1, VQ_n is super edge-connected for $n \geqslant 3$. □

References

[1] Xu Jun-ming. Theory and Application of Graphs[M]. Dordrecht/Boston/London: Kluwer Academic Publishers, 2003.

[2] Bauer D, Boesch F, Suffel C, et al. Connectivity extremal problems and the design of reliable probabilistic networks [C]//The Theory and Application of Graphs. New York: Wiley, 1981: 89-98.

[3] Esfahanian A H. Generalized measures of fault tolerance with application to n-cube networks [J]. IEEE Trans Comput, 1989, 38(11): 1 586-1 591.

[4] Esfahanian A H, Hakimi S L. On computing a conditional edge-connectivity of a graph [J]. Information Processing Letters, 1988, 27: 195-199.

[5] Balbuena C, Garcia-Vazquez R, Marcote X. Sufficient conditions for λ'-optimality in graphs with girth g[J]. J Graph Theory, 2006, 52(1): 73-86.

[6] Bonsma P, Ueffing N, Volkmann L. Edge-cuts leaving components of order at least three [J]. Discrete Mathematics, 2002, 256(1-2):431-439.

[7] Hellwig A, Volkmann L. Sufficient conditions for λ'-optimality in graphs of diameter 2 [J]. Discrete Mathematics, 2004, 283(1-3): 113-120.

[8] Lü Min, Chen Guo-liang, Xu Jun-ming. On super edge-connectivity of Cartesian product graphs [J]. Networks, 2007, 49(2): 135-157.

[9] Meng J X. Optimally super-edge-connected transitive graphs[J]. Discrete Mathematics, 2003, 260 (1-3): 239-248.

[10] Meng J X, Ji Y H. On a kind of restricted edge connectivity of graphs [J]. Discrete Applied Mathematics, 2002, 117: 183-193.

[11] Volkmann L. Restricted arc-connectivity of digraphs [J]. Information Processing Letters, 2007, 103(6): 234-239.

[12] Xu Jun-ming, Xu Ke-li. On restricted edge-connectivity of graphs [J]. Discrete Mathematics, 2002, 243 (1-3): 291-298.

[13] Fan Ying-mei, Xu Jun-ming, Lü Min. The restricted edge-connectivity of Kautz undirected graphs[J]. Ars Combinatoria, 2006, 81: 369-379.

[14] Ma Mei-jie, Liu Gui-zhen, Xu Jun-ming. The super connectivity of augmented cubes [J]. Information Processing Letters, 2008, 106 (2):59-63.

[15] Wu J, Guo G. Fault tolerance measures for m-ary n-dimensional hypercubes based on forbidden faulty sets [J]. IEEE Trans Comput, 1998, 47: 888-893.

[16] Xu Jun-ming, Xu Min, Zhu Qiang. The super connectivity of shuffle-cubes [J]. Information Processing Letters, 2005, 96(4): 123-127.

[17] Xu Jun-ming, Zhu Qiang, Xu Min. Fault-tolerant analysis of a class of networks [J]. Information Processing Letters, 2007, 103(6):222-226.

[18] Cheng Shou-yi, Chuang Jen-hui. Varietal hypercube: A new interconnection networks topology for large scale multicomputer [C]//Proceedings of the 1994 International Conference on Parallel and Distributed Systems. IEEE, 1994: 703-708.